MATH 104, HOMEWORK #2
Due Thursday, January 28

Remember, consult the Homework Guidelines for general instructions. It is not necessary to quote theorems for all your tiny arithmetic steps.

GRADED HOMEWORK:

1. Prove by directly verifying the axioms that $\mathbb{Q}(\sqrt{5}) = \{a+b\sqrt{5} : a, b \in \mathbb{Q}\}$ is an ordered field. Be sure to include A0 and M0 as clarified in the January 19 Daily Update. You may take it as a given that $\mathbb{R}$ is an ordered field. Note that when you are checking axioms such as M4 (multiplicative identities), you are not simply checking that each element of $\mathbb{Q}(\sqrt{5})$ *has* an inverse – you need to show that the inverse is *in the set* $\mathbb{Q}(\sqrt{5})$.

   Solution. First note that $\mathbb{R}$ is an ordered field and axioms A1, A2, A3, M1, M2, M3, O1, O2, O3, O4, and O5 are all simple equations or inequalities which work for all real numbers. Since $\mathbb{Q}(\sqrt{5})$ is a subset of $\mathbb{R}$, of course these hold in $\mathbb{Q}(\sqrt{5})$ as well.

   Next, let's look at A0 and M0. We need to know that addition and multiplication are binary operations on $\mathbb{Q}(\sqrt{5})$. We already know they are binary operations on $\mathbb{R}$, so given any $x, y \in \mathbb{Q}(\sqrt{5})$, we know that there is a unique real number $x + y$ and a unique real number xy. It remains to show that $x + y$ and xy are in $\mathbb{Q}(\sqrt{5})$. Let $x = a_x + b + x\sqrt{5}$ and $y = a_y + b_y\sqrt{5}$, with $a_x, b_x, a_y, b_y \in QQ$. Then $x + y = (a_x + b_x\sqrt{5}) + (a_y + b_y\sqrt{5}) = (a_x + a_y) + (b_x + b_y)\sqrt{5}$. Using the fact that $\mathbb{Q}$ is a field (which I allowed after a Piazza question), we know that $(a_x + a_y) \in \mathbb{Q}$ and $(b_x+b_y) \in \mathbb{Q}$, so $x+y$ does indeed lie in $\mathbb{Q}(\sqrt{5})$. Similarly, $xy = (a_x+b_x\sqrt{5})(a_y+b_y\sqrt{5}) = (a_xa_y + 5b_xb_y) + (a_xb_y + a_yb_x)\sqrt{5} \in \mathbb{Q}(\sqrt{5})$. Thus A0 and M0 hold for $\mathbb{Q}(\sqrt{5})$.

   Finally, we need to verify the axioms on inverses, namely A4 and M4. Given $x = a+b\sqrt{5} \in \mathbb{Q}(\sqrt{5})$ (where $a, b \in \mathbb{Q}$), we have $-x = -a - b\sqrt{5}$, which is clearly in $\mathbb{Q}(\sqrt{5})$ as well, since $\mathbb{Q}$ is a field and contains the additive inverses of a and b. Assuming $x \neq 0$, we also have $x^{-1} = \frac{1}{x} = \frac{1}{a+b\sqrt{5}} = \frac{1}{a+b\sqrt{5}} \cdot \frac{a-b\sqrt{5}}{a-b\sqrt{5}} = \frac{a-b\sqrt{5}}{a^2-5b^2} = \frac{a}{a^2-5b^2} - \frac{b}{a^2-5b^2}\sqrt{5}$. Note that the denominator is never zero for a rational pair $(a, b) \neq (0, 0)$, so this is sensible. Using the fact that $\mathbb{Q}$ is a field, we see that $x^{-1} \in \mathbb{Q}(\sqrt{5})$, as desired. Thus A4 and M4 hold for $\mathbb{Q}(\sqrt{5})$.

   This completes our list of axioms, so $\mathbb{Q}(\sqrt{5})$ is an ordered field, with the same ordering as $\mathbb{R}$. $\square$

   Notes. If you had not encountered the field axioms before this class, this may have seemed tough, as the book is slightly vague about, for example, axioms A4 and M4. I think (hope!) our Piazza conversations mostly cleared that up for those who paid attention. But note, for example, that the "clear" version of A4 for an arbitrary field

F would state "*For each $a \in F$, there is an element $(-a) \in F$ such that $a + (-a) = 0$.*" Axioms M4, A0, and M0 also need a bit of clarification. Make sure you know how it should go.

