ON THE NON-GENERIC PART OF THE L^2-COHOMOLOGY OF LOCALLY SYMMETRIC SPACES

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ABSTRACT. There have been a number of old and recent results on the cohomology of Shimura varieties and locally symmetric spaces with characteristic zero and torsion coefficients, sometimes leading to striking arithmetic applications. Often these results are obtained under a suitable local genericity or regularity hypothesis. Without such a hypothesis, we formulate a prediction on the range of vanishing cohomological degrees under a general local condition. This is based on an axiomatic formalism of Arthur parameters, which is conditionally available for classical groups, as well as the closure ordering conjecture from the p-adic Adams–Barbasch–Vogan theory and the Adams–Johnson theory of cohomological parameters at ∞. We start from the L^2-cohomology of locally symmetric spaces with complex coefficients and then proceed to consider the cohomology of Shimura varieties and their local analogues with torsion coefficients.

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1. Introduction

Locally symmetric spaces arise as the quotients of symmetric spaces for reductive Lie groups by arithmetic lattices. These spaces, equipped with Hecke correspondences, provide a geometric and topological incarnation of automorphic forms. When locally symmetric spaces are essentially Shimura varieties (modulo technical issues with the center), algebraic geometry is brought in to enrich the subject and lead to further progress.

A fundamental problem is to understand the cohomology of locally symmetric spaces, not only with $\mathbb{C}$-coefficients but also with torsion coefficients. In the case of certain Shimura varieties and GL_n-locally symmetric spaces, even partial answers took much effort but there was a great reward, namely outstanding progress in the Langlands program such as [Sch15, ACC$^+$23, CN]. Further, we refer to [CG18, GV18] for perspectives on the cohomology of locally symmetric spaces in the context of the Taylor–Wiles method and its generalization.

In fact, the cohomology with $\mathbb{C}$-coefficients is described in terms of automorphic forms by Franke's formula (subsuming the formulas due to Matsushima and Borel–Casselman). However, the formula is not always amenable to explicit calculations to control vanishing of cohomology. The situation is far more mysterious with torsion coefficients such as $\mathbb{F}_\ell$. For one thing, there is no general notion of automorphic forms with torsion coefficients (apart from the zero-dimensional case).

Our paper revolves around the following question: find a bound on the non-vanishing degrees for the cohomology of locally symmetric spaces under a prescribed local condition at a finite prime. Ideally we want that a universal recipe yields a bound that is tight or almost tight in all cases. The case of torsion coefficients is particularly deep and motivated by potential applications to automorphy lifting and local-global compatibility. Several results have been obtained on the vanishing of torsion cohomology of Shimura varieties by [CS17, CS24, Kosb, dSS, HL, DvHKZ] mostly under local genericity hypotheses (which differ slightly from each other) and by [Boy19, CT23] which cover some non-generic cases. Thus it is natural and timely to ask what would be an optimal genericity condition and what the bound would be in the non-generic case in general.

Our goal is to present reasonable conjectures for the L^2-cohomology and the Betti cohomology (with or without compact support) with characteristic zero coefficients under a general local condition, with a view towards the case of torsion coefficients under a non-genericity condition. The recipe is based on an axiomatic formalism of Arthur (a.k.a. A-)parameters and packets. It has been a well-known principle from the beginning [Art89], when the coefficient field is $\mathbb{C}$, that the SL_2-part of an Arthur parameter controls how the relevant part of the cohomology spreads out in different degrees in some precise way. We spell out what should happen under a local constraint, utilizing our knowledge of local Arthur packets at both non-archimedean and archimedean places. Moreover, a reinterpretation of our recipe in the case of Shimura varieties led us to come up with an analogous conjecture on non-vanishing cohomological degrees for moduli spaces of local shtukas in mixed characteristic with ℓ-adic coefficients. Guided by these considerations with characteristic zero coefficients, we make some predictions for cohomology with torsion coefficients locally and globally; see §1.11 below.

1.1. **L^2-cohomology of Shimura varieties.** We would like to illustrate the vanishing problem and basic ideas on the L^2-cohomology of Shimura varieties, where the cohomology is easiest to understand by means of automorphic representations.

Let (G,X) be a Shimura datum. For a sufficiently small open compact subgroup $K=\prod_v K_v\subset G(\mathbb{A}^\infty)$, let Sh_K denote the Shimura variety of level K. Though Sh_K admits a canonical model over a number field, we focus on Sh_K as a complex manifold. Consider a complex local system $\mathcal{E}_{\lambda,K}$ over Sh_K corresponding to an irreducible algebraic representation E_λ of $G_{\mathbb{C}}$ of highest weight λ. The L^2-cohomology

$$H^i_{(2)}(\mathrm{Sh},\mathcal{E}_\lambda):=\varinjlim_K H^i_{(2)}(\mathrm{Sh}_K,\mathcal{E}_{\lambda,K}) \tag{1.1}$$

is a $G(\mathbb{A}^\infty)$-module. Now fix a prime p and an irreducible smooth representation π_p^0 of $G(\mathbb{Q}_p)$. Here is a basic question.

Question 1.2. If π_p^0 appears as a subquotient of $H^i_{(2)}(\mathrm{Sh},\mathcal{E}_\lambda)$ then can we prove a bound on i in terms of a numerical invariant attached to π_p^0?

We can get off the ground thanks to an automorphic description by Borel–Casselman [BC83]. Write $\mathfrak{g}$ for the Lie algebra of $G_{\mathbb{C}}$, and K_∞ for the centralizer in $G(\mathbb{R})$ of an element of X (so that K_∞ is roughly a maximal compact subgroup of $G(\mathbb{R})$ modulo center). Then there is a $G(\mathbb{A}^\infty)$-module isomorphism

$$H^i_{(2)}(\mathrm{Sh},\mathcal{E}_\lambda)=\bigoplus_\pi m(\pi)\cdot\pi^\infty\otimes H^i(\mathfrak{g}_{\mathbb{C}},K_\infty,\pi_\infty\otimes E_\lambda), \tag{1.2}$$

where $\pi=\pi^\infty\otimes\pi_\infty$ runs over discrete automorphic representations of $G(\mathbb{A})$ (whose central character at ∞ is constrained by λ), and $m(\pi)\in\mathbb{Z}_{\geq 0}$ denotes the automorphic multiplicity of π. So Question 1.2 asks, if $\pi_p=\pi_p^0$, what we can say about the relative Lie algebra cohomology of π_∞.

Such a link between p and ∞ is best formulated by Arthur's conjectural endoscopic classification of automorphic representations by Arthur parameters (A-parameters), which takes the following rough form. There is a set of global A-parameters $\Psi(G)$ and a set of local A-parameters $\Psi(G_v)$ for each place v of $\mathbb{Q}$, equipped with localization maps $\Psi(G)\to\Psi(G_v)$, $\psi\mapsto\psi_v$, as well as an A-packet Π_{ψ_v} consisting of finitely many (isomorphism classes of) irreducible unitary representations of $G(\mathbb{Q}_v)$ attached to each ψ_v. Given $\psi\in\Psi(G)$, define Π_ψ^{aut} as the set of discrete automorphic representations $\pi=\otimes'_v\pi_v$ such that $\pi_v\in\Pi_{\psi_v}$. Then (1.2) becomes

$$H^i_{(2)}(\mathrm{Sh},\mathcal{E}_\lambda)=\bigoplus_{\psi\in\Psi(G)}\bigoplus_{\pi\in\Pi_\psi^{\mathrm{aut}}} m(\pi)\cdot\pi^\infty\otimes H^i(\mathfrak{g}_{\mathbb{C}},K_\infty,\pi_\infty\otimes E_\lambda). \tag{1.3}$$

Then Question 1.2 amounts to asking

Question 1.3. Let $\psi\in\Psi(G)$, $\pi\in\Pi_\psi^{\mathrm{aut}}$, and E_λ as above. If $\pi_p=\pi_p^0$, then what is the range of non-vanishing degrees for $H^i(\mathfrak{g}_{\mathbb{C}},K_\infty,\pi_\infty\otimes E_\lambda)$?

A key point for us is that each ψ (resp. each ψ_v) yields a morphism of algebraic groups $\mathrm{SL}_2\to\widehat{G}$ up to $\widehat{G}$-conjugacy, denoted by $\psi|_{\mathrm{SL}_2^A}$ (resp. $\psi_v|_{\mathrm{SL}_2^A}$), and that these associations are compatible with the localization map $\psi\mapsto\psi_v$. (The superscript A is a decoration for "Arthur" SL_2, to distinguish it from "Deligne" SL_2, denoted by SL_2^D.) In particular, if a global ψ localizes to ψ_p and ψ_∞ then

$$\psi_p|_{\mathrm{SL}_2^A}=\psi|_{\mathrm{SL}_2^A}=\psi_\infty|_{\mathrm{SL}_2^A}$$

(again up to $\widehat{G}$-conjugacy). Hence the question is divided into two parts.

(i) Given π_p^0, find all possible $\mathrm{SL}_2 \to \widehat{G}$ that arise as $\psi_p|_{\mathrm{SL}_2^A}$ for $\psi_p \in \Psi(G_p)$ whose A-packet contains π_p^0.

(ii) Given $\psi_\infty|_{\mathrm{SL}_2^A}$, the range of non-vanishing degrees for $H^i(\mathfrak{g}_{\mathbb{C}}, K_\infty, \pi_\infty \otimes E_\lambda)$.

Part (i) reflects the well-known subtlety that local A-packets are not disjoint, namely π_p^0 may appear in Π_{ψ_p} for multiple ψ_p. For our purpose, we are interested in identifying the parameter such that its restriction to SL_2 is the "largest" in some precise sense, which we denote by

$$\psi_p^0|_{\mathrm{SL}_2^A} : \mathrm{SL}_2 \to \widehat{G}.$$

(This should be considered to be analogous to "least" tempered representations.) The (unique) determination of $\psi_p^0|_{\mathrm{SL}_2^A}$ from π_p^0 would be a consequence of (a version of) the closure ordering conjecture, which suggests that $\psi_p^0|_{\mathrm{SL}_2^A}$ should equal the restriction to SL_2^D of the L-parameter

$$\phi_{\widehat{\pi}_p^0} : W_{\mathbb{Q}_p} \times \mathrm{SL}_2^D \to {}^L G$$

to SL_2^D for the Aubert dual of π_p^0. See §3.1 and §3.3 below for more details and references. In the special case when π_p^0 is A-generic, which by definition means $\phi_{\widehat{\pi}_p^0}|_{\mathrm{SL}_2^D}$ is trivial, the conjecture implies that an A-packet Π_{ψ_p} contains π_p^0 only if $\psi_p|_{\mathrm{SL}_2^A}$ is trivial. In other words, A-generic representations are conjectured to show up only in tempered A-packets.

Part (ii) is more classical and essentially understood from the works of Vogan–Zuckerman and Adams–Johnson [VZ84, AJ87] (for real reductive groups which need not come from Shimura data). A necessary condition is that π_∞ has infinitesimal character dual to that of E_λ, which imposes a constraint on $\psi_\infty|_{\mathrm{SL}_2^A}$. For example, if λ is a regular weight, then $\psi_\infty|_{\mathrm{SL}_2^A}$ must be trivial in order that $\pi_\infty \in \Pi_{\psi_\infty}$; if $\lambda = 0$ then there is no constraint on $\psi_\infty|_{\mathrm{SL}_2^A}$. With that said, we restrict ourselves to the case of constant coefficients for simplicity, namely when $\lambda = 0$. (The range of non-vanishing degrees for $\lambda = 0$ is valid for all λ but non-optimal in general. In the main text, we address the optimal range for general λ, at least for $\mathbb{C}$-coefficients.)

The answer to (ii) is complicated to describe in general, but a nice formulation is possible in the case of Shimura varieties following Arthur [Art89, §9]. Let $\mu : \mathbf{G}_m \to G_{\mathbb{C}}$ denote the Hodge cocharacter determined by (G, X), which gives rise to a weight character for the dual group $\widehat{G}$. Write $r_{-\mu}$ for the irreducible representation of $\widehat{G}$ with extreme weight $-\mu$. Given $\psi_\infty|_{\mathrm{SL}_2^A}$, we obtain $\psi_\infty|_{\mathbb{G}_m^A}$ by restricting from SL_2 to a maximal torus. (Here $\mathbb{G}_m^A$ means a maximal torus in SL_2^A.) Then the weights $w \in \mathbb{Z}$ of $\mathbf{G}_m$ on the representation $r_{-\mu} \circ \psi_\infty|_{\mathbb{G}_m^A}$ correspond to the non-vanishing degrees $i = w + \dim \mathrm{Sh}$. To put it differently, fix a Borel pair in $\widehat{G}$, write ρ for the half sum of positive coroots, and choose μ and $\psi_\infty|_{\mathbb{G}_m^A}$ to be dominant possibly after $\widehat{G}$-conjugation. Then $\dim \mathrm{Sh} = \langle 2\rho, \mu \rangle$ and the non-vanishing degrees i lie in the interval

$$\langle 2\rho - \psi_\infty|_{\mathbb{G}_m^A}, \mu \rangle \le i \le \langle 2\rho + \psi_\infty|_{\mathbb{G}_m^A}, \mu \rangle \tag{1.4}$$

In summary, a conditional answer to Questions 1.2 and 1.3 is (choosing $\phi_{\widehat{\pi}_p^0}|_{\mathbb{G}_m^D}$ to be dominant)

$$\langle 2\rho - \phi_{\widehat{\pi}_p^0}|_{\mathbb{G}_m^D}, \mu \rangle \le i \le \langle 2\rho + \phi_{\widehat{\pi}_p^0}|_{\mathbb{G}_m^D}, \mu \rangle, \tag{1.5}$$

assuming a classification result by local and global Arthur packets as well as the local Langlands correspondence by L-parameters and the closure ordering conjecture. For example, when π_p^0 is A-generic, (1.4) reads $i = \langle 2\rho, \mu \rangle$, namely the L^2-cohomology is concentrated in the middle degree. We expect that A-genericity is an optimal condition for this to happen.

The flow of information may be recapitulated as follows.

$$\pi_p^0 \underset{\text{local Langlands}}{\overset{\text{Aubert invol.}}{\rightsquigarrow}} \phi_{\widehat{\pi}_p^0}|_{\mathrm{SL}_2^D} \underset{\text{ordering}}{\overset{\text{closure}}{=\!=}} \psi_p^0|_{\mathrm{SL}_2^A} \overset{\text{via } \psi}{=\!=} \psi_\infty|_{\mathrm{SL}_2^A} \underset{\text{Vogan–Zuckerman}}{\overset{\text{Adams–Johnson}}{\rightsquigarrow}} \begin{array}{c}\text{vanishing range}\\ \text{of cohomology}\end{array} \qquad (1.6)$$

In the case of GL_n and quasi-split classical groups, a good deal is known regarding the classification result, the local Langlands correspondence, and the closure ordering conjecture. See §2.9 and §3.1 below for further details. It is worth remarking that it requires care to state the classification result for a few reasons: the Ramanujan conjecture is unknown, and the global Langlands group is only hypothetical. Keeping this in mind, we formulate in the main text a list of axioms distilled from what we really need from the classification.

From Question 1.2 one can try to generalize or extend in a few directions.

(I) Shimura varieties $\rightsquigarrow$ locally symmetric spaces.
(II) L^2-cohomology $H^i_{(2)}$ $\rightsquigarrow$ Betti cohomology (with compact support) H^i, H^i_c.
(III) Shimura varieties $\rightsquigarrow$ moduli of local shtukas such as local Shimura varieties.
(IV) $\mathbb{C}$-coefficients $\rightsquigarrow$ torsion coefficients (which makes sense for H^i, H^i_c). [1]

The goal of our paper is to take a first step in all of them by formulating precise questions/conjectures with little evidence and interpreting known results in an appropriate context. Let us elaborate a little more in the rest of the introduction: (I)–(II) in §1.4, (III) in §1.9, and (IV) in §1.11 below.

1.4. **Cohomology of locally symmetric spaces with $\mathbb{C}$-coefficients.** Let G be a connected reductive group over $\mathbb{Q}$. We make the simplifying hypothesis that the center of G is anisotropic over $\mathbb{Q}$. (We do not assume it in the main text.) We have a tower of locally symmetric spaces $Y_{G,K}$ indexed by sufficiently small open compact subgroups $K \subset G(\mathbb{A}^\infty)$; see (4.1). As in (1.2) we can define $G(\mathbb{A}^\infty)$-modules

$$H^i_{(2)}(Y_G, \mathcal{E}_\lambda), \qquad H^i(Y_G, \mathcal{E}_\lambda), \qquad H^i_c(Y_G, \mathcal{E}_\lambda).$$

Fix a prime p and an irreducible smooth representation π_p^0 of $G(\mathbb{Q}_p)$. We can ask the analogue of Question 1.2 about the cohomology spaces above. Again we will assume an axiomatic classification as formulated in §2.9 below. If we try to proceed as in the case of Shimura varieties, then we can again get started by applying the analogue of (1.2) due to Borel et al. but two differences stand out.

- The discrete automorphic spectra on proper Levi subgroups of G may contribute to the right hand side of (1.2).
- No clean formula such as (1.4) seems available for the non-vanishing degree at ∞.

As for the first point, our heuristics (to be confirmed by an argument) suggests that the contribution from proper Levi subgroups should not affect the bound on non-vanishing degrees. To address the second point, we introduce a numerical invariant in terms of $\psi_\infty|_{\mathrm{SL}_2^A} : \mathrm{SL}_2 \to \widehat{G}$ and the weight parameter λ, partly guided by the behavior of relative Lie algebra cohomology (Lemma 2.7):

$$a_G^{(2)}(\psi_\infty|_{\mathrm{SL}_2^A}, \lambda) := \max_{L \subset G_\mathbb{R}} q(L) \in \tfrac{1}{2}\mathbb{Z}_{\geq 0},$$

where L runs over the set of θ-stable Levi subgroups of $G_\mathbb{R}$ (see §2.4 for a reminder and further references) such that a regular unipotent element of $\widehat{L}$ lies in the closure of the unipotent orbit of $\psi_\infty|_{\mathrm{SL}_2^A}\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\right)$ and such that the centralizer of λ in $\widehat{G}$ contains a $\widehat{G}$-conjugate of $\widehat{L}$; here $q(L)$ is half the real dimension of the symmetric space associated with L. The condition on λ is an interpolation

[1] For $H^i_{(2)}$, we switch to the intersection cohomology via Zucker's conjecture in order to take torsion coefficients.

of the two extreme cases: if λ is trivial (i.e., when the cohomology is taken with constant coefficients) then no condition is imposed, while if λ is a regular weight, then the condition implies that L must be a fundamental maximal torus and that ψ_∞ must be a tempered parameter.

