MATH 104, HOMEWORK #2

Due Thursday, January 28

Remember, consult the Homework Guidelines for general instructions. It is not necessary to quote theorems for all your tiny arithmetic steps.

GRADED HOMEWORK:

1. Prove by directly verifying the axioms that $\mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} : a, b \in \mathbb{Q}\}$ is an ordered field. Be sure to include A0 and M0 as clarified in the January 19 Daily Update. You may take it as a given that $\mathbb{R}$ is an ordered field. Note that when you are checking axioms such as M4 (multiplicative identities), you are not simply checking that each element of $\mathbb{Q}(\sqrt{5})$ *has* an inverse – you need to show that the inverse is *in the set* $\mathbb{Q}(\sqrt{5})$.

2. The following problem is essentially Exercise 4.7 or Exercise 5.6. Allow for the possibility that $\sup S$ or $\sup T$ is ∞ or that $\inf S$ or $\inf T$ is $-\infty$, as in Section 5.

   (a) Suppose that S and T are nonempty subsets of $\mathbb{R}$ such that $S \subseteq T$. Prove that
   $$\inf T \leq \inf S \leq \sup S \leq \sup T.$$
   (b) Give an example of a pair of sets with $\inf T < \inf S = \sup S < \sup T$.

   (c) Give an example of a pair of sets with $S \subsetneq T$ but $\inf T = \inf S < \sup S = \sup T$.

   (d) Suppose A and B are nonempty subsets of $\mathbb{R}$. Prove that $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

3. In this problem, we'll do some more difficult business with inequalities.

   (a) First prove the following string of inequalities holds for all $n \in \mathbb{N}$. No need to do induction yet; algebraic manipulation is fine. *(Hint: probably do each half separately.)*
   $$2(\sqrt{n+1} - \sqrt{n}) < \frac{1}{\sqrt{n}} < 2(\sqrt{n} - \sqrt{n-1})$$
   (b) Next prove that for any integer $m \geq 2$, we have the following. *(Hint: try using part (a) a lot. Induction is not necessary, but I would love to see it if you do find a nice induction proof.)*
   $$2\sqrt{m} - 2 < \sum_{n=1}^{m} \frac{1}{\sqrt{n}}$$
   (c) Finally, prove by induction on m that for all integers $m \geq 2$,
   $$\sum_{n=1}^{m} \frac{1}{\sqrt{n}} < 2\sqrt{m} - 1.$$

UNGRADED HOMEWORK:

Note that you should pay special attention to starred problems; they are usually classics we will use many times, often important theorems hidden in the exercises. There are a lot of these; some are just quick computations, and even with the proofs, you should not feel compelled to write all of them up beautifully, but you should figure out how they work. There are some hints and (partial) solutions for many odd problems at the end of the book.

Section	Exercises
1	1, 2, 3, 4, 5, 6, 10, 11
2	1, 2, 3, 4, 5, 6, 7, 8
3	1, 2, 3, 4, 5*, 7*, 8*
4	1, 2, 5, 6, 8, 11*, 12, 14, 15, 16
5	1, 2, 3, 4