Just for fun: A wild thing to think about is that the standard ordering (i.e. matching $\mathbb{R}$) is not the only reasonable ordering which may be put on $\mathbb{Q}(\sqrt{5})$. Another option is to define a new relation $\preceq$ (the symbol is not important; I just want something different from standard $\leq$) as follows $0 \preceq a + b\sqrt{5}$ if and only if $0 \leq a - b\sqrt{5}$ in the usual ordering of $\mathbb{R}$. You can also convince yourself that knowing which numbers are positive and which are negative is sufficient to understand the full ordering. □

2. The following problem is essentially Exercise 4.7 or Exercise 5.6. Allow for the possibility that $\sup S$ or $\sup T$ is ∞ or that $\inf S$ or $\inf T$ is $-\infty$, as in Section 5.

   (a) Suppose that S and T are nonempty subsets of $\mathbb{R}$ such that $S \subseteq T$. Prove that

   $$\inf T \leq \inf S \leq \sup S \leq \sup T.$$

   Solution. First let's show that $\inf T \leq \inf S$. We know that either $\inf T = -\infty$ or $\inf T$ is a real number which is a lower bound for T. In the former case, we are done, since $-\infty \leq -\infty$, and $-\infty \leq r$ for all $r \in \mathbb{R}$. In the latter case, note that $\inf T$ is also a lower bound for S, since all elements in S belong to T. Then, since $\inf T$ is a lower bound for S and $\inf S$ is the greatest lower bound of S, we must have $\inf T \leq \inf S$.

   A symmetric argument proves $\sup S \leq \sup T$. We know that either $\sup T = \infty$ or $\sup T$ is a real number which is an upper bound for T. In the former case, we are done, since $\infty \leq \infty$, and $r \leq \infty$ for all $r \in \mathbb{R}$. In the latter case, note that $\sup T$ is also an upper bound for S, since all elements in S belong to T. Then, since $\sup T$ is an upper bound for S and $\sup S$ is the least upper bound of S, we must have $\sup S \leq \sup T$.

   Finally, we have the middle inequality. Let $x \in S$, which is possible since S is nonempty. Then since $\inf S$ is $-\infty$ or a lower bound of S, we have $\inf S \leq x$. Similarly, since $\sup S$ is ∞ or an upper bound for S, we have $x \leq \sup S$. By transitivity, $\inf S \leq \sup S$. □

   (b) Give an example of a pair of sets with $\inf T < \inf S = \sup S < \sup T$.

   Solution. Many examples are possible. One is $S = \{5\}$ and $T = \{5, 6, 7\}$. Note S should contain a single point, to satisfy the middle equality. □

   (c) Give an example of a pair of sets with $S \subsetneq T$ but $\inf T = \inf S < \sup S = \sup T$.

   Solution. Again, many examples are possible. One is $S = (0, 1)$, the open interval, and $T = [0, 1]$, the closed interval. □

(d) Suppose A and B are nonempty subsets of $\mathbb{R}$. Prove that $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Solution. There are several ways to untangle this one, but the easiest is to use part (a) to start. First, let's take care of the infinite case. Note that $A \cup B$ is unbounded above if and only if A is unbounded above or B is unbounded above, in which case we get $\infty = \infty$, which is fine. Next, assume $A \cup B$ is bounded above (and thus A and B are also bounded above.) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, we have $\sup A \leq \sup(A \cup B)$ and $\sup B \leq \sup(A \cup B)$, i.e. $\max\{\sup A, \sup B\} \leq \sup(A \cup B)$. Finally, note that $\max\{\sup A, \sup B\}$ is an upper bound of $A \cup B$, since $x \in A$ implies $x \leq \sup A \leq \max\{\sup A, \sup B\}$ and $x \in B$ implies $x \leq \sup B \leq \max\{\sup A, \sup B\}$. But $\sup(A \cup B)$ is the *least* upper bound of $A \cup B$, so $\sup(A \cup B) \leq \max\{\sup A, \sup B\}$. Combining with out previous inequality, we have $\sup(A \cup B) = \max\{\sup A, \sup B\}$. □

Notes. I will try to avoid making you write out symmetric proofs of things, e.g. a sup version of a statement and an inf version of the same idea, but given one, you should be able to come up with the other pretty easily. Here, for example, having done this problem, you could also easily prove that $\inf(A \cup B) = \min\{\inf A, \inf B\}$. □

3. In this problem, we'll do some more difficult business with inequalities.

(a) First prove the following string of inequalities holds for all $n \in \mathbb{N}$. No need to do induction yet; algebraic manipulation is fine. *(Hint: probably do each half separately.)*

$$2(\sqrt{n+1} - \sqrt{n}) < \frac{1}{\sqrt{n}} < 2(\sqrt{n} - \sqrt{n-1})$$

Solution. We'll deal with the left first. We have

$$2(\sqrt{n+1} - \sqrt{n}) = 2(\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{2((n+1) - n)}{\sqrt{n+1} + \sqrt{n}} = \frac{2}{\sqrt{n+1} + \sqrt{n}} < \frac{2}{\sqrt{n} + \sqrt{n}} = \frac{2}{2\sqrt{n}} = \frac{1}{\sqrt{n}},$$

with the inequality justified by the fact that $\sqrt{n+1} > \sqrt{n}$.