On the other hand, we still expect that $\phi_{\widehat{\pi}_p^0}|_{\mathrm{SL}_2^D} = \psi_\infty|_{\mathrm{SL}_2^A}$ as discussed in the previous subsection. Hence we are led to:

Conjecture 1.5 (Conjecture 4.4)**.** *If π_p^0 appears as a subquotient of $H^i_{(2)}(Y_G, \mathcal{E}_\lambda)$ then*

$$q(G_{\mathbb{R}}) - a_G^{(2)}(\phi_{\widehat{\pi}_p^0}|_{\mathrm{SL}_2^D}, \lambda) \leq i \leq q(G_{\mathbb{R}}) + a_G^{(2)}(\phi_{\widehat{\pi}_p^0}|_{\mathrm{SL}_2^D}, \lambda).$$

We stress that the statement of the conjecture relies only on the local Langlands correspondence for $G(\mathbb{Q}_p)$, although we arrived there via a conjectural classification in terms of local and global Arthur packets.

Shimura varieties as real manifolds are essentially locally symmetric spaces (modulo taking a finite quotient and issues with the center). In that case, Conjecture 1.5 is consistent with (1.5) in that $q(G_{\mathbb{R}})$ corresponds to $\dim \mathrm{Sh} = \langle 2\rho, \mu \rangle$ and $a_G^{(2)}(\phi_{\widehat{\pi}_p^0}|_{\mathrm{SL}_2^D}, \lambda)$ to $\langle \phi_{\widehat{\pi}_p^0}|_{\mathbb{G}_m^D}, \mu \rangle$.

We conditionally verify Conjecture 1.5.

Theorem 1.6 (made precise in Theorem 4.11)**.** *Assume an axiomatic classification and the closure ordering conjecture for π_p^0. Then Conjecture 1.5 holds true.*

The proof is straightforward for part of the cohomology that comes from discrete automorphic representations of $G(\mathbb{A})$, following the ideas of (1.6). The main point of the argument is to deal with the contribution from proper Levi subgroups via delicate computations of some numerical invariants.

Our axioms for endoscopic classification are known for inner forms of GL_n, and if we accept the twisted weighted fundamental lemma, also for quasi-split classical groups. The closure ordering conjecture is known for GL_n, and quasi-split classical groups. Hence we obtain the following, where "a suitable version" takes into account the problem with split tori in the center as well as whether the assignment of SL_2-morphism is compatible with the restriction (e.g., from GSp_{2n} to Sp_{2n}); see Corollaries 4.12, 4.13 for the precise statements.

Corollary 1.7. *A suitable version of Conjecture 1.5 is true if G is an inner form of GL_n that is split at p. Conditional on the twisted weighted fundamental lemma, it is also true for quasi-split classical groups as well as for G appearing in the Hilbert–Siegel and quasi-split unitary Shimura data.*

Examples of G as in the last case include the restriction of scalars $G = \mathrm{Res}_{\mathbb{Q}}^F \mathrm{GSp}_{2n}$ for a totally real field F. Even though the endoscopic classification is only partially known for such similitude groups [Xu18, Xu, Xu24], we deduce this case from the case of symplectic groups by relating the automorphic spectra via restriction.

It is more complicated to formulate conjectures and prove results on $H^i(Y_G, \mathcal{E}_\lambda)$ and $H^i_c(Y_G, \mathcal{E}_\lambda)$ due to increased complexity of Franke's formula [Fra98] compared to Borel–Casselman's formula for the L^2-cohomology. Accordingly we propose an a priori intricate definition of the invariant

$$a_G(\psi_\infty|_{\mathrm{SL}_2^A}, \lambda) \in \tfrac{1}{2}\mathbb{Z}_{\geq 0},$$

bounded below by $a_G^{(2)}(\psi_\infty|_{\mathrm{SL}_2^A}, \lambda)$ by definition. We often know or expect that $a_G(\psi_\infty|_{\mathrm{SL}_2^A}, \lambda) = a_G^{(2)}(\psi_\infty|_{\mathrm{SL}_2^A}, \lambda)$; see Question 3.15 below. We would like to advertise (cf. Question 4.3):

Question 1.8. If π_p^0 appears as a subquotient of $H^i(Y_G, \mathcal{E}_\lambda)$ (resp. $H_c^i(Y_G, \mathcal{E}_\lambda)$) then

$$i \geq q(G_{\mathbb{R}}) - a_G(\phi_{\widehat{\pi}_p^0}|_{\mathrm{SL}_2^D}, \lambda), \qquad \big(\text{resp.} \quad i \leq q(G_{\mathbb{R}}) + a_G(\phi_{\widehat{\pi}_p^0}|_{\mathrm{SL}_2^D}, \lambda)\ \big)\ ?$$

For Shimura varieties, analogously (cf. the discussion below Conjecture 1.5), if π_p^0 appears as a subquotient of $H^i(\mathrm{Sh}, \mathcal{E}_\lambda)$ then we may ask whether $i \geq \langle 2\rho - \phi_{\widehat{\pi}_p^0}|_{\mathbb{G}_m^D}, \mu\rangle$; in particular if π_p^0 is A-generic then we predict $i \geq \dim \mathrm{Sh}$. The non-vanishing of $H_c^i(\mathrm{Sh}, \mathcal{E}_\lambda)$ should work similarly as in Conjecture 6.8. We hope to discuss them for quasi-split classical groups, including relevant locally symmetric spaces, in a sequel.

In fact, we formulate unramified analogues of everything in this subsection, where the p-part of the level subgroup K is fixed to be a hyperspecial subgroup of $G(\mathbb{Q}_p)$ and π_p^0 is unramified. See Conjecture 4.2, Question 4.3, Theorem 4.9, and Corollary 4.12 below.

1.9. **Compactly supported cohomology of local Shimura varieties.** Let G be a connected reductive group over $\mathbb{Q}_p$. Write $\mathbb{C}_p$ for the completion of an algebraic closure of $\mathbb{Q}_p$. For a conjugacy class of (geometric) dominant cocharacter μ and an element b of the Kottwitz set $B(G, \mu^{-1})$, Scholze defines the tower of moduli spaces $\mathrm{Sht}(G, b, \mu)_K$ of local G-shtukas with open compact subgroups $K \subset G(\mathbb{Q}_p)$, which are diamonds in the sense of Scholze in general. For simplicity, assume μ is minuscule, in which case $\mathrm{Sht}(G, b, \mu)_K$ is a smooth rigid analytic variety of dimension $d = \langle 2\rho, \mu\rangle$, say over $\mathbb{C}_p$. In some special situations, these spaces are identified with the geometric generic fibers of Rapoport–Zink spaces. Fixing a prime $\ell \neq p$, we may define a $G(\mathbb{Q}_p)$-representation

$$H_c^i(\mathrm{Sht}(G, b, \mu), \overline{\mathbb{Q}}_\ell)$$

by taking a colimit over K as before; in fact, in the local setting, it extends to an action of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ for a certain inner form J_b of a Levi subgroup of the quasi-split form of G determined by b. (In the main text, we choose to write G_b in place of J_b.)

For an irreducible smooth representation of π^0 or π_b^0 of $G(\mathbb{Q}_p)$ or $J_b(\mathbb{Q}_p)$ respectively, we can again ask the analogue of Question 1.2 about the cohomology spaces above. For basic b's, we expect that the answer to this question is analogous to the case of compactly supported cohomology of global Shimura varieties. To formulate the statement, we assume a local Langlands correspondence for G and J_b by L-parameters with ℓ-adic coefficients, cf. **(ℓ-LLC)** in §6.20 below, and **(A2^{++})** in §5.1 and Theorem 5.6 for quasi-split classical groups. (We only need the restriction to $\mathbb{G}_m^D$ of the L-parameters.)

Conjecture 1.10. *Assume that b is basic, and let π_p^0 and π_b^0 be as above. If π_p^0 (resp. π_b^0) appears as a subquotient of $H_c^i(\mathrm{Sht}(G, b, \mu), \overline{\mathbb{Q}}_\ell)$ then*

$$d = \langle 2\rho, \mu\rangle \leq i \leq \langle 2\rho + \phi_{\widehat{\pi}^0}|_{\mathbb{G}_m^D}, \mu\rangle, \quad \textit{resp.} \quad d = \langle 2\rho, \mu\rangle \leq i \leq \langle 2\rho + \phi_{\widehat{\pi}_b^0}|_{\mathbb{G}_m^D}, \mu\rangle.$$

For general b and μ, the relevant cohomology is much more complicated, so the optimal bound must be modified and depend on b as well; refer to Conjecture 6.21 in the main text. These expectations come from some known cases, a relation with global Shimura varieties, and also some experience with the categorical local Langlands conjecture. Indeed, the conjectures for global and local Shimura varieties can be related via a suitable form of Mantovan's formula (Proposition 6.31).

1.11. **Torsion coefficients.** In both global and local settings, the case of torsion coefficients is far more difficult, especially in the case of locally symmetric spaces, and not much is known in general. Part of the difficulty is that there is no naïve analogue of the invariant like $\phi_{\widehat{\pi}}|_{\mathrm{SL}_2^D}$ due to lack of a precise enough mod ℓ formulation of the local Langlands correspondence and the Aubert involution for general reductive groups. However, given recent results as we mentioned, we are a little more optimistic about global (and local) Shimura varieties. We include a rather optimistic expectation (Conjecture 6.15) in the case of constant coefficients based on the idea that the bounds for characteristic 0 lifts might give a correct bound; this is the case at least for several generic cases discussed in these recent results.

1.12. **Organization.** In §2 we review the basics of local L- and A-parameters, Adams–Johnson's theory, the axiomatic formulation of local and global A-packets together with key examples. The goal of §3 is twofold. Firstly, we assign an SL_2-morphism, or a nilpotent conjugacy class, to an irreducible representation of a p-adic reductive group in light of the closure ordering conjecture. Secondly, we assign numerical invariants to a nilpotent conjugacy class for the purpose of measuring deviation from the middle degree when studying vanishing of cohomology. The next section §4 is devoted to formulating vanishing conjectures on the cohomology of locally symmetric spaces with $\mathbb{C}$-coefficients and proving results in the case of L^2-cohomology. To switch from $\mathbb{C}$-coefficients to ℓ-adic or torsion coefficients, we set up representation-theoretic preliminaries in §5. Finally, §6 formulates conjectures on the vanishing range for the cohomology of Shimura varieties and their local analogues with both ℓ-adic and mod ℓ coefficients. We also discuss examples and the relationship between the global and local conjectures.

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1.14. **Notation.** Let k be a field. Let $\overline{k}$ denote an algebraic closure of k, and Γ_k the full Galois group over k. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. When there is a distinguished prime p (as in §6) we also fix $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\overline{\mathbb{Q}}_p \cong \mathbb{C}$ compatibly with it.

Given an algebraic group G over k, write $G_{\overline{k}}$ for its base change to $\overline{k}$. The center of G is denoted by Z_G. Let $\mathrm{Cent}_G(h)$ denote the centralizer of h in G, where h is either a subgroup of G or a morphism into G. Set $\mathfrak{a}_G^* := X_k^*(G) \otimes_{\mathbb{Z}} \mathbb{R}$, where $X_k^*(G)$ is the group of characters $G \to \mathbf{G}_m$ over k. When G is connected reductive over k, choose a minimal k-rational parabolic subgroup P_0 with Levi decomposition $P_0 = M_0N_0$ and take A_0 to be the maximal k-split torus A_0 in Z_{M_0}. We may choose a maximal torus T and a Borel subgroup B of $G_{\overline{k}}$ such that $A_{0,\overline{k}} \subset T \subset B \subset M_{0,\overline{k}}$; see [SZ18, 2.3.2]. Let $\widehat{G}$ denote the Langlands dual group of G equipped with a Γ_k-invariant pinning. When k'/k is a finite field extension and G is a group over k', write $\mathrm{Res}_k^{k'} G$ for the k-group obtained from G by the Weil restriction of scalars.

Now let k be a local or global field. Write W_k for the Weil group over k. For G as above, we adopt the Weil form of the L-group ${}^LG := \widehat{G} \rtimes W_k$, where the action of W_k on $\widehat{G}$ is determined by the pinning of $\widehat{G}$.

When k is a number field, we fix hyperspecial subgroups at almost all finite places v of k such that G_{k_v} is unramified as in [Art81, p.9]. The choice of hyperspecial subgroups is usually suppressed although it determines the notion of unramified Hecke algebras and unramified representations. (When k is a non-archimedean local field and G is unramified over k, we fix an arbitrary hyperspecial subgroup.) Set $G_\infty := (\mathrm{Res}^k_{\mathbb{Q}} G) \times_{\mathbb{Q}} \mathbb{R}$ and $[G] := G(k)\backslash G(\mathbb{A}_k)/A_G(\mathbb{R})^\circ$. Denote by $L^2([G])$ the space of square-integrable functions on $[G]$ as a $G(\mathbb{A}_k)$-module under the right translation action, and $L^2_{\mathrm{disc}}([G])$ the discrete spectrum. Let A_G denote the maximal $\mathbb{Q}$-split torus in $\mathrm{Res}^k_{\mathbb{Q}} Z_G$. When $\chi : A_G(\mathbb{R})^\circ \to \mathbb{C}^\times$ is a (possibly non-unitary) character, it can be viewed as an automorphic character of $G(\mathbb{A}_k)$ by pulling back along the projection $G(\mathbb{A}_k) \to A_G(\mathbb{R})^\circ$. Write $L^2_\chi([G])$ for the $G(\mathbb{A}_k)$-module obtained by twisting $L^2([G])$ by χ.

Let $\mathrm{diag}(x, y)$ denote the diagonal 2×2 matrix with diagonal entries x, y. We will often decorate the group SL_2 as SL_2^A or SL_2^D for "Arthur" or "Deligne". Write $\mathbf{G}_m^A$ and $\mathbf{G}_m^D$ for the diagonal maximal tori therein.

Often a representation or a parameter means an isomorphism class of representations or parameters by abuse of language. When G is over a local field k, write $\mathrm{Irr}(G)$ for the set of (isomorphism classes of) irreducible admissible representations of $G(k)$.

2. Arthur packets

2.1. **Local parameters.** Let G be a connected reductive group over a local field k. Adopt the notation and convention from §1.14; in particular, $A_0 \subset M_0 \subset P_0 \subset G$ are fixed as well as a Γ_k-invariant pinning for $\widehat{G}$. Write $|\cdot| : k^\times \to \mathbb{R}^\times_{>0}$ for the absolute value character (normalized to send a uniformizer to the inverse of the residue field cardinality if k is nonarchimedean), which induces a character $W_k \to \mathbb{R}^\times_{>0}$, still denoted by $|\cdot|$. We define $|\cdot|^{1/2}$ by taking the positive square root values of $|\cdot|$.

The local Langlands group $\mathcal{L}_k$ is defined to be W_k if k is archimedean and $W_k \times \mathrm{SL}_2^D(\mathbb{C})$ otherwise. We define an embedding $i_W : W_k \to \mathcal{L}_k$ to be the identity map if k is archimedean, and the following map otherwise:

$$i_W : W_k \to \mathcal{L}_k = W_k \times \mathrm{SL}_2^D(\mathbb{C}), \qquad x \mapsto (x, \mathrm{diag}(|x|^{1/2}, |x|^{-1/2})).$$

Denote by $\Psi^+(G)$ the set of isomorphism classes of continuous morphisms over W_k

$$\psi : \mathcal{L}_k \times \mathrm{SL}_2^A(\mathbb{C}) \to {}^L G(\mathbb{C}) = \widehat{G}(\mathbb{C}) \rtimes W_k$$

such that (i) $\psi|_{\mathrm{SL}_2^A}$ is algebraic and (ii) $\psi|_{\mathcal{L}_k}$ is semisimple. The isomorphism is given by $\widehat{G}$-conjugation. (The superscripts D and A are inserted to distinguish the two copies of SL_2. We drop them when there is no danger of confusion.)

A subset $\Psi(G) \subset \Psi^+(G)$ is defined by imposing the condition on $\psi \in \Psi^+(G)$ that the 1-cocycle $\zeta_\psi : W_k \to \widehat{G}$ given by ψ (so that $\psi(w) = \zeta_\psi(w) \rtimes w$ for $w \in W_k$) sends W_k into a bounded subset of $\widehat{G}$; the condition depends only on the isomorphism class of ψ. A member of $\Psi(G)$ is often called an (isomorphism class of) local A-parameter in the literature; a member of $\Psi^+(G)$ may be thought of as a generalized A-parameter. We define the subsets $\Phi_{\mathrm{bdd}}(G) \subset \Psi(G)$ and $\Phi(G) \subset \Psi^+(G)$ by the condition that $\psi|_{\mathrm{SL}_2^A}$ is trivial. Members in $\Phi_{\mathrm{bdd}}(G)$ and $\Phi(G)$ are referred to as (isomorphism classes of) bounded L-parameters and L-parameters, respectively. Given $\psi \in \Psi^+(G)$, the associated L-parameter $\phi_\psi \in \Phi(G)$ is defined to be the pullback of ψ along the map

$$i_{\mathcal{L}} : \mathcal{L}_{F_v} \to \mathcal{L}_{F_v} \times \mathrm{SL}_2^A(\mathbb{C}), \qquad x \mapsto (x, \mathrm{diag}(|x|^{1/2}, |x|^{-1/2})),$$

where $|\cdot| : \mathcal{L}_{F_v} \twoheadrightarrow W_{F_v} \to \mathbb{R}^\times_{>0}$ is the absolute value character. The map

$$\Psi^+(G) \to \Phi(G), \qquad \psi \mapsto \phi_\psi$$

is not injective (unless G is a torus) but it is injective when restricted to $\Psi(G)$ by [CFM$^+$22, Lem. 3.3] or [Art84, Prop. 1.3.1]. In §2.9 below, we will introduce an intermediate set $\Psi'(G)$ on which the map remains injective.

Let P be a standard k-rational parabolic subgroup of G, which has a unique Levi decomposition $P = MN$ such that $M \supset M_0$. Put $\mathfrak{a}_M^* := X_k^*(M) \otimes_{\mathbb{Z}} \mathbb{R}$. Each $\nu \in \mathfrak{a}_M^*$ determines an L-morphism $\varphi_\nu : W_k \to (Z_{\widehat{M}}^{\Gamma_k})^\circ$; see [SZ18, 4.8]. Write $\mathfrak{a}_M^{*,+}$ for the open chamber in $\mathfrak{a}_M^*$ determined by P as in [SZ18, 1.3]. Now define the set of triples

$$\Phi_{\text{std.tri}}(G) := \{(P, \phi_M, \nu)\},$$

where $P \subset G$ is a standard k-rational parabolic with unique Levi decomposition $P = MN$ as above, $\phi_M \in \Phi_{\text{bdd}}(M)$, and $\nu \in \mathfrak{a}_M^{*,+}$. We also define a variant $\Psi_{\text{std.tri}} := \{(P, \psi_M, \nu)\}$ using $\psi_M \in \Psi(M)$ in place of ϕ_M. The Langlands classification theorem for representations admits the following analogue for parameters.

