MATH 104, HOMEWORK #3 SOLUTIONS

DUE THURSDAY, FEBRUARY 4

Remember, consult the Homework Guidelines for general instructions.

GRADED HOMEWORK:

1. Give direct proofs for the two following limits. Do not use any of the shortcut theorems from Section 9 (e.g. limit of a sum, limit of a quotient).

   (a) Determine $\lim_{n\to\infty} \frac{3n^3+n^2}{4n^3-1001}$ and give a formal $\epsilon - N$ proof that your answer is correct.

   (b) Prove that $\lim_{n\to\infty} \frac{3n^3+n^2}{4n^2-1001} = +\infty$, using Definition 9.8 about sequences diverging to $\pm\infty$. (You might call this one an $M - N$ proof, though that is not a universal name.)

Solution. We will show that $\lim_{n\to\infty} \frac{3n^3+n^2}{4n^3-1001} = \frac{3}{4}$, i.e. for any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that $n > N$ implies

$$\left|\frac{3n^3+n^2}{4n^3-1001} - \frac{3}{4}\right| < \epsilon.$$

Proof: (a) Let $\epsilon > 0$ and take $N = \max\{10, \frac{35}{\epsilon}\}$. Let $n > N$, so that $n > 10$ and $n > \frac{35}{\epsilon}$. We have

$$\left|\frac{3n^3+n^2}{4n^3-1001} - \frac{3}{4}\right| = \left|\frac{4n^2+3003}{16n^3-4004}\right| = \frac{4n^2+3003}{16n^3-4004},$$

with the last statement true since $n > 10$, forcing both the numerator and denominator to be positive.

Next, we claim that

$$\frac{4n^2+3003}{16n^3-4004} < \frac{35n^2}{n^3} = \frac{35}{n}.$$

This holds because $4n^2 + 3003 < 35n^2$ when $n > 10$, since $31n^2 > 3100 > 3003$ (so we have a smaller numerator), AND $16n^3 - 4004 > n^3$ when $n > 10$, since $15n^2 > 1500 > 4004$ (so we have a larger denominator).

Then, since $n > \frac{35}{\epsilon}$, we see that $n\epsilon > 35$ and thus $\frac{35}{n} < \epsilon$. Putting everything together using transitivity, we have

$$\left|\frac{3n^3+n^2}{4n^3-1001} - \frac{3}{4}\right| < \epsilon$$

when $n > N$, so $\lim \frac{3n^3+n^2}{4n^3-1001} = \frac{3}{4}$.

(b)We wish to show that for any $M > 0$, there exists $N \in \mathbb{R}$ such that $n > N$ implies

$$\lim_{n\to\infty} \frac{3n^3+n^2}{4n^2-1001} > M.$$

Proof: Let $M > 0$ and take $N = \frac{4M}{3}$. Let $n > N$, so $n > \frac{4M}{3}$. Since all quantities here are positive, we may rearrange this to get $\frac{3n}{4} > M$. We claim that

$$\frac{3n^3 + n^2}{4n^2 - 1001} > \frac{3n^3}{4n^2} = \frac{3n}{4}.$$

This holds since $\frac{3n^3+n^2}{4n^2-1001}$ has a larger numerator and a smaller denominator than $\frac{3n}{4}$, as follows. First $3n^3 + n^2 > 3n^3$, since $n^2 > 0$ for all n. Next, $4n^2 - 1001 < 4n^2$ since $-1001 < 0$.

Finally, by transitivity, we have

$$\frac{3n^3 + n^2}{4n^2 - 1001} > \frac{3n}{4} > M$$

whenever $n > N$, so

$$\lim_{n\to\infty} \frac{3n^3 + n^2}{4n^2 - 1001} = +\infty.$$

□

Notes. (a) You could possibly have gone smaller than 10 for your first restriction, but I chose 10 since it made mental arithmetic extremely simple. (And of course it is large enough; you won't always be able to just take 10.) Notice that you can pull the absolute value bars off for $n > 6$, but then you run into trouble trying to bound the denominator. So, it's not even typically ideal to try to aim for the smallest N possible. Also, other values are possible for c and d in $\frac{cn^2}{dn^3}$. I like taking $d = 1$ when I can, since it keeps my fractions neater.