Next, let's see the right half. It is nearly identical in spirit.

$$2(\sqrt{n} - \sqrt{n-1}) = 2(\sqrt{n} - \sqrt{n-1}) \cdot \frac{\sqrt{n} + \sqrt{n-1}}{\sqrt{n} + \sqrt{n-1}}$$

$$= \frac{2(n - (n-1))}{\sqrt{n} + \sqrt{n-1}} = \frac{2}{\sqrt{n} + \sqrt{n-1}} > \frac{2}{\sqrt{n} + \sqrt{n}} = \frac{2}{2\sqrt{n}} = \frac{1}{\sqrt{n}},$$

with the inequality justified using $\sqrt{n-1} < \sqrt{n}$. □

(b) Next prove that for any integer $m \geq 2$, we have the following. *(Hint: try using part (a) a lot. Induction is not necessary, but I would love to see it if you do find a nice induction proof.)*

$$2\sqrt{m} - 2 < \sum_{n=1}^{m} \frac{1}{\sqrt{n}}$$

Solution. Add up the left inequalities from part (a) (see notes below) for $n = 1$, $n = 2$, $\ldots$, $n = m$:

$$2(\sqrt{2} - \sqrt{1}) + 2(\sqrt{3} - \sqrt{2}) + \cdots + 2(\sqrt{m+1} - \sqrt{m}) < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{m}}.$$

Notice the left is a telescoping sum, simplifying to

$$2(\sqrt{m+1} - 1) < \sum_{n=1}^{m} \frac{1}{\sqrt{n}}.$$

This is actually more powerful than what we were asked for, but we can finish it off by noticing that $\sqrt{m} < \sqrt{m+1}$, so

$$2\sqrt{m} - 2 < 2\sqrt{m+1} - 2 < \sum_{n=1}^{m} \frac{1}{\sqrt{n}}.$$

□

Notes. It's not directly obvious from the order axioms that you can legitimately add inequalities, which state you may add the same thing to both sides, but two applications of O4 can show this quickly. If $a < b$ and $c < d$, then $a + c < b + c$ and $b + c < b + d$, so using transitivity, we get $a + c < b + d$ as desired. This one actually works for $m = 1$ as well. □

(c) Finally, prove by induction on m that for all integers $m \geq 2$,

$$\sum_{n=1}^{m} \frac{1}{\sqrt{n}} < 2\sqrt{m} - 1.$$

Solution. First we check our base case $m = 2$, which case we verify that $1.71 \approx \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} < 2\sqrt{2} - 1 \approx 1.83$. (A quick calculator approximation is sufficient here.)

For our inductive step, we assume the result hold for some $m \in \mathbb{N}$, i.e. $\sum_{n=1}^{m} \frac{1}{\sqrt{n}} < 2\sqrt{m} - 1$. We wish to show that the result holds for $m + 1$, namely that $\sum_{n=1}^{m+1} \frac{1}{\sqrt{n}} < 2\sqrt{m+1} - 1$.

Now, the right hand side of part (a) implies that $\frac{1}{\sqrt{m+1}} < 2(\sqrt{m+1} - \sqrt{m})$, so we can combine this with our inductive hypothesis, yielding.

$$\left(\sum_{n=1}^{m} \frac{1}{\sqrt{n}}\right) + \frac{1}{\sqrt{m+1}} < 2\sqrt{m} - 1 + 2(\sqrt{m+1} - \sqrt{m}) = 2\sqrt{m+1} - 1,$$

that is $\displaystyle\sum_{n=1}^{m+1} \frac{1}{\sqrt{n}} < 2\sqrt{m+1} - 1$, which completes the inductive step, and thus our result holds for $m \geq 2$..

$\square$

UNGRADED HOMEWORK:

Note that you should pay special attention to starred problems; they are usually classics we will use many times, often important theorems hidden in the exercises. There are a lot of these; some are just quick computations, and even with the proofs, you should not feel compelled to write all of them up beautifully, but you should figure out how they work. There are some hints and (partial) solutions for many odd problems at the end of the book.

Section	Exercises
1	1, 2, 3, 4, 5, 6, 10, 11
2	1, 2, 3, 4, 5, 6, 7, 8
3	1, 2, 3, 4, 5*, 7*, 8*
4	1, 2, 5, 6, 8, 11*, 12, 14, 15, 16
5	1, 2, 3, 4