Lemma 2.2. *The map $\Phi_{\text{std.tri}}(G) \to \Phi(G)$ induced by $(P, \phi_M, \nu) \mapsto \varphi_\nu \cdot \phi_M$ (composed with ${}^LM \hookrightarrow {}^LG$) is a bijection. The map $\Psi_{\text{std.tri}}(G) \to \Psi^+(G)$, $(P, \psi_M, \nu) \mapsto \varphi_\nu \cdot \psi_M$, is also a bijection.*

Proof. See [SZ18, Thm. 1.4, §4.8, §A.2] for the first assertion. The same argument applies to the second assertion. □

It is convenient to introduce the subset $\Phi_{\text{e.bdd}}(G) \subset \Phi(G)$ of essentially bounded L-parameters by the condition that $\zeta_\phi(W_k) \subset \widehat{G}$ has bounded image in $\widehat{G}/Z_{\widehat{G}}^{\Gamma_k}$, where $\zeta_\phi : w_k \to \widehat{G}$ is the 1-cocycle defined by ϕ. Lemma 2.2 tells us that this subset exactly corresponds to the subset of $\Phi_{\text{std.tri}}(G)$ determined by $P = G$ under the bijection, i.e., $\phi \in \Phi(G)$ is essentially bounded exactly when $\phi = \varphi_\nu \phi_0$ for $\nu \in \mathfrak{a}_G^*$ and $\phi_0 \in \Phi_{\text{bdd}}(G)$.

Likewise, let $\Psi_{\text{e}}(G) \subset \Psi(G)$ denote the subset of ψ such that $\psi|_{\mathcal{L}_k}$ belongs to $\Phi_{\text{e.bdd}}(G)$. So $\psi \in \Psi_{\text{e}}(G)$ if and only if $\psi = \varphi_\nu \psi_0$ for $\nu \in \mathfrak{a}_G^*$ and $\psi_0 \in \Psi(G)$. Henceforth, the word "essentially" and the subscript "e" stand for "up to twist" by $\nu \in \mathfrak{a}_G^*$ or the corresponding parameter/character.

Now suppose that k is non-archimedean. We define the Aubert involution $\psi \mapsto \widehat{\psi}$ on $\Psi^+(G)$ by switching SL_2^D and SL_2^A, i.e., the Aubert dual parameter is given by

$$\widehat{\psi} : W_k \times \mathrm{SL}_2^D \times \mathrm{SL}_2^A \to {}^LG(\mathbb{C}), \qquad (w, h_1, h_2) \mapsto \psi(w, h_2, h_1).$$

Clearly $\Psi(G)$ is invariant under the involution. When G is unramified over k, a parameter $\psi \in \Psi^+(G)$ is said to be *unramified* if ψ is trivial on SL_2^D as well as the inertia subgroup of W_k.

When k is archimedean and $\psi \in \Psi^+(G)$, the parameter $\zeta_\psi := \phi_\psi \circ i_W|_{W_\mathbb{C}}$ is said to be the *infinitesimal character* of ψ. Since $W_\mathbb{C} = \mathbb{C}^\times$ is commutative, we can conjugate ζ_ψ to have image contained in a maximal torus $\widehat{T}$ of $\widehat{G}$. If the centralizer of ζ_ψ is $\widehat{T}$ then ψ is said to be *regular*. We say ψ is *C-algebraic* if ζ_ψ is of the form $z \mapsto \lambda_1(z)\lambda_2(\overline{z})$, where $\lambda_1, \lambda_2 \in X_*(\widehat{G})_\mathbb{Q}$ such that $\lambda_1 - \rho_G \in X_*(\widehat{T})$ for the half sum of positive coroots ρ_G of $\widehat{T}$ in $\widehat{G}$. The regularity and C-algebraicity are invariant under $\widehat{G}$-conjugacy and independent of the choices, cf. [BG14, §2.3]. Define $\Psi_{\text{ra}}(G)$, $\Psi_{\text{era}}(G)$, $\Psi_{\text{ra}}^+(G)$ to be the subset of regular C-algebraic parameters in $\Psi(G)$, $\Psi_{\text{e}}(G)$, $\Psi^+(G)$, respectively. We remark that our definition of $\Psi_{\text{ra}}(G)$ coincides with [NP21, §7, Def. 3]. Finally let $\Psi_{\text{e.ra}}(G) \subset \Psi^+(G)$ denote the subset of $\psi = \varphi_\nu \psi_0$ over all $\nu \in \mathfrak{a}_G^*$ and all $\psi_0 \in \Psi_{\text{ra}}(G)$. This subset extends $\Psi_{\text{ra}}(G)$ by character twists, and we have $\Psi_{\text{ra}}(G) \subset \Psi_{\text{era}}(G) \subset \Psi_{\text{e.ra}}(G)$. We apologize for the possibly confusing notation. Only $\Psi_{\text{era}}(G)$ matters for our discussion of cohomology as opposed to $\Psi_{\text{e.ra}}(G)$, which is introduced only for a marginally expository purpose.

2.3. **Local parameters valued in C-groups.** Later we will need to know how the assignment of local L-packets and A-packets changes under the $\mathrm{Aut}(\mathbb{C})$-action on the coefficient field. For this purpose, we introduce C-groups and local parameters valued in C-groups. This subsection will not play a role until §5 below.

We maintain the setting of §2.1. Write $\widehat{G}^{\mathrm{ad}}$ for the adjoint group of $\widehat{G}$, equipped with a pinning induced by that of $\widehat{G}$. In particular $\widehat{T}^{\mathrm{ad}} \subset \widehat{G}^{\mathrm{ad}}$ is the image of $\widehat{T} \subset \widehat{G}$. We have $\rho^{\mathrm{ad}} \in X_*(\widehat{T}^{\mathrm{ad}})$, the half sum of positive coroots of $\widehat{T}^{\mathrm{ad}}$ in $\widehat{G}^{\mathrm{ad}}$, so that ρ^{ad} is the image of $\rho_G \in X^*(T)_{\mathbb{Q}} = X_*(\widehat{T})_{\mathbb{Q}}$ under $\widehat{T} \to \widehat{T}^{\mathrm{ad}}$. By $\mathrm{Ad}\rho^{\mathrm{ad}}$ we denote the action of $\mathbf{G}_m$ on $\widehat{G}$ where $t \in \mathbf{G}_m$ acts by the adjoint action of $\rho^{\mathrm{ad}}(t)$. Thereby we define a semi-direct product group

$$ {}^C G := {}^L G \rtimes_{\mathrm{Ad}\rho^{\mathrm{ad}}} \mathbf{G}_m, $$

so that the group law reads $(g_1 \rtimes t_2)(g_2 \rtimes t_2) = g_1 \rho^{ad}(t_2)(g_2) \rtimes t_1 t_2$. Let $\mathrm{pr} : {}^C G \to \mathbf{G}_m$ denote the natural projection map. Equivalently, ${}^C G$ is isomorphic to the quotient of ${}^L G \times \mathbf{G}_m$ by the central element $(2\rho_G(-1), -1) \in Z(\widehat{G})^{W_k} \times \mathbf{G}_m$ of order 2. The projection ${}^L G \to W_k$ induces a projection ${}^C G \to W_k$. Our definition of C-groups follows [Zhu], cf. [Shi24, §3], which is equivalent to the original definition [BG14, Def. 5.38, Prop. 5.39].

Denote by ${}^C\Psi^+(G)$ the set of $\widehat{G}$-conjugacy classes of continuous morphisms $\psi : \mathcal{L}_k \times \mathrm{SL}_2^A(\mathbb{C}) \to {}^C G(\mathbb{C})$ satisfying (i), (ii) of §2.1 such that $\mathrm{pr} \circ \psi = |\cdot|^{-1}$. Following the same procedure as in §2.1 we define ${}^C\Psi_{\mathrm{e}}(G)$, ${}^C\Phi(G)$, and ${}^C\Phi_{\mathrm{e.bdd}}(G)$. We have a bijection

$$ \Psi^+(G) \stackrel{\sim}{\to} {}^C\Psi^+(G), \qquad \psi \mapsto {}^C\psi, $$

given by the formula ${}^C\psi(x,h) = \psi(x,h) 2\rho_G(|x|^{1/2}) \rtimes |x|^{-1}$, $x \in \mathcal{L}_k$, $h \in \mathrm{SL}_2^A$. This bijection restricts to bijections $\Psi_{\mathrm{e}}(G) \stackrel{\sim}{\to} {}^C\Psi_{\mathrm{e}}(G)$, $\Phi(G) \stackrel{\sim}{\to} {}^C\Phi(G)$, and $\Phi_{\mathrm{e.bdd}}(G) \stackrel{\sim}{\to} {}^C\Phi_{\mathrm{e.bdd}}(G)$.

2.4. **Cohomological Arthur packets à la Adams–Johnson.** Let G be a connected reductive group over $\mathbb{R}$. Choose a Cartan involution θ on G and compatibly a maximal compact subgroup K of G; the choice is unique up to $G(\mathbb{R})$-conjugacy. Let A be an $\mathbb{R}$-split subtorus of Z_G. Set

$$ \begin{aligned} q(G) &:= \tfrac{1}{2}(\dim G - \dim K), \qquad & q^\flat(G) &:= q(G/A) = q(G) - \tfrac{1}{2}\dim A \quad && \in \tfrac{1}{2}\mathbb{Z}_{\geq 0}, \\ l_0(G) &:= \mathrm{rk}(G) - \mathrm{rk}(K), \qquad & l_0^\flat(G) &:= l(G/A) = l_0(G) - \mathrm{rk}(A) \quad && \in \mathbb{Z}_{\geq 0}, \\ q_0(G) &:= q(G) - \tfrac{1}{2} l_0(G) = q^\flat(G) - \tfrac{1}{2} l_0^\flat(G) && && \in \mathbb{Z}_{\geq 0}. \end{aligned} $$

These invariants do not depend on the choice of θ and K. Our definition follows [BW00, III.4.3] but allows for a flexible choice of A, which we often quotient out for Shimura varieties and locally symmetric spaces; see Example 2.5 and (4.1) below. Given a connected reductive subgroup $L \subset G$ over $\mathbb{R}$, define

$$ q^\flat(L) := q(L/A) = q(L) - \tfrac{1}{2}\dim A \quad \in \tfrac{1}{2}\mathbb{Z}_{\geq 0}. $$

We have the notion of θ-stable parabolic subalgebras Q of $\mathfrak{g}_{\mathbb{C}}$ and analogously θ-stable parabolic subgroups of $G_{\mathbb{C}}$ as in [NP21, Def. 6]; in particular, Q determines a subgroup $L_Q \subset G$ over $\mathbb{R}$ such that $L_{Q,\mathbb{C}}$ is a Levi of $G_{\mathbb{C}}$. The subgroups of G that arise in this way are called θ-stable Levi subgroups. For example, if $G = \mathrm{GL}_N$ then θ-stable Levi subgroups have the form $L = \mathrm{GL}_{n_0} \times \prod_{i=1}^r \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_{n_i}$, where $r, n_0, ..., n_r \in \mathbb{Z}_{\geq 0}$ and $n_0 + 2\sum_{i=1}^r n_i = N$. See [NP21, §9] for more examples.

Fix pinnings for G and $\widehat{G}$ as well as a θ-stable fundamental maximal torus $T_\infty \subset G$ and a θ-stable Borel subgroup $B \subset G_{\mathbb{C}}$ containing T_∞; such a B always exists by [NP21, Prop. 11]. Write $\mathfrak{g} := \mathrm{Lie}\, G \otimes_{\mathbb{R}} \mathbb{C}$; likewise fraktur letters stand for the complex Lie algebras arising from Lie groups. By $\omega_{\widehat{G}}$ we mean the longest element of the Weyl group of $\widehat{T}$ in $\widehat{G}$, or its lift to $\widehat{G}$.

Example 2.5. Suppose that our real group arises from a Shimura datum (G, X). Then T_∞ is an elliptic maximal torus over $\mathbb{R}$. Write $\mu \in X_*(T_\infty)$ for the dominant minuscule cocharacter over $\mathbb{C}$ arising from the Shimura datum, and $\rho_G \in X^*(T_\infty)$ for the half sum of all positive roots for G. In this case, take $A := A_{G_\mathbb{R}}$ to be the maximal $\mathbb{R}$-split torus in $Z_{G_\mathbb{R}}$ and write

$$\overline{q}(G),\ \overline{l}_0(G),\ \overline{q}(L) \quad \text{for} \quad q^\flat(G),\ l_0^\flat(G),\ q^\flat(L).$$

Then $\overline{q}(G) = \langle 2\rho_G, \mu\rangle$, which equals the complex dimension of X as well as the dimension of the Shimura variety; $\overline{l}_0(G) = 0$ in this case. If we define ρ_L similarly for a θ-stable Levi subgroup L then $\overline{q}(L) = \langle 2\rho_L, \mu\rangle$.

Remark 2.6. For G as in the preceding example, a typical choice of A for the locally symmetric space is not $A_{G_\mathbb{R}}$ but the one coming from the maximal $\mathbb{Q}$-split torus in Z_G. This is why we want a flexible choice of A in the definitions.

Let $\psi \in \Psi_{\mathrm{ra}}(G)$. Conjugating ψ we may and will assume that $\psi|_{W_\mathbb{C}\times T_{\mathrm{SL}_2}}$ has image in $\widehat{T}$ and that $\phi_\psi(z) = \lambda_1(z)\lambda_2(\overline{z})$ with $\lambda_1, \lambda_2 \in X_*(\widehat{T})_\mathbb{Q}$ such that $\lambda_1 - \rho_G$ equals the highest weight $\lambda \in X^*(T) = X_*(\widehat{T})$ of an irreducible algebraic representation E_λ of $G_\mathbb{C}$. Then $\widehat{L} := Z_{\widehat{G}}(\psi(W_\mathbb{C}))$ is a Levi subgroup of $\widehat{G}$, and $\psi|_{\mathrm{SL}_2}$ is a principal morphism into $\widehat{L}$. Moreover, $\widehat{L}$ is invariant under $\omega_{\widehat{G}} \cdot j \in {}^L G$, $\omega_{\widehat{G}} \cdot j(\lambda) = -\lambda$, and λ pairs trivially with every root of $\widehat{T}$ in $\widehat{L}$ by [NP21, Thm. 5]. Then there is a finite nonempty set $\mathfrak{Q}$ of θ-stable standard parabolic subgroups Q such that $\widehat{L}_Q$ (as a standard Levi of $\widehat{G}$) equals $\widehat{L}$. Each $Q \in \mathfrak{Q}$ gives rise to an irreducible unitary representation $\pi(\psi, Q)$ of $G(\mathbb{R})$ obtained by cohomological induction from a one-dimensional representation of $L_Q(\mathbb{R})$. The Adams–Johnson packet is defined to be

$$\Pi_\psi^{\mathrm{AJ}} := \{\pi(\psi, Q) : Q \in \mathfrak{Q}\}.$$

See [NP21, Thm. 6] for other parametrizations of Π_ψ^{AJ}. We remark that members of Π_ψ^{AJ} share the same infinitesimal character, which corresponds to ζ_ψ defined in §2.1.

Since the highest weight representation E_λ may have a non-unitary central character, it is useful to extend the construction to $\psi \in \Psi_{\mathrm{era}}(G)$. In fact Π_ψ^{AJ} can be constructed for $\psi \in \Psi_{\mathrm{e.ra}}(G)$ in the same way as above via cohomological induction from a one-dimensional representation; what we lose is the unitarity of the members $\pi(\psi, Q)$. Note that Π_ψ^{AJ} was considered in [Kot90, p.194] not only for $\psi \in \Psi_{\mathrm{ra}}(G)$ but for $\psi \in \Psi_{\mathrm{e.ra}}(G)$ (assuming G contains an elliptic maximal torus).

In the following, we take $K' := K^\circ A(\mathbb{R})^\circ$ so that $\mathfrak{k}' = \mathfrak{k} \oplus \mathfrak{a}$ holds for their Lie algebras.

Lemma 2.7. *Let $\psi \in \Psi_{\mathrm{era}}(G)$. For each $\pi = \pi(\psi, Q) \in \Pi_\psi^{\mathrm{AJ}}$, we have*

$$H^i(\mathfrak{g}, K', \pi \otimes E_\lambda^\vee) = 0, \quad i \notin [q^\flat(G) - q^\flat(L_Q),\ q^\flat(G) + q^\flat(L_Q)].$$

Proof. We have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and a Levi decomposition $\mathfrak{q} = \mathrm{Lie}\, Q = \mathfrak{l} \oplus \mathfrak{u}$. Since the latter is a θ-stable decomposition, we have $\mathfrak{l} = (\mathfrak{l} \cap \mathfrak{k}) \oplus (\mathfrak{l} \cap \mathfrak{p})$ and $\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{k}) \oplus (\mathfrak{u} \cap \mathfrak{p})$. It follows from [VZ84, Thm. 5.5], cf. [NP21, Thm. 6 (2)], that

$$H^i(\mathfrak{g}, \mathfrak{k}, \pi \otimes E_\lambda^\vee) = 0, \qquad i \notin [\dim(\mathfrak{u} \cap \mathfrak{p}),\ \dim(\mathfrak{u} \cap \mathfrak{p}) + \dim(\mathfrak{l} \cap \mathfrak{p})].$$

One has $\dim(\mathfrak{u} \cap \mathfrak{p}) = \dim \mathfrak{p} - \dim(\mathfrak{l} \cap \mathfrak{p}) = (\dim \mathfrak{g} - \dim \mathfrak{k}) - (\dim \mathfrak{l} - \dim(\mathfrak{l} \cap \mathfrak{k})) = q^\flat(G) - q^\flat(L)$. Similarly $\dim(\mathfrak{u} \cap \mathfrak{p}) + \dim(\mathfrak{l} \cap \mathfrak{p}) = (\dim \mathfrak{g} - \dim \mathfrak{k}) + (\dim \mathfrak{l} - \dim(\mathfrak{l} \cap \mathfrak{k})) = q^\flat(G) + q^\flat(L) + \dim A$. On the other hand, we may assume that $\pi \otimes E_\lambda^\vee$ has trivial central character on $A(\mathbb{R})^\circ$, since $H^i(\mathfrak{g}, \mathfrak{k}, \pi \otimes E_\lambda^\vee) = 0$ otherwise. Then we apply the Künneth formula for the compatible decompositions $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{a}$ and $\mathfrak{k}' = \mathfrak{k} \oplus \mathfrak{a}$ to obtain

$$H^\bullet(\mathfrak{g}, \mathfrak{k}, \pi \otimes E_\lambda^\vee) = H^\bullet(\mathfrak{g}_0, \mathfrak{k}, \pi \otimes E_\lambda^\vee) \otimes H^\bullet(\mathfrak{a}, \mathbf{1}) = H^\bullet(\mathfrak{g}, \mathfrak{k}', \pi \otimes E_\lambda^\vee) \otimes H^\bullet(\mathfrak{a}, \mathbf{1}),$$

where the last equality is satisfied by the definition of the relative Lie algebra cohomology. Since $H^j(\mathfrak{a},\mathbf{1}) = \wedge^j \mathfrak{a}^*$ is nonzero in $j \in [0, \dim A]$, the lemma follows from the non-vanishing range of $H^i(\mathfrak{g},\mathfrak{k},\pi\otimes E_\lambda^\vee)$. $\square$

Lemma 2.8. *Let $\psi \in \Psi_{\mathrm{era}}(G)$. For $\pi = \pi(\psi, Q) \in \Pi_\psi^{\mathrm{AJ}}$, if $H^i(\mathfrak{g}, K', \pi \otimes E_\lambda^\vee) \neq 0$ for some $i \in \mathbb{Z}_{\geq 0}$ then a $\widehat{G}$-conjugate of $\widehat{L}_Q$ is contained in the centralizer* $\mathrm{Cent}_{\widehat{G}}(\lambda)$.