(b) Other choices may work, but this is by far the simplest way to go on this one. Note that we didn't even need to specify a numerical starting point for N, since our bounds for the numerator and denominator work for all positive integers. □

2. Prove the following, which is basically Exercise 9.12 in Ross (the book provides some hints).

   Let (a_n) be a sequence of real numbers such that $a_n \neq 0$ for all n. Assume that the limit $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists (that is, it's a real number or $\pm\infty$).

   (a) Show that if $L < 1$, then $\lim_{n\to\infty} a_n = 0$.

   (b) Show that if $L > 1$, then $\lim_{n\to\infty} |a_n| = +\infty$.

   (c) *Think about, but don't turn in: find examples of sequences which show that the $L = 1$ case is inconclusive for this test, i.e (a_n) might converge or diverge.*

Solution. (a) Clearly $L \geq 0$, since 0 is a lower bound for $\left|\frac{a_{n+1}}{a_n}\right|$, so we need not worry about the possibility that $L = -\infty$. Set $\epsilon_1 = \frac{1-L}{2}$ and $b = L + \epsilon_1$. (The book uses b instead of a, but I foolishly renamed the sequence to (a_n) without double checking their notation.) Then since $L < 1$, we see that $\epsilon_1 > 0$ and $L < b < 1$. Then since $\left|\frac{a_{n+1}}{a_n}\right|$ converges to L, we know there exists some integer N_1 such that $n \geq N_1$ (see notes on why this can be a weak inequality) implies

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < \epsilon_1,$$

or in other words (ummm, inequalities), we have

$$L - \epsilon_1 < \left|\frac{a_{n+1}}{a_n}\right| < L + \epsilon_1 = b.$$

We don't always specify that N_1 is an integer, but of course it is possible (round up if you didn't already have an integer), and here it will help with the next part.

Rearranging, we see that $|a_{n+1}| < b|a_n|$ for all $n > N_1$, and this completes the first hint.

On to the next hint: we wish to show that $|a_n| < b^{n-N_1}|a_{N_1}|$ for all $n > N_1$. To show this, we use the previous result multiple times. Since $N_1, N_1+1, N_1+2, \cdots, n-1, n$ are all at least as large as N_1, we have $\left|\frac{a_{N+1}}{a_N}\right| < b$, $\left|\frac{a_{N_1+2}}{a_{N_1+1}}\right| < b$, and so on, up to $\left|\frac{a_n}{a_{n-1}}\right| < b$. Multiplying all of these inequalities together (all consisting of positive quantities), we obtain

$$\left|\frac{a_n}{a_N}\right| = \left|\frac{a_{N+1}}{a_N}\right| \cdot \left|\frac{a_{N_1+2}}{a_{N_1+1}}\right| \cdots \cdot \left|\frac{a_{n-1}}{a_{n-2}}\right| \cdot \left|\frac{a_n}{a_{n-1}}\right| < b^{n-N_1}.$$

Rearranging, we conclude $|a_n| < b^{n-N_1}|a_{N_1}|$ for all $n \geq N_1$, which completes the second hint.

It remains to show how to complete the proof that $\lim |a_n| = 0$, using these hints. Intuitively, the idea is that we can shrink b^{n-N_1} as close to 0 as we like if we go far out in the sequence, and $|a_{N_1}|$ is a constant, so we can trap $|a_n|$ close to 0 far out in the sequence. Let's make it rigorous. Let $\epsilon > 0$, and let N_1 be defined as in our previous work. Choose $N = \max\{N_1, \log_b\left(\frac{\epsilon \cdot b^{N_1}}{|a_{N_1}|}\right)\}$, and let $n > N$. Thus we have

$$n > \log_b\left(\frac{\epsilon \cdot b^{N_1}}{|a_{N_1}|}\right),$$

and since $0 < b < 1$, exponentiation reverses the inequality, yielding

$$b^n < \frac{\epsilon \cdot b^{N_1}}{|a_{N_1}|}.$$

All quantities here are positive, so we can easily rewrite this as

$$b^{n-N_1}|a_{N_1}| = \frac{b^n \cdot |a_{N_1}|}{b^{N_1}} < \epsilon.$$

Combining with the inequality from the second book hint (which holds since $n > N \geq N_1$), we have

$$|a_n| < b^{n-N_1}|a_{N_1}| < \epsilon$$

whenever $n > N$.