Proof. Up to $\widehat{G}_\infty$-conjugation, [NP21, Prop. 6] allows us to assume that $\psi|_{W_\mathbb{C}}$ has image in $\widehat{T}$, that the centralizer of $\psi|_{W_\mathbb{C}}$ in $\widehat{G}$ equals $\widehat{L}_Q$, and that $\psi|_{W_\mathbb{C}}$ is given by

$$\psi|_{W_\mathbb{C}} : W_\mathbb{C} \to \widehat{T}, \qquad z \mapsto (z/\overline{z})^{\lambda+\rho_G-\rho_{L_Q}},$$

where we identify ρ_G (resp. ρ_{L_Q}) with the half sum of positive coroots in $\widehat{G}$ (resp. $\widehat{L}_Q$), and $\lambda \in X_*(\widehat{T})$ is dominant. In particular, every root α of $\widehat{T}$ in $\widehat{L}_Q$ satisfies $\langle \alpha, \lambda + \rho^\vee_{\widehat{G}} - \rho^\vee_{\widehat{L}_Q}\rangle = 0$. It follows that $\langle \alpha, \lambda\rangle = 0$, hence $\widehat{L}_Q \subset \mathrm{Cent}_{\widehat{G}}(\lambda)$. $\square$

2.9. **Axioms.** We state a minimalistic list of three axioms (A1)–(A3) for local and global Arthur packets as working hypotheses for our paper. Their extensions inclusive of Levi subgroups will be given a superscript +. The reader is referred to [Art89] for refined conjectures in the original form on the internal parametrization of each local Arthur packet, endoscopic character identities, the global Arthur multiplicity formula, etc. We also drew inspiration from [Clo07, Lect. 2] and [Art13]. Notable results on verifying the axioms are highlighted in §2.15 with further references below.

Let G be a connected reductive group over a global field F. For each place v, we write G_v for G_{F_v}. Our first axiom is the existence of a set of global A-parameters (up to isomorphism):

(A1) There exists a set $\Psi(G)$ equipped with a localization map at every place v of F:

$$\Psi(G) \to \Psi^+(G_v), \qquad \psi \mapsto \psi_v. \tag{2.1}$$

Moreover, there exists an intermediate subset

$$\Psi(G_v) \subset \Psi'(G_v) \subset \Psi^+(G_v),$$

which depends only on G_v over F_v, such that the map (2.1) factors through $\Psi'(G_v)$ and such that the map $\Psi'(G_v) \to \Phi(G_v)$, $\psi_v \mapsto \phi_{\psi_v}$, is injective. For each infinite place y, if $\psi \in \Psi(G)$ and the localization ψ_y lies in $\Psi^+_{\mathrm{ra}}(G_y)$ then $\psi_y \in \Psi_{\mathrm{ra}}(G_y)$.

Given axiom (A1), we extend $\Psi(G)$ by certain "character twists" to define

$$\Psi_{\mathrm{e}}(G) := \big\{(\psi,\chi) \,|\, \psi \in \Psi(G),\ \chi : A_G(\mathbb{R})^\circ \to \mathbb{R}^\times_{>0}\big\}, \tag{2.2}$$

where χ runs over (possibly non-unitary) continuous characters. Each χ determines an element $\nu_\chi \in \mathfrak{a}^*_G$ via the natural linear pairing between $\mathrm{Lie}\, A_G(\mathbb{R})^\circ$ and $\mathfrak{a}^*_G$. Write $\nu_{\chi,v}$ for the image of ν_χ under $\mathfrak{a}^*_G \to \mathfrak{a}^*_{G_v}$. Then we can extend (2.1) by setting (recall $\varphi_{\nu_{\chi,v}}$ from §2.1)

$$\Psi_{\mathrm{e}}(G) \to \Psi^+(G_v), \qquad \underline{\psi} = (\psi,\chi) \mapsto \underline{\psi}_v = \varphi_{\nu_{\chi,v}} \cdot \psi_v. \tag{2.3}$$

The image lies in $\Psi'_{\mathrm{e}}(G_v) := \{\varphi_{\nu_v}\cdot\psi' \,|\, \nu_v \in \mathfrak{a}^*_{G_v},\ \psi' \in \Psi'(G_v)\}$.

Now we introduce a strengthening of axiom (A1):

(A1$^+$) For all Levi subgroups M of G, axiom (A1) holds true; moreover there exists a map $\Psi(M) \to \Psi(G)$ which makes the following diagram commute at every place v:

$$\begin{array}{ccc} \Psi(M) & \longrightarrow & \Psi(G) \\ {\scriptstyle (2.1)}\downarrow & & \downarrow{\scriptstyle (2.1)} \\ \Psi^+(M_v) & \longrightarrow & \Psi^+(G_v), \end{array}$$

where the bottom map is induced by the natural map ${}^L M_v \to {}^L G_v$. Moreover the image of $\Psi'(M_v)$ is contained in $\Psi'(G_v)$.

The image of (2.1) should be contained in $\Psi(G_v)$ by a suitable generalization of the Ramanujan conjecture, namely we should be able to take $\Psi'(G_v) = \Psi(G_v)$. However this is a wide open problem. Hence we often take $\Psi'(G_v)$ to be strictly larger.

Each local A-parameter should be assigned a local A-packet, as initially conjectured by Arthur in [Art89, Conj. 6.1, 6.2]. We formulate a coarse axiom tailored to our purpose.

(A2) For every v and $\psi_v \in \Psi(G_v)$, there exists a finite set Π_{ψ_v} consisting of irreducible unitary representations of $G(F_v)$. (The set Π_{ψ_v} is allowed to be empty if G_v is not quasi-split.) The set of isomorphism classes of irreducible tempered representations of $G(F_v)$ is partitioned into Π_{ϕ_v} as ϕ_v runs over $\Phi_{\mathrm{bdd}}(G_v)$. Moreover if v is finite then for each $\psi_v \in \Psi(G_v)$,

- (A2a) suppose G_{F_v} is unramified; then ψ_v is unramified if and only if Π_{ψ_v} contains an unramified representation (with respect to a fixed hyperspecial subgroup, cf. §1.14). If so, the unramified representation in Π_{ψ_v} is unique and has L-parameter ϕ_{ψ_v}.
- (A2b) the Aubert involution $\pi_v \mapsto \widehat{\pi}_v$ (denoted by $\pi_v^{\#}$ in [Aub95, Cor. 3.9]) induces a bijection $\Pi_{\psi_v} \stackrel{\sim}{\to} \Pi_{\widehat{\psi}_v}$.

At each infinite place y the following hold:

- (A2c) every $\pi_y \in \Pi_{\psi_y}$ has infinitesimal character given by ϕ_{ψ_y},
- (A2d) if $\psi_y \in \Psi_{\mathrm{ra}}(G_y)$, then $\Pi_{\psi_y} = \Pi^{\mathrm{AJ}}_{\psi_y}$.

Once Axiom (A2) is granted, its analogue for "essential" Arthur parameters $\psi_v \in \Psi_{\mathrm{e}}(G_v)$ is easily obtained by character twists. More precisely, we take a decomposition $\psi_v = \varphi_\nu \psi_{v,0}$ for $\nu \in \mathfrak{a}^*_{G_v}$ and $\psi_{v,0} \in \Psi(G_v)$, write $\chi_\nu : G(F_v) \to \mathbb{R}^\times_{>0}$ for the character determined by ν, and then define $\Pi_{\psi_v} := \{\pi \otimes \chi_\nu : \pi \in \Pi_{\psi_{v,0}}\}$. Using compatibility of various representation-theoretic operations with character twists, one verifies that the set of isomorphism classes of irreducible essentially tempered representations of $G(F_v)$ is partitioned into Π_{ψ_v} over $\psi_v \in \Phi_{\mathrm{e.bdd}}(G_v)$, and that (A2a)–(A2d) hold true for $\Psi_{\mathrm{e}}(G_v)$; e.g., (A2d) is upgraded to the equality $\Pi_{\psi_y} = \Pi^{\mathrm{AJ}}_{\psi_y}$ for $\psi_y \in \Psi_{\mathrm{era}}(G_y)$. It is useful to strengthen (A2) as follows.

(A2$^+$) For each v, axiom (A2) holds for every Levi subgroup M_v of G_v in place of G_v, inclusive of (A2a)–(A2d); moreover the following holds for each $\psi_{M,v} \in \Psi(M_v)$:

- **(A2e)** all irreducible constituents of the normalized parabolic induction of each $\pi_{M,v} \in \Pi_{\psi_{M,v}}$ to G_v are contained in Π_{ψ_v}, where $\psi_v \in \Psi(G_v)$ is the image of $\psi_{M,v}$.

Remark 2.10. The C-algebraicity at an infinite place is only compatible with the un-normalized parabolic induction. For $\psi_{M,v} \mapsto \psi_v$ as in (A2e), when v is an infinite place, what is preserved is not the C-algebraicity but the essential C-algebraicity.

Upon accepting (A2$^+$), we can assign a local packet to a general parameter $\psi_v \in \Psi^+(G_v)$ following [Art13, p.45]. (For Axiom (A3) below, only the local packet for $\psi_v \in \Psi'(G_v)$ matters.) The parameter ψ_v corresponds to a standard parabolic subgroup $P_v = M_v N_v$, a parameter $\psi_{M,v} \in \Psi(M_v)$, and $\nu_v \in \mathfrak{a}^{*,+}_{M_v}$ by Lemma 2.2. Write $\chi_{\nu_v} : M(F_v) \to \mathbb{R}^\times_{>0}$ for the character given by ν_v, which is unramified if v is non-archimedean; e.g., see [SZ18, §1.2, §A.1]. Then Π_{ψ_v} is defined to consist of irreducible subquotients of $\text{n-ind}^{G(F_v)}_{P_v(F_v)}(\pi_v \otimes \chi_{\nu_v})$ as π_v runs over $\Pi_{\psi_{M,v}}$. In particular, if $\psi_v \in \Psi^+(G_v)$ is unramified then Π_{ψ_v} contains a unique unramified representation $\pi^0_{\psi_v}$ (with respect to a fixed hyperspecial subgroup), whose L-parameter is ϕ_{ψ_v}. Conversely, if Π_{ψ_v} contains an unramified representation then π_v must be unramified. Then $\psi_{M,v}$ is unramified by (A2$^+$), thereby ψ_v is also unramified.

On the other hand, if $\phi_v \in \Phi(G_v)$ is a general L-parameter then the L-packet Π_{ϕ_v} is defined by a similar procedure: again we find $P_v = M_v N_v$, $\phi_{M,v} \in \Phi_{\text{bdd}}(M_v)$, and $\nu_v \in \mathfrak{a}^*_{M_v}$ by Lemma 2.2. Then Π_{ϕ_v} consists of the unique irreducible quotients of $\text{n-ind}^{G(F_v)}_{P_v(F_v)}(\pi_v \otimes \chi_{\nu_v})$ by definition. By the Langlands quotient theorem (and axiom (A2) above), the L-packets Π_{ϕ_v} form a partition of $\text{Irr}(G_v)$ as ϕ_v runs over $\Phi(G_v)$.

Before moving on to the third axiom, we record a useful extension of (A2b).

Lemma 2.11. *Assume axioms (A2) and (A2$^+$). Let $\psi_v \in \Psi^+(G_v)$, to which we assign a packet Π_{ψ_v} as above. Then the involution $\pi_v \mapsto \widehat{\pi}_v$ induces a bijection $\Pi_{\psi_v} \xrightarrow{\sim} \Pi_{\widehat{\psi}_v}$.*

Proof. We have $\psi_{M,v} \in \Psi(M_v)$ and $\nu_v \in \mathfrak{a}^*_{M_v}$ from ψ_v as above. Then the twist of $\widehat{\psi}_{M,v} \in \Psi(M_v)$ by ν_v maps to $\widehat{\psi}_v$. By (A2b), $\Pi_{\widehat{\psi}_{M,v}} = \{\widehat{\pi}_{M,v} : \pi_{M,v} \in \Pi_{\psi_{M,v}}\}$. Hence $\Pi_{\widehat{\psi_v}}$ consists of irreducible subquotients of $\text{n-ind}^{G(F_v)}_{P_v(F_v)}(\widehat{\pi}_{M,v})$ as $\pi_{M,v}$ runs over $\Pi_{\psi_{M,v}}$. Since Aubert duality induces a bijection between irreducible subquotients of $\text{n-ind}^{G(F_v)}_{P_v(F_v)}(\pi_{M,v})$ and those of $\text{n-ind}^{G(F_v)}_{P_v(F_v)}(\widehat{\pi}_{M,v})$ (see [Aub95, p.2189]), we are done. ☐

Given $\psi \in \Psi(G)$, utilizing (A1) and (A2$^+$), define a global A-packet

$$\Pi_\psi := \textstyle\prod'_v \Pi_{\psi_v} = \{\otimes'_v \pi_v \,:\, \pi_v \in \Pi_{\psi_v},\ \pi_v = \pi^0_{\psi_v} \text{ for all but finitely many } v\}. \tag{2.4}$$

Not every member of Π_ψ is to be automorphic in this formulation. Nevertheless our axiom (A3) postulates that the L^2-discrete automorphic spectrum of $G(\mathbb{A}_F)$ should be exhausted by Π_ψ.

(A3) We have a $G(\mathbb{A}_F)$-module decomposition for suitable coefficients $m_\pi \in \mathbb{Z}_{\geq 0}$:

$$L^2_{\text{disc}}([G]) = \bigoplus_{\psi \in \Psi(G)} \bigoplus_{\pi \in \Pi_\psi} m_\pi \pi. \tag{2.5}$$

(A3$^+$) Axiom (A3) is true for all Levi subgroups M of G, presupposing (A1$^+$) and (A2$^+$).

Axiom (A3) implies the following variant. Let $\chi : A_G(\mathbb{R})^\circ \to \mathbb{R}^\times_{>0}$ be a continuous character. Write $\Psi_\chi(G) \subset \Psi_{\text{e}}(G)$ for the preimage of χ under the second projection map, cf. (2.2). Via the projection map $G(F)\backslash G(\mathbb{A}_F) \to A_G(\mathbb{R})^\circ$, we can view χ as an automorphic character and decompose $\chi = \prod_v \chi_v$ over the places v of F. The character $\chi_v : G(F_v) \to \mathbb{R}^\times_{>0}$ is nothing but the character corresponding to the element $\nu_{\chi,v} \in \mathfrak{a}^*_{G_v}$ below Axiom (A1). For $\underline{\psi} = (\psi, \chi) \in \Psi_\chi(G)$, we can define $\Pi_{\underline{\psi}} := \prod'_v \Pi_{\underline{\psi}_v} = \prod'_v \Pi_{\psi_v} \otimes \chi_v$. Twisting (2.5) by χ we obtain

$$L^2_{\text{disc},\chi}([G]) = \bigoplus_{\underline{\psi} \in \Psi_\chi(G)} \bigoplus_{\pi \in \Pi_{\underline{\psi}}} m_\pi \pi, \qquad m_\pi \in \mathbb{Z}_{\geq 0}. \tag{2.6}$$

Remark 2.12. Since we are mainly concerned with cohomology of locally symmetric spaces, we do not need the full strength of the above axioms. Rather, it suffice to understand the packets satisfying the regular C-algebraicity condition at infinity.

Remark 2.13. Even though endoscopic character identities are crucial in the study of A-packets, they do not enter our axioms as we do not make any direct use of them. Similarly we do not ask for parametrization of members of a packet. Axiom (A3) is much weaker than Arthur's multiplicity formula because nothing is said about m_π (e.g., $m_\pi > 0$ for which π); one can think of (A3) as an upper bound on the set of π occurring in the automorphic spectrum.

Example 2.14. For a sanity check we verify axioms (A1)–(A3) when G is a torus. In this case take $\Psi(G)$ to be the continuous cohomology group $H^1(W_F, \widehat{G})$ modulo the locally trivial cohomology group $\ker^1(W_F, \widehat{G})$. (The definition can be rephrased in terms of morphisms $W_F \to {}^LG$.) At each place v we have $\Psi(G_v) = \Phi_{\mathrm{bdd}}(G_v)$ by definition; put $\Psi^+(G_v) := \Phi_{\mathrm{bdd}}(G_v)$ and $\Psi'(G_v) := \Phi_{\mathrm{bdd}}(G_v)$. As explained in [Lan97, Thm. 2], class field theory assigns a unique character $\chi(\phi_v) : G(F_v) \to \mathbb{C}^\times$ to the packets Π_{ϕ_v} for each $\phi_v \in \Phi(G_v)$, in particular for $\phi_v \in \Phi_{\mathrm{bdd}}(G_v)$; each $\phi \in \Psi(G)$ is associated with a character $\chi(\phi) : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}^\times$ such that $\chi(\phi) = \prod_v \chi(\phi_v)$ if ϕ_v denotes the localization of ϕ at each v. Given this, axioms (A1)–(A3) are readily checked.

2.15. **Arthur's formalism for general linear groups and classical groups.** In this subsection we discuss notable cases where the axioms of §2.9 are known. We assume that the global field F is a number field for two reasons: results are more complete in this case, and only this case is relevant to our later discussions where archimedean theory plays a key role.

2.15.1. *Inner forms of* GL_n. We begin with $G = \mathrm{GL}_n$ over F. Following [Art13, §1.4], we define $\Psi(G) = \Psi(\mathrm{GL}_n)$ to be the set consisting of formal sums $\psi = \boxplus_{i=1}^r(\pi_i \boxtimes \nu_i)$, where π_i is a cuspidal automorphic representation of $\mathrm{GL}_{m_i}(\mathbb{A}_F)$, ν_i is an irreducible n_i-dimensional algebraic representation of SL_2 such that $\sum_{i=1}^r m_i n_i = n$. Given ψ and a place v of F, we define $\psi_v = \oplus_{i=1}^r \phi_{\pi_{i,v}} \otimes \nu_i$, where $\phi_{\pi_{i,v}} : \mathcal{L}_{F_v} \to \mathrm{GL}_{m_i}(\mathbb{C})$ is the L-parameter of $\pi_{i,v}$; this is the first part of (A1). For the second part, we take the local parameter set at each v as follows:

$$\Psi'(\mathrm{GL}_n) := \Big\{(\oplus_{i\in I}\psi_i)\oplus\Big(\oplus_{j\in J}\psi_j \otimes (|\cdot|_{W_F}^{\epsilon_j} \oplus |\cdot|_{W_F}^{-\epsilon_j})\Big) : \psi_i \in \Psi(\mathrm{GL}_{n_i}),\ \psi_j \in \Psi(\mathrm{GL}_{n_j}),\ \epsilon_j \in (0, \tfrac{1}{2})\Big\},$$

where the obvious constraint that $\sum_i n_i + \sum_j n_j = n$ is in place. (The sets I and J are allowed to be empty.) Then the injectivity of $\Psi'(\mathrm{GL}_n) \to \Phi(\mathrm{GL}_n)$ is a straightforward combinatorial exercise. The image of (2.1) is contained in $\Psi'(\mathrm{GL}_n)$ by the deep results on the unitary dual of GL_n over local fields by Tadić and Vogan [Tad86, Vog86]. For the last assertion of (A1), it is enough to observe that if ψ_y belongs to $\Psi'(\mathrm{GL}_n)$ as above and if the index set J is non-empty then ψ_y cannot be C-algebraic (nor L-algebraic); if J is empty then the parameter obviously lies in $\Psi(\mathrm{GL}_n)$. Here regularity is not used and only algebraicity matters for the verification. As for (A2), the local A-packets Π_{ψ_v} are defined to be the L-packet $\Pi_{\phi_{\psi_v}}$ given by the local Langlands for GL_n for $\psi_v \in \Psi(G_v)$. Even when ψ_v lies in the larger set $\Psi'(G_v)$ defined above, the packet Π_{ψ_v} (defined in §2.9) equals $\Pi_{\phi_{\psi_v}}$, and Π_{ψ_v} is a singleton whose unique member is a unitary representation by [Tad86, Vog86] mentioned above. Property (A2b) in this case was shown in [Zel80, §9]. (Zelevinsky's involution coincides with the involution of (A2b) by [Aub95, Thm. 2.3].) By definition ψ_v is unramified if and only if ϕ_{ψ_v} is unramified, and (A2c) is clear from the corresponding property of $\Pi_{\phi_{\psi_y}}$. Property (A2d) should be implied by $\Pi_{\phi_{\psi_y}} \subset \Pi^{\mathrm{AJ}}_{\psi_y}$; this can also be seen from the explicit description of Π_{ψ_y} and $\Pi^{\mathrm{AJ}}_{\psi_y}$, cf. [AMR18, §3.2, §11.1] and [NP21, §12]. Axiom (A3) is implied by [MW89], which a fortiori tells us, combined with the multiplicity one theorem, that $m_\psi = 1$ for every ψ and π as in (A3).

Now let G be an inner form of GL_n. The local and global parameter sets remain unchanged (by allowing the corresponding packets to be empty), so (A1) is the same as before. Axioms (A2) and (A3) are consequences of the main theorems of [Bad08, BR10], if the local A-packets for inner forms are defined to be singletons or possibly the empty set as determined via character identities by the local theorem in the introduction of [Bad08] in the p-adic case and [BR10, Thm. 1.2] in the real case. They also show unitarity; in our formulation it means that the unique member of each local A-packet Π_{ψ_v} (if nonempty) is unitary for each v and each $\psi_v \in \Psi(G_v)$ (in fact also for $\psi_v \in \Psi'(G_v)$).

For GL_n and its inner forms, it is straightforward to upgrade (A1)–(A3) to (A1$^+$)–(A3$^+$). For example (A2e) is easily verified from the compatibility of the Langlands correspondence with parabolic induction (and the fact that the parabolic induction of an irreducible unitary representation is always irreducible).

2.15.2. *Quasi-split classical groups.* The case of even special orthogonal group is postponed to the end of §2.15.2. When G is a symplectic or odd special orthogonal group, it is a twisted endoscopic group of GL_N for a suitable $N \in \mathbb{Z}_{\geq 1}$, the set $\Psi(G)$ is realized within the subset of self-dual parameters in $\Psi(\mathrm{GL}_N)$ via the standard embedding ${}^LG \to {}^L\mathrm{GL}_N$. When G is a unitary group with respect to a quadratic extension E/F, it is a twisted endoscopic group of $\mathrm{Res}_{E/F}\mathrm{GL}_N$, and $\Psi(G)$ lies in the subset of conjugate self-dual parameters in $\Psi(\mathrm{GL}_{N,E})$. For either type of G, axioms (A1$^+$)–(A3$^+$) including the characterization of $\Psi(G)$ are proved in [Art13, Mok15] in much more precise forms, except that (A2d) is verified in [AMR18]. We remark that (A1$^+$)–(A3$^+$) for quasi-split classical groups include (A1$^+$)–(A3$^+$) for GL_n since every proper Levi subgroup has general linear groups as direct factors.

Some further explanation may be appropriate regarding (A2e), (A2c), (A2b), (A2a), and the last two assertions of (A1) as they are hard to locate in the main theorems of [Art13, Mok15]. (Once they are understood for classical groups G, it is not hard to extend to Levi subgroups.) Axiom (A2e) is verified in the proof of [Art13, Prop. 2.4.3]. Even though axiom (A2c) is not explicitly stated, it results from the twisted endoscopic character identity, which reduces the axiom to the analogue for general linear groups since the twisted transfer is compatible with infinitesimal characters, cf. the argument in [Taï19, p.867]. Axiom (A2b) can be extracted from [Art13, §7.1] and its adaptation to unitary groups. The "if" part and the uniqueness in (A2a) are only implicit in [Art13, Mok15] but these can be derived from *loc. cit.* as explained in [AHKO, App. C], [Taï17, Lemma 4.1.1]. The last assertion of (A2a) is [Art13, Prop. 7.4.1] and [Mok15, Prop. 8.4.1]. We add that once (A2a) is verified for one hyperspecial subgroup, it is true for all hyperspecial subgroups because the natural action of $G^{\mathrm{ad}}(F_v)$ is transitive on all hyperspecial subgroups while $G^{\mathrm{ad}}(F_v)$ induces a permutation of Π_{ψ_v} since $G^{\mathrm{ad}}(F_v)$ preserves the stable character associated with Π_{ψ_v}. As for (A1), we take $\Psi'(G_v)$ to be the preimage of $\Psi'(\mathrm{GL}_N)$ (see §2.15.1) under $\Psi^+(G_v) \to \Psi^+(\mathrm{GL}_N)$. Our $\Psi'(G_v)$ is the same as $\widetilde{\Psi}^+_{\mathrm{unit}}(G_v)$ in [Art13, p.45], and the localization lies is in this set. The injectivity of $\Psi'(G_v) \to \Phi(G_v)$ reduces to the case of GL_N since the map $\Psi'(G_v) \to \Psi'(\mathrm{GL}_N)$ is injective. For the last part of (A1), if $\psi_y \in \Psi'(G_y)$ is localized from ψ then ψ_y transfers to a parameter $\psi_y^\#$ in $\Psi'(\mathrm{GL}_n)$ (over $\mathbb{R}$ or $\mathbb{C}$) by the endoscopic classification. Then the same argument in §2.15.1 shows that $\psi_y^\# \in \Psi(\mathrm{GL}_n)$, hence ψ_y also belongs to $\Psi(G_y)$.

Returning to even special orthogonal groups, everything above is true if we work consistently with the outer automorphism orbits of various objects. (For the last assertion of (A1) the transferred parameter $\psi_y^\#$ is only L-algebraic and possibly non-regular but the argument of §2.15.1 for that part still works, as we explained there.) The reader is referred to [Art13] for the precise formulation and statements.

It should be noted that the above results of [Art13, Mok15] are *conditional* (only) on the validity of the twisted weighted fundamental lemma in general; see [AGI$^+$] and the discussion in §0.4 therein.

2.15.3. *Non-quasi-split classical groups.* At the outset let us mention our convention to take the local and global parameter sets to be the same as their counterparts for quasi-split inner forms. Instead of imposing the relevance condition for parameters, we allow packets to be empty.

Axioms (A1)–(A3) should follow from the stronger but provisional results and conjectures stated in [Art13, Ch. 9] and [KMSW, §§1.6–1.7], which have yet to be established. Axiom (A3) is a coarser version of the main global theorem. There is nothing new concerning (A1) and (A2a) in that they are assertions about the quasi-split case. Since the provisional local A-packets are going to satisfy endoscopic character identities, we can check (A2c). The same identities are a starting point for (A2d), but verification requires serious work; this is complete for pure inner forms by [MR19, MR20], which obtain an explicit description of A-packets for all A-parameters (which may not be regular C-algebraic). The upgrade to (A1$^+$)–(A3$^+$) should go in the same way as in the quasi-split case; e.g., the proof of (A2e) can be adapted from the quasi-split case (see §2.15.2 above) to the non-quasi-split case as in [KMSW, §2.6]. In summary, once the assertions in [Art13, Ch. 9] and [KMSW, §§1.6–1.7] are proven, all of (A1$^+$)–(A3$^+$) are going to be true at least for pure inner forms of quasi-split classical groups.

3. On Arthur SL_2-morphisms

3.1. **Assignment of a nilpotent conjugacy class.** Let $\widehat{G}$ denote the Langlands dual group of a connected reductive group G over a local field k. We have a natural bijection between the following sets:

(i) $\widehat{G}(\mathbb{C})$-conjugacy classes of algebraic morphisms $\mathrm{SL}_2 \to \widehat{G}$,
(ii) $\widehat{G}(\mathbb{C})$-conjugacy classes of Lie algebra morphisms $\mathfrak{sl}_2 \to \mathrm{Lie}\,\widehat{G}$,
(iii) $\widehat{G}(\mathbb{C})$-conjugacy classes of nilpotent elements in $\mathrm{Lie}\,\widehat{G}(\mathbb{C})$.

The map from (i) to (ii) is induced by differentiation, and (ii) to (iii) by evaluation at $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. In fact the bijection works for an arbitrary connected reductive group over a field of characteristic 0 in place of $\widehat{G}$ over $\mathbb{C}$; e.g., see [BMIY, Thm. 3.2]. A morphism $\mathrm{SL}_2 \to \widehat{G}$ is said to be *principal* if it corresponds to the conjugacy class of regular nilpotent elements in (iii).

Write $\mathit{Nilp}_{\widehat{G}}$ for the set consisting of (iii) above. This set is equipped with a partial ordering $\le$ such that $N_1 \le N_2$ if the closure of N_2 contains N_1 when viewing N_1 and N_2 as locally closed subset in the nilpotent cone of $\widehat{G}$.

Given a parameter $\psi \in \Psi^+(G)$, we have a nilpotent conjugacy class

$$\mathcal{N}_\psi \in \mathit{Nilp}_{\widehat{G}}$$

corresponding to the restriction $\psi|_{\mathrm{SL}_2^A}$. On the other hand, if $\widehat{L} \subset \widehat{G}$ is a connected reductive subgroup (possibly well-defined up to $\widehat{G}$-conjugacy), we define

$$\mathcal{N}(\widehat{L}) \in \mathit{Nilp}_{\widehat{G}}$$

to be the nilpotent conjugacy class corresponding to some (thus any) principal morphism $\mathrm{SL}_2 \to \widehat{L}$. We remark that the assignment $\widehat{L} \mapsto \mathcal{N}(\widehat{L})$ is injective when $\widehat{L}$ runs over $\widehat{G}$-conjugacy classes of Levi subgroups by [CM93, Thm. 8.1.1]. When $\widehat{L}$ arises from $\psi \in \Psi_{\mathrm{ra}}(G)$ as in §2.4, we have $\mathcal{N}_\psi = \mathcal{N}(\widehat{L})$ by definition. For $\psi_\alpha \in \Psi_{\mathrm{era}}(G)$ obtained by twisting ψ, we still have $\mathcal{N}_{\psi_\alpha} = \mathcal{N}(\widehat{L})$.

From here, assume that k is non-archimedean and that axiom (A2$^+$) holds true for G over k in place of G_v over F_v there. Each $\pi \in \mathrm{Irr}(G)$ admits the Aubert–Zelevinsky dual $\widehat{\pi} \in \mathrm{Irr}(G)$ by

[Aub95]. As a consequence of axiom (A2$^+$), as explained in the paragraph below (A2$^+$), we have a unique L-parameter $\phi_{\widehat{\pi}} : W_k \times \mathrm{SL}_2^D(\mathbb{C}) \to {}^L G$ whose packet contains $\widehat{\pi}$. Hence $\phi_{\widehat{\pi}}|_{\mathrm{SL}_2^D(\mathbb{C})}$ determines a nilpotent conjugacy class, to be denoted

$$\mathcal{N}(\pi) \in \mathit{Nilp}_{\widehat{G}}.$$

The significance of this definition stems from our belief in the following axiom:

(CO(π)) For every $\psi \in \Psi(G)$ such that $\pi \in \Pi_\psi$, we have $\mathcal{N}_\psi \le \mathcal{N}(\pi)$.

Indeed, this goes back to Clozel's ideas [Clo11, p.103], one precise version of which was confirmed by Mœglin [Mœg09] for classical groups, cf. §3.3 below. Axiom (CO(π)) is also closely related to the closure ordering conjecture [Xu24, Conj. 3.1], which is natural from the perspective of p-adic Adams–Barbasch-Vogan packets as in [CFM$^+$22, Vog93]. More precisely, axiom (CO(π)) follows from (a version of) the closure ordering conjecture for the Aubert duals $\widehat{\pi}$ and $\Pi_{\widehat{\psi}}$ together with axiom (A2b).[2]

In fact, it is reasonable to expect the following extension of (CO(π)) is true when axiom (A1) is available, which provides us with the set $\Psi'(G)$:

(CO$'$(π)) For every $\psi \in \Psi'(G)$ such that $\pi \in \Pi_\psi$, we have $\mathcal{N}_\psi \le \mathcal{N}(\pi)$.

There is another way to assign a nilpotent conjugacy class in the unramified case. Assume that G is an unramified group over k. Let $\Psi'(G)$ be a set such that $\Psi(G) \subset \Psi'(G) \subset \Psi^+(G)$ and such that the map $\Psi'(G) \to \Phi(G)$, $\psi \mapsto \phi_\psi$, is injective. As above, $\Psi'(G)$ naturally comes from axiom (A1). We continue to assume (A2$^+$) for G over k. Now consider $\pi \in \mathrm{Irr}(G)$ that is unramified, and *assume* that $\pi \in \Pi_{\psi_\pi}$ for some $\psi_\pi \in \Psi'(G)$. Let

$$\mathcal{N}^{\mathrm{ur}}(\pi) \in \mathit{Nilp}_{\widehat{G}}$$

denote the nilpotent conjugacy class corresponding to $\psi_\pi|_{\mathrm{SL}_2^A}$. For well-definedness of $\mathcal{N}^{\mathrm{ur}}(\pi)$, it is enough to see that $\pi \in \Pi_{\psi_\pi}$ for a unique ψ_π. Observe that such a ψ_π is unramified, and the L-parameter of π is ϕ_{ψ_π} by (A2a) and its extension to $\Psi^+(G)$; this was explained in §2.9. Then ψ_π is uniquely determined by the L-parameter of π since the map $\Psi'(G) \to \Phi(G)$ in (A1) is injective.

Lemma 3.2. *In the unramified setting of the preceding paragraph, $\mathcal{N}^{\mathrm{ur}}(\pi) = \mathcal{N}(\pi)$.*

Proof. By (A2b) and its extension to $\Psi'(G)$ (see §2.9), we obtain $\widehat{\pi} \in \Pi_{\widehat{\psi}_\pi}$ from $\pi \in \Pi_{\psi_\pi}$. Since ψ_π is unramified, it is trivial on SL_2^D, so $\widehat{\psi}_\pi$ is an L-parameter, i.e., trivial on SL_2^A. Hence, by definition, $\mathcal{N}(\pi)$ corresponds to $\widehat{\psi}_\pi|_{\mathrm{SL}_2^D}$, but the latter equals nothing but $\psi_\pi|_{\mathrm{SL}_2^A}$. ☐

In practice, the local data above arise from axioms (A1$^+$)–(A3$^+$) by localizing a group G over a global field F at a finite place v. In that case it is not a serious restriction to require an unramified π_v to be in Π_{ψ_v} for a $\Psi'(G)$. If not, we know from the axioms that π_v does not show up in the spectral decomposition of (A3) so such a π_v is negligible in global applications.

3.3. **Axiom (CO$'$(π)) for GL_n and classical groups.** Let k be a p-adic field. Let us verify axiom (CO$'$(π)) for $G = \mathrm{GL}_n$ over k; we will shortly see that the equality holds. Suppose $\pi \in \Pi_\psi$ for $\psi \in \Psi'(G)$. By (A2b) $\widehat{\pi} \in \Pi_{\widehat{\psi}} = \Pi_{\phi_{\widehat{\psi}}}$. Hence the L-parameter $\phi_{\widehat{\pi}}$ of $\widehat{\pi}$ is equal to $\phi_{\widehat{\psi}}$. So we have

$$\phi_{\widehat{\pi}}|_{\mathrm{SL}_2^D} = \phi_{\widehat{\psi}}|_{\mathrm{SL}_2^D} = \widehat{\psi}|_{\mathrm{SL}_2^D} = \psi|_{\mathrm{SL}_2^A}, \tag{3.1}$$

[2]In [HLLZ], there are several versions of closure ordering conjectures. Roughly in their notation, the one we need here is $\phi_{\widehat{\pi}} \ge_D \widehat{\psi}$, equivalently $\psi \ge_A \widehat{\phi}_{\widehat{\pi}}$ ($\widehat{\phi}_{\widehat{\pi}}$ is taken as a generalized A-parameter), while [Xu24, Conj. 3.1] claims a stronger inequality $\phi_{\widehat{\pi}} \ge_C \phi_{\widehat{\psi}}$. Refer to [HLLZ, Rem. 4.6] for the relation between the different orderings.

where the last two equalities hold by definition (see §2.1). It follows that $\mathcal{N}(\pi) = \mathcal{N}_\psi$.