(b) Consider the sequence defined by $t_n = |\frac{1}{a_n}|$ (well-defined, since $a_n \neq 0$ for all n). Then the sequences of absolute values of ratios satisfy

$$\left|\frac{t_{n+1}}{t_n}\right| = \frac{1}{|\frac{a_{n+1}}{a_n}|},$$

so taking limits, we have

$$\lim \left|\frac{t_{n+1}}{t_n}\right| = \lim \frac{1}{|\frac{a_{n+1}}{a_n}|} = \frac{1}{L},$$

by our rule for the limit of a quotient. Then, since $L > 1$, we know $0 < \frac{1}{L} < 1$, so (t_n) satisfies the conditions required to use part (a), and we see that $\lim |t_n| = 0$. Finally, we proved in class that for sequences of positive reals, the limit of a sequence is 0 if and only if the limit of its reciprocal sequence is $+\infty$. Thus, $\lim |a_n| = +\infty$. □

Notes. (a) Note usually, we would say $n > N_1$ in our convergence definition, but since $n \in \mathbb{Z}$, we know $n > N_1$ if and only if $n \geq N_1 - 1$, so by slightly modifying our choice of N_1, it turns out it is fine to use weak inequalities.

Perhaps you could have used induction to prove the second hint, but as long as you take N_1 to be an integer, it is not strictly necessary.

Note that even though you "know" $b^n \to 0$ as $n \to \infty$ when $|b| < 1$, we have not rigorously showed it yet, so it does require a proof. This is in the Section 9 ungraded HW, by the way. □

3. Let $t_1 = 1$ and for $n \geq 1$, let $t_{n+1} = \frac{(t_n)^2+6}{3t_n}$. Assuming that (t_n) converges, find $\lim_{n\to\infty} t_n$.

Solution. Since (t_n) converges, we know $\lim t_n = T$ for some real number T. Then

$$T = \lim_{n\to\infty} t_n = \lim_{n\to\infty} t_{n+1} = \lim_{n\to\infty} \frac{(t_n)^2 + 6}{3t_n},$$

since our limit as we go to infinity is not affected by shifting the terms by one. Then, applying our limit shortcuts for limit of a quotient, limit of a sum, and limit of a product, we see (with all limits below having $n \to \infty$):

$$\lim \frac{(t_n)^2 + 6}{3t_n} = \frac{\lim (t_n)^2 + 6}{\lim 3t_n} = \frac{\lim (t_n)^2 + \lim 6}{3 \lim t_n} = \frac{(\lim t_n)^2 + 6}{3 \lim t_n} = \frac{T^2 + 6}{3T}.$$

Thus, by transitivity, we have $T = \frac{T^2+6}{3T}$. Solving for T, we obtain $3T^2 = T^2 + 6$, i.e. $2T^2 = 6$, i.e. $T^2 = 3$. This equation has two real solutions $\pm\sqrt{3}$, but as our terms are all positive, we know we must have $T = +\sqrt{3}$. □

Notes. Nothing crazy on this one. To get started, you really needed to notice that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} t_{n+1}$, but the rest is straightforward. Even though the problem statement doesn't say to prove your answer is correct, written explanations are ALWAYS required in homework, unless it explicitly says otherwise. □

Math 104 Name:

UNGRADED HOMEWORK:

Note that you should pay special attention to starred problems; they are usually classics we will use many times, often important theorems hidden in the exercises. There are a lot of these; some are just quick computations, and even with the proofs, you should not feel compelled to write all of them up beautifully, but you should figure out how they work. There are some hints and (partial) solutions for many odd problems at the end of the book. When practicing $\epsilon - N$ limit proofs, you may not need to do ALL of these, especially on multi-part problems; the point is to do enough that you are confident you can do most similar ones.

Section	Exercises
7	1, 2, 3, 4, 5
8	1, 2, 3, 4*, 5*, 7, 8, 9*
9	1, 2, 3, 6, 7, 8, 9*, 10, 11

PS: Ex. 8.5 called the Squeeze Theorem, not the Squeeze Lemma.