We expect $(\mathrm{CO}'(\pi))$ to be true for inner forms of GL_n but do not check it here. We merely observe that the equality does not always hold. For example, if $G = D^\times$ for a central division algebra D over k and π is the trivial representation, then $\widehat{\pi} = \pi$ and $\phi_{\widehat{\pi}}$ is principal on SL_2^D. On the other hand, $\pi \in \Pi_\psi$ when $\psi = \phi_{\widehat{\pi}}$ is viewed as an A-parameter. Then $\mathcal{N}(\psi) = \{0\} < \mathcal{N}(\pi)$. In this case, the preceding argument for GL_n does not apply since $\Pi_{\phi_{\widehat{\psi}}}$ is empty.

Now consider a quasi-split classical group G over k. Recall from §2.15.2 that we have the set $\Psi'(G)$ of generalized A-parameters. (This set is stable under the involution $\psi \mapsto \widehat{\psi}$, as this is the case for each of $\Psi'(\mathrm{GL}_n)$ and $\Psi(G)$.) The reader is also reminded that Π_ψ is defined for each $\psi \in \Psi'(G)$. Hence axiom $(\mathrm{CO}'(\pi))$ has an unequivocal meaning. This is proven by Mœglin in [Mœg09, §6.3] for $\Psi(G)$; she was inspired by Clozel's ideas, and not by the p-adic ABV theory. The case of $\Psi'(G)$ reduces to the case of $\Psi(G)$: the members of $\Pi_\psi, \psi \in \Psi'(G)$ are defined after **(A2$^+$)**, and they are irreducible subquotients of parabolic induction of twists of members of Π_{ψ_M} for certain $\psi_M \in \Psi(M)$ for a Levi subgroup M. However, these parabolic inductions are irreducible, thanks to [Mœg11, Prop. 5.1], and their L-parameters are induced from those of M. Therefore, it reduces to checking the claim for ψ_M, which is covered by Mœglin as already mentioned.

Let us also note that a stronger version, i.e., $\phi_{\widehat{\pi}} \geq_C \widehat{\phi_\psi}$ using the closure relation $\geq_C$ in the Vogan variety, is verified when G is the split group SO_{2n+1} or Sp_{2n} in a recent paper [HLLZ, Thm. 1.3, Rem. 2.5], after the works of Mœglin, Xu, and Atobe on A-packets.

Example 3.4. Assume G is the split group SO_{2n+1}, O_{2n}, or Sp_{2n}. One interesting example is the case π is *generic* in the sense that it has a Whittaker model. In this case, [HLLZ, Lem. 7.11] implies that, if π belongs to Π_ψ for some $\psi \in \Psi'(G)$, then $\mathcal{N}(\pi) = \{0\}$. The extra assumption is negligible in global applications. Combined with axiom $(\mathrm{CO}'(\pi))$, this means that every $\psi \in \Psi'(G)$ such that $\pi \in \Pi_\psi$ is *generic* in the sense that $\psi|_{\mathrm{SL}_2^A}$ is trivial, i.e., $\psi = \phi_\psi$. In particular, if $\psi \in \Psi(G)$ then ψ is tempered. This statement is known as the enhanced Shahidi conjecture [LS25, Conj. 1.6].

In this vein, it is natural to call π *A-generic* when $\mathcal{N}(\pi) = \{0\}$ in our framework. (This notion of genericity has to do with A-parameters or Aubert-dual, hence the terminology.) As above, axiom $(\mathrm{CO}'(\pi))$ implies that, for such a π, any $\psi \in \Psi'(G)$ such that $\pi \in \Pi_\psi$ is *generic*. And [HLLZ, Lem. 7.11] implies that any generic π is A-generic.

For general $\phi \in \Phi(G)$ (not necessarily of the form ϕ_ψ for $\psi \in \Psi'(G)$), there is the notion of open L-parameters [Sol, CDFZ]; see [DHKM, Prop. 6.10] for equivalent characterizations. For quasi-split groups, it is expected that ϕ is open if and only if its L-packet Π_ϕ contains a generic representation, and it is known for classical groups [CDFZ, Cor. 4.8].

3.5. **Extension along central morphisms.** Recall that the definition of $\mathcal{N}(\pi)$ relies on local Langlands, but the local Langlands parametrization does not fully propagate along central morphisms. (E.g., our understanding of local Langlands for GSp_{2n} is more limited than that for Sp_{2n}.) The purpose of this section is to extend the definition of $\mathcal{N}(\pi)$ along central morphisms nevertheless.

Suppose (A2$^+$) holds for G. Let $G_{\mathrm{der}} \to H$ be a central morphism over a non-archimedean local field k inducing an isogeny $G_{\mathrm{der}} \to H_{\mathrm{der}}$. Let π be an irreducible unitary representation of $H(k)$. By [LS19, Prop. 4.1.3], its pullback to $G_{\mathrm{der}}(k)$ is the direct sum of finitely many irreducible unitary representations π_i of $G_{\mathrm{der}}(k)$ that are $H(k)$-conjugate. Again by [LS19, Prop. 4.1.3], each π_i appears in the restriction of some irreducible unitary representation $\widetilde{\pi}_i$ of $G(k)$, unique up to certain character twists. According to the desiderata for the local Langlands correspondence regarding the adjoint action and character twists [Xu16, pp.1802–1804], $\mathcal{N}(\widetilde{\pi}_i)$ should be independent of lifts of π_i and i. In this case, we obtain a well-defined nilpotent conjugacy class $\mathcal{N}(\pi) \in \mathit{Nilp}_{\widehat{H}}$. We may temporarily consider all possible $\mathcal{N}(\widetilde{\pi}_i)$ and corresponding nilpotent conjugacy classes of $\widehat{H}$.

Example 3.6. In this example, we use results from §2.15.1.

(i) If $G = \mathrm{GL}_n$ and $H = G_{\mathrm{der}} = \mathrm{SL}_n$, it follows from [HS12, Ch.12] that $\mathcal{N}(\widetilde{\pi})$ is well-defined. For instance, π is generic only if $\mathcal{N}(\widetilde{\pi})$ is trivial as π has to be the restriction of a generic representation of $\mathrm{GL}_n(k)$.

(ii) Let $G = \mathrm{GL}_n$ and $H = \mathrm{PGL}_n$. We may regard π as an irreducible representation of $\mathrm{GL}_n(k)$ as the natural map $\mathrm{GL}_n(k) \to \mathrm{PGL}_n(k)$ is surjective. Its restriction to $\mathrm{SL}_n(k)$ gives rise to π_i above. Therefore, $\mathcal{N}(\widetilde{\pi}_i)$ is well-defined and independent of i, and it is the same class as $\mathcal{N}(\pi)$ when we regard π as an irreducible representation of $\mathrm{GL}_n(k)$.

(iii) Suppose $G = \mathrm{Res}^{F'}_{\mathbb{Q}} \mathrm{GL}_n$ for a totally real field F'. Let $H \subset G$ be the inverse image of $\mathbf{G}_m$ under the determinant map $G \to \mathrm{Res}^{F'}_{\mathbb{Q}} \mathbf{G}_m$. (The dual group $\widehat{H}$ can be described as in [DLP19, Prop. 2.6].) If $k = \mathbb{Q}_v$ for a finite place v and π is as above, then $\mathcal{N}(\widetilde{\pi})$ is again well-defined. This example is related to Hilbert modular varieties when $n = 2$.

Example 3.7. We make use of the results described in Section 2.15.2. Consider quasi-split $H = \mathrm{GSp}_{2n}, \mathrm{GO}^{\eta}_{2n}, \mathrm{GU}_n$ with $G = \mathrm{Sp}_{2n}, \mathrm{SO}^{\eta}_{2n}, \mathrm{U}_n$. Let π be an irreducible representation of $H(k)$. All the irreducible summands of the restriction of π to $G(k)$ are contained in a single L-packet Π_ϕ, up to $\mathrm{O}^{\eta}_{2n}(k)$-conjugation in the second case as in [Art13]. (Since the irreducible summands are in the same $H(k)$-orbit, this follows from the fact that the L-packet of π, up to $\mathrm{O}^{\eta}_{2n}(k)$-conjugation in the orthogonal case, is preserved under conjugation by $H(k)$-conjugation. Indeed, the stable character of an L-packet for $G(k)$ does not change under the $H(k)$-conjugation.) As the restriction from $H(k)$ to $G(k)$ commutes with the Aubert involution, $\mathcal{N}(\pi)$ is well-defined and determined by the common L-parameter of the summands of the restriction of $\widehat{\pi}$. (For unitary groups, $G \neq H_{\mathrm{der}}$ but this construction provides the same nilpotent conjugacy class as before.)

Suppose moreover that G and H are unramified over v and that $G_{\mathrm{der}} \to H$ extends to a map of reductive models over the ring of integers $\mathcal{O}_k$. If π is unramified, there exists unique i such that π_i is unramified.[3] There exists an unramified lift $\widetilde{\pi}_i$, unique up to an unramified twist. The nilpotent conjugacy class $\mathcal{N}(\widetilde{\pi}_i)$ is independent of unramified lifts, so it would be reasonable to define $\mathcal{N}(\pi)$ to be the corresponding nilpotent conjugacy class of H.

3.8. **Definition of the main invariants.** In this subsection we define certain (half-)integral invariants, which measure the distance from the middle degree (which can be a half-integer in general) to the degree of interest in our vanishing conjectures below.

Let G a connected reductive group over a number field F. Fix a minimal F-rational parabolic subgroup P_0 and its Levi subgroup M_0. Let T be a maximal torus contained in M_0. Define

$$G_\infty := (\mathrm{Res}^F_{\mathbb{Q}} G) \times_{\mathbb{Q}} \mathbb{R} = \textstyle\prod_{y|\infty} G_y,$$

and define T_∞ likewise. Write A_G for the maximal $\mathbb{Q}$-split torus in the center of $\mathrm{Res}^F_{\mathbb{Q}} G$, and $A_{G,\mathbb{R}}$ for its base change to $\mathbb{R}$. We take $A = A_{G,\mathbb{R}}$ to define invariants such as $q^\flat(G_\infty) = q(G_\infty / A_{G,\mathbb{R}})$ in §2.4. Let $\mathcal{N} \in \mathit{Nilp}_{\widehat{G}_\infty}$ and $\lambda \in X^*(T_\infty)^+ = X_*(\widehat{T}_\infty)^+$. We define an invariant

$$a^{(2)}_G(\mathcal{N}, \lambda) := \max_{L \subset G_\infty} q^\flat(L) \in \tfrac{1}{2}\mathbb{Z}_{\geq 0}, \tag{3.2}$$

where $L = \prod_{y|\infty} L_y$ runs over the θ-stable Levi subgroups of $G_\infty = \prod_{y|\infty} G_y$ such that $\mathcal{N}(\widehat{L}_y) = \mathcal{N}$ for all $y|\infty$ and such that $\mathrm{Cent}_{\widehat{G}_\infty}(\lambda)$ contains a $\widehat{G}_\infty$-conjugate of $\widehat{L}$. The above number will enter

[3] As all the summands are conjugate under $G(k)$, each summand is unramified with respect to some hyperspecial subgroup.

our conjecture on the range of non-vanishing degrees for the L^2-cohomology of locally symmetric spaces whose coefficient sheaf is determined by λ. In case there is no such L, we set $a^{(2)}_G(\mathcal{N}) := -\infty$.

For Shimura varieties, we choose $A = A_{G_{\mathbb{R}}}$ as in Example 2.5; this differs from the above choice of A in general. Using the notation from that example, we define a variant

$$\overline{a}^{(2)}_G(\mathcal{N},\lambda) := \max_{L\subset G_\infty} \overline{q}(L) = \max_{L\subset G_\infty} q(L/A_{G_{\mathbb{R}}}) \in \tfrac{1}{2}\mathbb{Z}_{\geq 0}. \tag{3.3}$$

Remark 3.9. The definitions (3.2) and (3.3) are motivated by the relative Lie algebra computation in Lemma 2.7. Compare with Theorem 4.9 below.

We return to locally symmetric spaces and the choice $A = A_{G,\mathbb{R}}$. The analogue of $a^{(2)}_G(\mathcal{N},\lambda)$ for ordinary or compactly supported cohomology is motivated by Franke's formula and requires more preparation. Write $W_{G_\infty} = N_{G_\infty}(T_\infty)/T_\infty$ for the (absolute) Weyl group. For each $w \in W_{G_\infty}$ and $\lambda \in X_*(\widehat{T}_\infty)^+$ set

$$w \star \lambda := w(\lambda+\rho) - \rho.$$

Given an F-rational Levi M of G, write $W^{M_\infty}_{G_\infty}$ for the set of minimal-length representatives in the Weyl group W_{G_∞} for $N_{G_\infty}(M_\infty)/M_\infty$. So $W^{M_\infty}_{G_\infty}$ consists of $w \in W_{G_\infty}$ such that w^{-1} sends positive roots in M_∞ into the set of positive roots of G_∞. Define

$$a_G(\mathcal{N},\lambda) := \max_{(M,L_M,w)} \left(a^{(2)}_M(\mathcal{N}_M, w\star\lambda) + q(G_\infty/A_{G,\mathbb{R}}) - q(M_\infty/A_{M,\mathbb{R}}) - l(w)\right) \in \tfrac{1}{2}\mathbb{Z}_{\geq 0}, \tag{3.4}$$

where M is an F-rational Levi of G, $L_M = \prod_{y|\infty} L_{M,y}$ is a θ-stable Levi of M_∞, and $w \in W^{M_\infty}_{G_\infty}$ such that

- (i) $\mathcal{N}_M(\widehat{L}_{M,y}) \mapsto \mathcal{N}$ under the natural map $\mathit{Nilp}_{\widehat{M}_\infty} \to \mathit{Nilp}_{\widehat{G}_\infty}$ for all $y|\infty$,
- (ii) $\mathrm{Cent}_{\widehat{M}_\infty}(w\star\lambda)$ contains an $\widehat{M}_\infty$-conjugate of $\widehat{L}_M$,
- (iii) $-\mathrm{Re}\,([w(\lambda+\rho)])_{\mathfrak{a}^*_{M,\mathbb{C}}}$ belongs to the set $W(\lambda+\rho)_{\mathfrak{c}}$ defined in [Wal97, 4.6–4.7]; in particular $[w(\lambda+\rho)]_{\mathfrak{a}^*_M}$ pairs non-positively with each positive root in G,
- (iv) the restriction of $w(\lambda+\rho)$ to the maximal split subtorus of each fundamental maximal torus in M^{der}_∞ is trivial.

Again if no triple (M, L_M, w) as above exists, then set $a_G(\mathcal{N},\lambda) := -\infty$.

Remark 3.10. The definition (3.4) is motivated by Franke's spectral sequence [Fra98, Thm. 19.I]. See also [Wal97, p.151, Cor.]; condition (iv) is a necessary condition for the cohomology in the formula of *loc. cit.* to be non-vanishing. We also mention that the above definition could be tweaked by requiring that $\mathcal{N}_M(\widehat{L}_{M,y}) \leq \mathcal{N}$ in (i) if $\mathcal{N}_M(\widehat{L}_{M,y})$ is viewed as an element of $\mathit{Nilp}_{\widehat{G}_\infty}$, to account for the fact that the assignment of (generalized) A-parameters need not commute with parabolic induction of non-unitary representations. However we do not know of an example where this changes the value of $a_G(\mathcal{N},\lambda)$.

Lemma 3.11. *Assume that either $\mathcal{N}=0$ or that λ is regular. Then*

$$a_G(\mathcal{N},\lambda) = a^{(2)}_G(\mathcal{N},\lambda) = \tfrac{1}{2} l^\flat_0(G_\infty).$$

Proof. By assumption, the only L allowed in (3.2) is a fundamental maximal torus, say T_∞. Hence $a^{(2)}_G(\mathcal{N},\lambda) = q^\flat(T_\infty) = \frac{1}{2}l^\flat_0(G_\infty)$.

To see that $a_G(\mathcal{N},\lambda) = a^{(2)}_G(\mathcal{N},\lambda)$, we need to verify that the quantity in (3.4) for each (M, L_M, w) is bounded by $\frac{1}{2}l^\flat_0(G_\infty)$. If $\mathcal{N}=0$ then $\mathcal{N}_M = 0$ and L_M must be a fundamental maximal torus of M_∞. The same holds for $\mathcal{N}_M$ and L_M if λ is regular for G_∞, since $w\star\lambda$ is then regular for M by [Sch94, Lem. 4.9]. It follows that $a_M^{(2)}(\mathcal{N}_M, \lambda) = \frac{1}{2} l_0^\flat(M_\infty)$. Hence we will be done if we show

$$\tfrac{1}{2} l_0^\flat(M_\infty) + q(G_\infty/A_{G,\mathbb{R}}) - q(M_\infty/A_{M,\mathbb{R}}) - l(w) \le l_0^\flat(G_\infty),$$

or equivalently, by the definition of $q_0(\cdot)$,

$$l(w) \ge q_0(G_\infty) - q_0(M_\infty).$$

This is exactly [LS04, (4.1)], and their proof carries over after reconciling the notation (e.g., our $G_\infty, M_\infty, \lambda$ correspond to their G, MA, Λ) and the conditions: their condition (a) is satisfied by our (iii) above, and while their condition (b) is not applicable in our setting, what we need is only its consequence that their ν' vanishes; the latter is provided by our condition (iv). □

Example 3.12. In case the nilpotent orbit $\mathcal{N}$ is regular, only $M = G$ can contribute to (3.4). We need that $\mathrm{Cent}_{\widehat{G}}(\lambda)$ meets $\mathcal{N}$, for the formulas in (3.2) and (3.4) to be non-vacuous. Since the centralizer of a regular unipotent element is a product of Z_G and a unipotent subgroup (see [Spr66, Lem. 4.3]), it follows that λ is a central cocharacter of $\widehat{G}$. If so, $a_G^{(2)}(\mathcal{N}, \lambda) = a_G(\mathcal{N}, \lambda) = q^\flat(G_\infty)$.

Example 3.13. In this example and the next, let $G = \mathrm{GL}_n$ over $F = \mathbb{Q}$ with P_0 the upper triangular Borel subgroup and $M_0 = T$ the diagonal maximal torus; we concentrate on $\lambda = 0$. Here we take $n = 3$, and $\mathcal{N}$ to be the orbit of a regular nilpotent element in the Levi subgroup $\mathrm{GL}_2 \times \mathrm{GL}_1$. The θ-stable Levi subgroups of GL_3 are either GL_3 itself or of the form $\mathrm{GL}_1 \times \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_1$, so $a_G^{(2)}(\mathcal{N}, 0) = -\infty$. The maximum in (3.4) is realized by $M = L_M = \mathrm{GL}_2 \times \mathrm{GL}_1$ and w such that $w \star 0 = w\rho - \rho = (0, -1, 1) \in X_*(T_\infty)$ in the standard coordinate. Namely $a_G(\mathcal{N}, 0)$ equals

$$a_M^{(2)}(\mathcal{N}_M, w \star \lambda) + q(G_\infty/A_{G,\mathbb{R}}) - q(M_\infty/A_{M,\mathbb{R}}) - l(w) = 0 + \tfrac{5}{2} - 1 - 1 = \tfrac{1}{2}.$$

Example 3.14. Now let $G = \mathrm{GL}_4$. Consider $\mathcal{N}$ arising from a regular unipotent element in $\mathrm{GL}_2 \times \mathrm{GL}_2$. Then $L = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_2$ contributes to (3.2) and we can compute $a_G^{(2)}(\mathcal{N}, 0) = q^\flat(L) = \frac{3}{2}$. In (3.4) the maximum is realized by not only $(M, L_M, w) = (G, L, 1)$ but also by $M = L_M = \mathrm{GL}_2 \times \mathrm{GL}_2$ and w is a length 2 element such that $w \star 0 \in \{(0, -2, 1, 1), (-1, -1, 2, 0)\}$. Indeed, in the latter case we have

$$a_M^{(2)}(\mathcal{N}_M, w \star \lambda) + q(G_\infty/A_{G,\mathbb{R}}) - q(M_\infty/A_{M,\mathbb{R}}) - l(w) = 1 + \tfrac{9}{2} - 2 - 2 = \tfrac{3}{2}.$$

Question 3.15. If $a_G^{(2)}(\mathcal{N}, \lambda) \ge 0$, i.e., if the maximum in (3.2) is not vacuous, then do we have $a_G(\mathcal{N}, \lambda) = a_G^{(2)}(\mathcal{N}, \lambda)$?

In other words, the question is whether the maximum in (3.4) is attained by the $M = G$ case. Small evidence is provided by Lemma 3.11 and Examples 3.12–3.14, but the authors do not know in general.

4. Vanishing range for the L^2-cohomology of locally symmetric spaces

4.1. **The conjectures: char 0 coefficients.** We maintain the notation from §3.8. Fix a maximal compact subgroup K_∞ of $G_\infty(\mathbb{R})$. For neat open compact subgroups $K \subset G(\mathbb{A}_F^\infty)$ define the locally symmetric space of level K:

$$Y_{G,K} := G(F)\backslash G(\mathbb{A}_F)/K K_\infty^\circ A_G(\mathbb{R})^\circ. \tag{4.1}$$

Let $\mathfrak{p}$ be a prime of F where G is unramified, and $K_\mathfrak{p} \subset G(F_\mathfrak{p})$ a hyperspecial subgroup. Write $\mathbb{T}_\mathfrak{p} := \mathbb{C}[K_\mathfrak{p}\backslash G(F_\mathfrak{p})/K_\mathfrak{p}]$ for the unramified Hecke algebra. Write $H^i_{(2)}$, H^i, and H^i_c for the L^2-cohomology, the ordinary (singular) cohomology, and cohomology with compact support, respectively, in degree $i \in \mathbb{Z}_{\ge 0}$.

Fix a Borel subgroup B and a maximal torus T of $(\mathrm{Res}^F_{\mathbb{Q}} G)\times_{\mathbb{Q}}\overline{\mathbb{Q}}$ such that $B\supset T$. Let $\lambda\in X^*(T)^+$ be a dominant weight of a maximal torus, which gives rise to an irreducible highest weight representation E_λ and a complex local system $\mathcal{E}_\lambda$ over the system of locally symmetric spaces $Y_{G,K}$. Then the colimit

$$H^i_\star(Y_{G,K_{\mathfrak{p}}},\mathcal{E}_\lambda):=\varinjlim_{K^{\mathfrak{p}}} H^i_\star(Y_{G,K_{\mathfrak{p}}K^{\mathfrak{p}}},\mathcal{E}_\lambda),\qquad \star\in\{(2),\emptyset,c\}$$

over sufficiently small open compact subgroups $K^{\mathfrak{p}}\subset G(\mathbb{A}_F^{\mathfrak{p},\infty})$ is a $\mathbb{T}_{\mathfrak{p}}$-module with a commuting $G(\mathbb{A}^{\mathfrak{p},\infty})$-action. Similarly we define a $G(\mathbb{A}_F^\infty)$-module $H^i_\star(Y_G,\mathcal{E}_\lambda)$ by taking colimit over $K_{\mathfrak{p}}\subset G(F_{\mathfrak{p}})$ as well as $K^{\mathfrak{p}}$.

Let $\mathfrak{m}_{\mathfrak{p}}$ be a maximal ideal of $\mathbb{T}_{\mathfrak{p}}$. This corresponds to a $\mathbb{C}$-algebra morphism $\mathbb{T}_{\mathfrak{p}}\to\mathbb{C}$, an unramified L-parameter $\phi^0_{\mathfrak{p}}:W_{F_{\mathfrak{p}}}\to {}^LG$, and an unramified representation $\pi^0_{\mathfrak{p}}$ of $G(F_{\mathfrak{p}})$. Set

$$\mathcal{N}_{\mathfrak{m}_{\mathfrak{p}}}:=\mathcal{N}(\pi^0_{\mathfrak{p}}).$$

Suppose that $\phi^0_{\mathfrak{p}}=\phi_{\psi^0_{\mathfrak{p}}}$ for some $\psi^0_{\mathfrak{p}}\in\Psi_{\mathrm{e}}(G_{\mathfrak{p}})$, or more generally for some $\psi^0_{\mathfrak{p}}\in\Psi'_{\mathrm{e}}(G_{\mathfrak{p}})$ if axiom (A1) is in effect (which provides us with the set $\Psi'_{\mathrm{e}}(G_{\mathfrak{p}})$); this is a harmless assumption as we remarked in §3.1. Then we can define $\mathcal{N}^{\mathrm{ur}}(\pi^0_{\mathfrak{p}})$ in terms of $\psi^0_{\mathfrak{p}}|_{\mathrm{SL}_2^A}$ as in §3.1. Lemma 3.2 (and its extension via character twists) tells us that $\mathcal{N}_{\mathfrak{m}_{\mathfrak{p}}}=\mathcal{N}^{\mathrm{ur}}(\pi^0_{\mathfrak{p}})$.

Write $\mathcal{L}_{\mathrm{fun}}$ for the set of F-rational Levi subgroups M up to conjugacy such that M_∞ contains a fundamental maximal torus of G_∞; if G_∞ contains an elliptic maximal torus then $\mathcal{L}_{\mathrm{fun}}=\{G\}$.

As in §3.8 we use $A=A_{G,\mathbb{R}}$ to define $q^\flat(G_\infty)=q(G_\infty/A_{G,\mathbb{R}})$.

Conjecture 4.2 (unramified at $\mathfrak{p}$). *If $H^i_{(2)}(Y_{G,K_{\mathfrak{p}}},\mathcal{E}_\lambda)_{\mathfrak{m}_{\mathfrak{p}}}\neq 0$ then*

$$q^\flat(G_\infty)-a_G(\mathcal{N}_{\mathfrak{m}_{\mathfrak{p}}},\lambda)\le i\le q^\flat(G_\infty)+a^{(2)}_G(\mathcal{N}_{\mathfrak{m}_{\mathfrak{p}}},\lambda).$$

For H^i and H^i_c, a naive expectation is that the inequality holds only on one side. Since we have not gathered enough evidence, we state it as a question; see also Conjecture 6.8 below.

Question 4.3. Are the following true?

$$\begin{aligned}H^i(Y_{G,K_{\mathfrak{p}}},\mathcal{E}_\lambda)_{\mathfrak{m}_{\mathfrak{p}}}=0\qquad &\text{for}\quad i<q^\flat(G_\infty)-a_G(\mathcal{N}_{\mathfrak{m}_{\mathfrak{p}}},\lambda),\\ H^i_c(Y_{G,K_{\mathfrak{p}}},\mathcal{E}_\lambda)_{\mathfrak{m}_{\mathfrak{p}}}=0\qquad &\text{for}\quad i>q^\flat(G_\infty)+a_G(\mathcal{N}_{\mathfrak{m}_{\mathfrak{p}}},\lambda)\end{aligned}$$

Now let us consider an arbitrary prime $\mathfrak{p}$, an *arbitrary* irreducible smooth representation $\pi^0_{\mathfrak{p}}$ of $G(F_{\mathfrak{p}})$ and arbitrary level at $\mathfrak{p}$. Set

$$a^{(2)}_G(\pi^0_{\mathfrak{p}},\lambda):=\max_{\mathcal{N}\le\mathcal{N}(\pi^0_{\mathfrak{p}})} a^{(2)}_G(\mathcal{N},\lambda).$$

We are not insisting on $\mathcal{N}=\mathcal{N}(\pi^0_{\mathfrak{p}})$ because the local A-parameter at $\mathfrak{p}$ that is Aubert dual to $\phi_{\widehat{\pi}^0_{\mathfrak{p}}}$ (in the notation of §3.1) need not globalize to a global A-parameter which is cohomological at ∞ (and has infinitesimal character determined by λ); other local A-parameters $\psi_{\mathfrak{p}}$ such that $\pi^0_{\mathfrak{p}}\in\Pi_{\psi_{\mathfrak{p}}}$ are expected to satisfy $\mathcal{N}_{\psi_{\mathfrak{p}}}\le\mathcal{N}(\pi^0_{\mathfrak{p}})$ in light of (CO$'(\pi)$).

Conjecture 4.4 (arbitrary at $\mathfrak{p}$). *If $\pi^0_{\mathfrak{p}}\in\mathrm{Irr}(G_{\mathfrak{p}})$ appears as a subquotient of $H^i_{(2)}(Y_G,\mathcal{E}_\lambda)$ as a $G(F_{\mathfrak{p}})$-module then*

$$q^\flat(G_\infty)-a^{(2)}_G(\pi^0_{\mathfrak{p}},\lambda)\le i\le q^\flat(G_\infty)+a^{(2)}_G(\pi^0_{\mathfrak{p}},\lambda).$$

Remark 4.5. (i) One can contemplate an analogous conjecture by localizing at a maximal ideal of either the Bernstein center of $G_{F_{\mathfrak{p}}}$ or the spectral Bernstein center à la Fargues–Scholze (the latter localization is equivalently done with respect to a semisimple L-parameter), but the resulting inequalities would be coarser in general.

(ii) If G is unramified at $\mathfrak{p}$ and $\pi_{\mathfrak{p}}^0$ is unramified with respect to a hyperspecial subgroup $K_{\mathfrak{p}}$, then Conjecture 4.4 for $\pi_{\mathfrak{p}}^0$ is true if and only if Conjecture 4.2 for the maximal ideal corresponding to $\pi_{\mathfrak{p}}^0$ is true by taking $K_{\mathfrak{p}}$-invariants.

Example 4.6. Conjecture 4.4 and Lemma 3.11 imply the following statements.

(i) If λ is regular, $H^i_{(2)}(Y_G, \mathcal{E}_\lambda) = 0$ for $i \notin [q_0(G_\infty), q_0(G_\infty) + l_0^\flat(G)]$. This special case can be shown unconditionally by using Vogan–Zuckerman's classification of unitary cohomological representations; no axioms on A-parameters are needed. Indeed, when λ is regular, the classification [VZ84, Thm. 5.3] tells us that only automorphic representations which are essentially tempered at ∞ contribute. (More precisely, in the notation of *loc. cit.*, the Lie algebra $\mathfrak{l}$ must be a Cartan subalgebra, which implies $A_{\mathfrak{q}}(\lambda)$ is tempered.) Then we obtain the desired degree bound from either [VZ84, Thm. 5.5] or [BW00, III.5]. (For the contribution from the continuous spectrum, one directly verifies (4.8) without recourse to A-parameters.)

(ii) If $\pi_{\mathfrak{p}}^0$ is A-generic and appears in $H^i_{(2)}(Y_G, \mathcal{E}_\lambda)$, then $i \in [q_0(G_\infty), q_0(G_\infty) + l_0^\flat(G)]$.

Remark 4.7. In [LS04, Gro13], it is shown that if λ is regular, $H^i(Y_G, \mathcal{E}_\lambda) = 0$ for $i < q_0(G_\infty)$. Hence, Question 4.3 has a positive answer in this case. The A-generic case of Question 4.3 also has an affirmative answer for all quasi-split classical groups, conditionally on twisted weighted fundamental lemma, cf. [Kosa] for an announcement of a related result.

4.8. **L^2-cohomology in the case of classical groups.** Let $\mathfrak{m}_{\mathfrak{p}}$, $\pi_{\mathfrak{p}}^0$, $\phi_{\mathfrak{p}}^0$ be as in the paragraph above Conjecture 4.2.

Theorem 4.9. *Assume that $H^i_{(2)}(Y_{G,K_{\mathfrak{p}}}, \mathcal{E}_\lambda)_{\mathfrak{m}_{\mathfrak{p}}} \neq 0$ and that axioms (A1$^+$)–(A3$^+$) hold for G. Then*

(i) There exists a unique $\psi_{\mathfrak{p}} \in \Psi'_e(G_{\mathfrak{p}})$ such that $\pi_{\mathfrak{p}}^0 \in \Pi_{\psi_{\mathfrak{p}}}$ and $\mathcal{N}_{\psi_{\mathfrak{p}}} = \mathcal{N}_{\mathfrak{m}_{\mathfrak{p}}}$.

(ii) $q^\flat(G_\infty) - a_G^{(2)}(\mathcal{N}_{\mathfrak{m}_{\mathfrak{p}}}, \lambda) \leq i \leq q^\flat(G_\infty) + a_G^{(2)}(\mathcal{N}_{\mathfrak{m}_{\mathfrak{p}}}, \lambda)$.

Proof. We point out at the outset that the uniqueness of $\psi_{\mathfrak{p}}$ in (i) follows from the axioms. So part (i) is mainly about the existence of $\psi_{\mathfrak{p}}$ as in the statement.

Write $\chi : A_G(\mathbb{R})^\circ \to \mathbb{R}^\times_{>0}$ for the inverse of the central character of E_λ restricted to $A_G(\mathbb{R})^\circ$. Put $\widetilde{K}_\infty^\circ := A_G(\mathbb{R})^\circ K_\infty^\circ$. From [BC83, 4.2, Eq.(5),(6)] (adapted from semisimple groups in the classical setting to reductive groups in the adelic setting as in [Fra98, Thm. 3 and 4], taking ρ there to be a positive constant function) we have a $\mathbb{T}_{\mathfrak{p}}$-equivariant decomposition

$$H^*_{(2)}(Y_{G,K_{\mathfrak{p}}}, \mathcal{E}_\lambda) = H^*(\mathfrak{g}, \widetilde{K}_\infty^\circ; L^2_{\mathrm{disc},\chi}([G])^{K_{\mathfrak{p}}} \otimes E_\lambda) \oplus H^*(\mathfrak{g}, \widetilde{K}_\infty^\circ; L^2_{\mathrm{cont},\chi}([G])^{K_{\mathfrak{p}}} \otimes E_\lambda).$$

We simply write $H^*_{\mathrm{disc},\lambda}$ and $H^*_{\mathrm{cont},\lambda}$ for the two summands. By assumption at least one of them is nonzero when localized at $\mathfrak{m}_{\mathfrak{p}}$. Suppose $(H^*_{\mathrm{disc},\lambda})_{\mathfrak{m}_{\mathfrak{p}}} \neq 0$. Decomposing $L^2_{\mathrm{disc},\chi}([G])$ into irreducibles, we obtain

$$0 \neq (H^*_{\mathrm{disc},\lambda})_{\mathfrak{m}_{\mathfrak{p}}} = \bigoplus_{\pi \subset L^2_{\mathrm{disc},\chi}([G])} m_\pi \pi^{\mathfrak{p},\infty} \otimes (\pi_{\mathfrak{p}}^{K_{\mathfrak{p}}})_{\mathfrak{m}_{\mathfrak{p}}} \otimes H^*(\mathfrak{g}, \widetilde{K}_\infty^\circ, \pi_\infty \otimes E_\lambda). \tag{4.2}$$

The relative Lie algebra cohomology is nonzero only if π_∞ has the infinitesimal character dual to that of E_λ; in particular π_∞ must be regular C-algebraic. Applying axiom (A3) in the form of

(2.6), we can replace the sum over π with a double sum over $\psi \in \Psi_\chi(G)$ and $\pi \in \Pi_\psi$. (To simplify notation we do not underline ψ.) Consider each pair (ψ, π) such that the corresponding summand is nonzero. By axiom (A2), we have $\psi_y \in \Psi_{\mathrm{era}}(G_y)$ and $\pi_y \in \Pi_{\psi_y} = \Pi^{\mathrm{AJ}}_{\psi_y}$ for all $y|\infty$. So we can write $\pi_y = \pi(\psi_y, Q_y)$ in the notation of §2.4. Then $L := \prod_{y|\infty} L_{Q_y}$ is a θ-stable Levi of G_∞. By Lemma 2.7,

$$i \in \left[q^\flat(G_\infty) - q^\flat(L),\ q^\flat(G_\infty) + q^\flat(L)\right]. \tag{4.3}$$

We saw in §2.4 that the unipotent conjugacy class in $\widehat{G}$ arising from a regular unipotent element of $\widehat{L}_{Q_y}$ coincides with the one arising from $\psi_y|_{\mathrm{SL}_2^A}$; denote the unipotent conjugacy class by $\mathcal{N}_{\psi_y}$. On the other hand, $(\pi_{\mathfrak{p}}^{K_{\mathfrak{p}}})_{\mathfrak{m}_{\mathfrak{p}}} \neq 0$ implies that $\pi_{\mathfrak{p}} = \pi_{\mathfrak{p}}^0$. Since $\pi_{\mathfrak{p}} \in \Pi_{\psi_{\mathfrak{p}}}$, we have $\pi_{\mathfrak{p}}^0 \in \Pi_{\phi_{\psi_{\mathfrak{p}}}}$ by (A2a). By definition $\pi_{\mathfrak{p}}^0 \in \Pi_{\phi_{\mathfrak{p}}^0}$ so $\phi_{\psi_{\mathfrak{p}}} = \phi_{\mathfrak{p}}^0$. By the injectivity in (A1), $\mathcal{N}_{\psi_{\mathfrak{p}}} = \mathcal{N}_{\mathfrak{m}_{\mathfrak{p}}}$; assertion (i) is proved at this point. Since $\psi|_{\mathrm{SL}_2^A}$, $\psi_{\mathfrak{p}}|_{\mathrm{SL}_2^A}$, and $\psi_y|_{\mathrm{SL}_2^A}$ are $\widehat{G}$-conjugate for each infinite place y, cf. axiom (A1), we see that $\mathcal{N}_{\psi_y} = \mathcal{N}_{\mathfrak{m}_{\mathfrak{p}}}$. By the definition of $a_G^{(2)}(\cdot,\cdot)$ and Lemma 2.8 (the lemma ensures that L contributes to the right hand side of (3.2)), we have

$$0 \le q^\flat(L) \le a_G^{(2)}(\mathcal{N}_{\psi_y}, \lambda) = a_G^{(2)}(\mathcal{N}_{\mathfrak{m}_{\mathfrak{p}}}, \lambda). \tag{4.4}$$

Together with (4.3), this tells us that the bound in (ii) holds true.

Now suppose that $(H^*_{\mathrm{cont},\lambda})_{\mathfrak{m}_{\mathfrak{p}}} \neq 0$. As in [BC83, Prop. 4.4] we decompose

$$H^i_{\mathrm{cont},\lambda} = \oplus_{(P,\pi_M)} m_{\pi_M} H^i(\mathfrak{g}, \widetilde{K}^\circ_\infty; L^\infty_{P,\pi_M} \otimes E_\lambda)^{K_{\mathfrak{p}}}, \tag{4.5}$$

where P runs over a set of conjugacy classes of proper cuspidal parabolic subgroups with Levi decomposition $P = MN$ such that M_∞ contains a fundamental maximal torus of G_∞ (i.e., $M \in \mathcal{L}_{\mathrm{fun}}$), π_M is a subspace of $L^2_{\mathrm{disc},\chi_M}([M])$ for a character $\chi_M : A_M(\mathbb{R})^\circ \to \mathbb{R}^\times_{>0}$ to be defined in a moment, and L^∞_{P,π_M} is a family of normalized parabolic inductions from π_M twisted by a family of unitary characters. To define χ_M, note that there is a natural map $A_M(\mathbb{R})^\circ \to A_G(\mathbb{R})^\circ$ dual to the map $X^*_F(G) \to X^*_F(M)$ induced by $M \hookrightarrow G$. We obtain χ_M from χ via $A_M(\mathbb{R})^\circ \to A_G(\mathbb{R})^\circ \stackrel{\chi}{\to} \mathbb{R}^\times_{>0}$. (The reader may compare with a similar construction in [Art13, p.122].)

Let us introduce some more notation. Put $A^\circ_{M_\infty} := A_{M_\infty}(\mathbb{R})^\circ$. We have a product decomposition $M_\infty = A^\circ_{M_\infty} \times {}^0M_\infty$, cf. [BC83, §3.3]. (Our $A^\circ_{M_\infty}, {}^0M_\infty$ correspond to their A, M.) Let K_{M_∞} be a maximal compact subgroup of $M_\infty(\mathbb{R})$, thus contained in ${}^0M_\infty(\mathbb{R})$. Let $\mathfrak{h}$ be a Cartan subalgebra of ${}^0\mathfrak{m}_\infty$, so that $\mathfrak{h} \oplus \mathfrak{a}_{M_\infty}$ is the Lie algebra of a maximal torus T_∞ of G_∞. We fix a Borel subgroup B of $G_{\infty,\mathbb{C}}$ in $P_{\infty,\mathbb{C}}$ containing $T_{\infty,\mathbb{C}}$. We may assume that $\lambda \in X^*(T_\infty)$ is B-dominant. Denote by $\rho \in X^*(T_\infty)_{\mathbb{Q}}$ the half sum of B-positive roots. Write $\mathfrak{a}^G_{M_\infty}$ for the kernel of the map $\mathfrak{a}_{M_\infty} \to \mathfrak{a}_G$ dual to $X^*_F(G) \to X^*_{\mathbb{R}}(G_\infty) \to X^*_{\mathbb{R}}(M_\infty)$. Similarly put $A^{\circ,G}_{M_\infty} := \ker(A^\circ_{M_\infty} \to A_G(\mathbb{R})^\circ)$. Denote by $\mathfrak{a}^*_{M_\infty/G}$ the linear dual of $\mathfrak{a}^G_{M_\infty}$, and $\mathfrak{a}^{*,+}_{M_\infty/G}$ for the positive chamber with respect to P. We have a continuous family of unitary characters $\mathbb{C}_{i\mu}$ for $\mu \in \mathfrak{a}^{*,+}_{M_\infty/G}$, and $\mathfrak{a}^{*,+}_{M_\infty/G}$ is equipped with a measure $d\mu$; see [BC83, §4.3]. (The precise definition does not matter to us.)

Going back to (4.5), if the summand for (P, π_M) is nonzero then [BC83, Thm. 3.4] tells us that there exists a unique element $w \in W^{P_\infty}$ satisfying

$$w(\lambda+\rho)|_{A^{\circ,G}_{M_\infty}} = 0, \quad \chi_{\pi_{M,\infty}} = \chi_{-w(\lambda+\rho)|_{\mathfrak{h}}}, \tag{4.6}$$

which has length $l(w)=\frac{1}{2}\dim N_\infty$, and that

$$\begin{aligned} & H^{\bullet+l(w)}(\mathfrak{g},\widetilde{K}^\circ_\infty; L^\infty_{P,\pi_M}\otimes E_\lambda) \\ = \;& \text{n-ind}^G_P(\pi^\infty_M)\otimes H^\bullet({}^0\mathfrak{m},K^\circ_{M_\infty},\pi_{M,\infty}\otimes E^{M_\infty}_{w\star\lambda})\otimes H^\bullet(\mathfrak{a}^G_{M_\infty},\textstyle\int^{\oplus}_{\mathfrak{a}^{*+}_{M_\infty/G}}\mathbb{C}_{i\mu}d\mu),\end{aligned}$$

The cohomology of $\mathfrak{a}^G_{M_\infty}$ is supported on degrees $[0,\dim\mathfrak{a}_{M_\infty}-\dim\mathfrak{a}_G]$. In order to prove that the non-vanishing degrees of $(H^i_{\text{cont},\lambda})_{\mathfrak{m}_\mathfrak{p}}$ satisfy (ii) of the theorem: for each π_M as above satisfying the two conditions

$$\text{(a) } \big(\text{n-ind}^{G_\mathfrak{p}}_{P_\mathfrak{p}}(\pi_{M,\mathfrak{p}})^{K_\mathfrak{p}}\big)_{\mathfrak{m}_\mathfrak{p}}\neq 0,\qquad \text{(b) } H^j({}^0\mathfrak{m},K^\circ_{M_\infty},\pi_{M,\infty}\otimes E^{M_\infty}_{w(\lambda+\rho)-\rho})\neq 0,$$

it suffices to show that

$$q^\flat(G_\infty)-a^{(2)}_G(\mathcal{N}_{\mathfrak{m}_\mathfrak{p}},\lambda)-l(w)\le j\le q^\flat(G_\infty)+a^{(2)}_G(\mathcal{N}_{\mathfrak{m}_\mathfrak{p}},\lambda)-l(w)-(\dim A_{M_\infty}-\dim A_G). \quad (4.7)$$

Thanks to (4.6), the formula of [BW00, III.3.3] (after reconciling the notation) tells us that

$$H^{\bullet+l(w)}(\mathfrak{g},\widetilde{K}^\circ_\infty,\text{n-ind}^{G_\infty}_{P_\infty}(\pi_{M,\infty})\otimes E_\lambda)=H^\bullet({}^0\mathfrak{m},K^\circ_{M_\infty},\pi_{M,\infty}\otimes E^{M_\infty}_{w\star\lambda})\otimes\wedge^\bullet\mathfrak{a}^*_{M_\infty/G},$$

so the proof of (4.7) boils down to verifying the implication that

$$\begin{aligned} & H^i(\mathfrak{g},\widetilde{K}^\circ_\infty,\text{n-ind}^{G_\infty}_{P_\infty}(\pi_{M,\infty})\otimes E_\lambda)\neq 0 \\ \Longrightarrow\;\; & i\in\big[q^\flat(G_\infty)-a^{(2)}_G(\mathcal{N}_{\mathfrak{m}_\mathfrak{p}},\lambda),\ q^\flat(G_\infty)+a^{(2)}_G(\mathcal{N}_{\mathfrak{m}_\mathfrak{p}},\lambda)\big].\end{aligned} \quad (4.8)$$

From Lemma 2.7 and the definition of $a^{(2)}_G(\mathcal{N}_{\psi_\mathfrak{p}},\lambda)$, we will be done if the unitary induction $\text{n-ind}^{G_\infty}_{P_\infty}(\pi_{M,\infty})$ contains a constituent of the form $\pi(\psi_\infty,Q_\infty)$ for some ψ_∞ and $Q_\infty=\prod_{y|\infty}Q_y$ such that

- $\mathcal{N}(\widehat{L}_{Q_y})=\mathcal{N}_{\mathfrak{m}_\mathfrak{p}}$ for each $y|\infty$,
- a $\widehat{G}_\infty$-conjugate of $\widehat{L}_{Q_\infty}$ is contained in $\text{Cent}_{\widehat{G}_\infty}(\lambda)$.

Since the second bullet point follows from Lemma 2.8, it is enough to show that $\text{n-ind}^{G_\infty}_{P_\infty}(\pi_{M,\infty})$ nontrivially intersects the A-packet Π_{ψ_∞} for some $\psi_\infty=\prod_{y|\infty}\psi_y\in\Psi_{\text{era}}(G_\infty)$ such that $\mathcal{N}_{\psi_y}=\mathcal{N}_{\mathfrak{m}_\mathfrak{p}}$ for all $y|\infty$. Indeed, (A2d) tells us that $\Pi_{\psi_\infty}=\Pi^{\text{AJ}}_{\psi_\infty}$, and every member $\pi(\psi_\infty,Q_\infty)$ of $\Pi^{\text{AJ}}_{\psi_\infty}$ satisfies $\mathcal{N}(\widehat{L}_{Q_y})=\mathcal{N}_{\psi_y}$ for $y|\infty$ (see §2.4).

To this end, write $\psi_M\in\Psi_{\chi_M}(M)$ for a global parameter whose packet contains π_M; here we are appealing to axiom (A3$^+$). In particular $\pi_{M,\mathfrak{p}}\in\Pi_{\psi_{M,\mathfrak{p}}}$ and $\pi_{M,y}\in\Pi_{\psi_{M,y}}$, and $\mathcal{N}_{\psi_{M,\mathfrak{p}}}=\mathcal{N}_{\psi_{M,y}}$ for each $y|\infty$. We deduce from (a) above that $\psi_{M,\mathfrak{p}}\in\Psi'_{\text{e}}(M_\mathfrak{p})$ maps to $\psi_\mathfrak{p}$ under ${}^LM\hookrightarrow{}^LG$ and that $\pi^0_\mathfrak{p}\in\Pi_{\psi_\mathfrak{p}}$; thus $\mathcal{N}_{\psi_{M,\mathfrak{p}}}$ maps to $\mathcal{N}_{\psi_\mathfrak{p}}$, which equals $\mathcal{N}_{\mathfrak{m}_\mathfrak{p}}$ in view of (a). We have shown assertion (i) at this point.

Take $\psi_\infty\in\Psi(G_\infty)$ to be the image of the parameter $\psi_{M,\infty}$. Then $\mathcal{N}_{\psi_{M,y}}$ maps to $\mathcal{N}_{\psi_y}$, hence $\mathcal{N}_{\mathfrak{m}_\mathfrak{p}}=\mathcal{N}_{\psi_y}$ for $y|\infty$. By axiom (A2e), all constituents of $\text{n-ind}^{G_\infty}_{P_\infty}(\pi_{M,\infty})$ show up in Π_{ψ_∞}. The non-vanishing condition in (4.8) tells us that Π_{ψ_∞} contains a cohomological representation. Hence $\psi_\infty\in\Psi_{\text{era}}(G_\infty)$ by axiom (A2c), so the proof of part (ii) is finished. □

Remark 4.10. The argument is much simpler if $\mathcal{L}_{\text{fun}}(G)=\{G\}$, e.g., if G_∞ contains an elliptic maximal torus. In that case, $H^*_{\text{cont},\lambda}=0$ as can be seen from (4.5) since G contains no proper cuspidal parabolic subgroup, cf. [BC83, Thm. 3.4(c)].

Theorem 4.11. *Assume that axioms (A1$^+$)–(A3$^+$) hold for G. Let $\mathfrak{p}$ be a finite prime of F and suppose that $\pi^0_{\mathfrak{p}} \in \mathrm{Irr}(G_{\mathfrak{p}})$ satisfies axiom $(\mathrm{CO}'(\pi^0_{\mathfrak{p}}))$. If $\pi^0_{\mathfrak{p}}$ appears as a subquotient of $H^i_{(2)}(Y_G, \mathcal{E}_\lambda)$ as a $G(F_{\mathfrak{p}})$-module for $i \in \mathbb{Z}_{\geq 0}$ then we have*

$$q^\flat(G) - a^{(2)}_G(\pi^0_{\mathfrak{p}}, \lambda) \leq i \leq q^\flat(G) + a^{(2)}_G(\pi^0_{\mathfrak{p}}, \lambda).$$

Proof. Since the argument is mostly the same as for Theorem 4.9, we simply point out the main differences. Again we decompose $H^*_{(2)}(Y_G, \mathcal{E}_\lambda) = H^*_{\mathrm{disc},\lambda} \oplus H^*_{\mathrm{cont},\lambda}$ (without taking the $K_{\mathfrak{p}}$-invariants). If $\pi^0_{\mathfrak{p}}$ appears in $H^*_{\mathrm{disc},\lambda}$, we have a global parameter ψ such that $\pi^0_{\mathfrak{p}} \in \Pi_{\psi_{\mathfrak{p}}}$ and such that Π_{ψ_∞} contains $\pi_\infty = \otimes_{y|\infty} \pi(\psi_y, Q_y)$ which satisfies the bound (4.3) for $L = \prod_y L_{Q_y}$. Now the key point is that $\mathcal{N}_{\psi_{\mathfrak{p}}} \leq \mathcal{N}(\pi^0_{\mathfrak{p}})$ by $(\mathrm{CO}'(\pi^0_{\mathfrak{p}}))$, therefore we obtain the following analogue of (4.4):

$$0 \leq q^\flat(L) \leq a^{(2)}_G(\mathcal{N}_{\psi_y}, \lambda) \leq a^{(2)}_G(\pi^0_{\mathfrak{p}}, \lambda).$$

This finishes the proof when $\pi^0_{\mathfrak{p}}$ shows up in $H^*_{\mathrm{disc},\lambda}$.

In the other case when $\pi^0_{\mathfrak{p}}$ is a subquotient of $H^*_{\mathrm{cont},\lambda}$, we change condition (a) to the new condition (a$'$) that $\pi^0_{\mathfrak{p}}$ is a subquotient of $\text{n-ind}^{G_{\mathfrak{p}}}_{P_{\mathfrak{p}}}(\pi_{M,\mathfrak{p}})$. Other than that, the argument remains unchanged until the second last paragraph in the proof of Theorem 4.9, except that $a^{(2)}_G(\mathcal{N}_{\psi_{\mathfrak{p}}}, \lambda)$ should be changed to $a^{(2)}_G(\pi_{\mathfrak{p}}, \lambda)$ and that we need to show $\mathcal{N}_{\psi_y} \leq \mathcal{N}(\pi^0_{\mathfrak{p}})$ instead of $\mathcal{N}_{\psi_y} = \mathcal{N}_{\psi_{\mathfrak{p}}}$.

To obtain $\mathcal{N}_{\psi_y} \leq \mathcal{N}(\pi^0_{\mathfrak{p}})$, we proceed as in the last paragraph in the proof of Theorem 4.9 but apply axiom (A2e) also at $\mathfrak{p}$ to observe that $\pi^0_{\mathfrak{p}} \in \Pi_{\psi_{\mathfrak{p}}}$, where $\psi_{\mathfrak{p}}$ is the parameter for $G_{\mathfrak{p}}$ coming from $\psi_{M,\mathfrak{p}}$. We still have the relation $\mathcal{N}_{\psi_{\mathfrak{p}}} = \mathcal{N}_{\psi_y}$ for $y|\infty$. In addition, $\mathcal{N}_{\psi_{\mathfrak{p}}} \leq \mathcal{N}(\pi^0_{\mathfrak{p}})$ from $(\mathrm{CO}'(\pi^0_{\mathfrak{p}}))$. Hence $\mathcal{N}_{\psi_y} \leq \mathcal{N}(\pi^0_{\mathfrak{p}})$ as desired. ☐

Corollary 4.12. *Conjectures 4.2 and 4.4) hold true if G is an inner form of GL_n that is split at $\mathfrak{p}$. Assuming that the twisted weighted fundamental lemma is valid (see §2.15.2), Conjectures 4.2 and 4.4 are true for all quasi-split classical groups.*

Proof. For inner forms of GL_n and quasi-split classical groups, axioms (A1$^+$)–(A3$^+$) hold true as explained in §2.15.1 and §2.15.2. Hence Conjecture 4.2 is an immediate consequence of Theorem 4.9. As for Conjecture 4.4, we need to check $(\mathrm{CO}'(\pi_{\mathfrak{p}}))$ in addition. Since the latter is true for quasi-split classical groups (see §3.3), Conjecture 4.4 is verified in these cases. ☐

One can extend above results along central morphisms. Let us discuss only the following simple instance; for relevant examples, see Example 6.12 below.

Corollary 4.13. *Let G be a connected reductive group over $\mathbb{Q}$ such that $G_{\mathrm{der}} \simeq \mathrm{Res}^F_{\mathbb{Q}}\mathrm{Sp}_{2n}$ for a totally real field F and $n \in \mathbb{Z}_{\geq 1}$; if $n \geq 2$, assume that the twisted weighted fundamental lemma is valid. If $\pi^0_p \in \mathrm{Irr}(G_p)$ appears as a subquotient of $H^i_{(2)}(Y_G, \mathbb{C})$ as a $G(\mathbb{Q}_p)$-module for $i \in \mathbb{Z}_{\geq 0}$ then*

$$q^\flat(G) - \max_{\pi^0_{p,j}} \max_{\mathcal{N} \leq \mathcal{N}(\pi^0_{p,j})} a^{(2)}_G(\mathcal{N}, 0) \leq i \leq q^\flat(G) + \max_{\pi^0_{p,j}} \max_{\mathcal{N} \leq \mathcal{N}(\pi^0_{p,j})} a^{(2)}_G(\mathcal{N}, 0),$$

where $\pi^0_{p,j}$ are irreducible summands of the restriction of π^0_p to $G_{\mathrm{der}}(\mathbb{Q}_p)$ as in §3.5. The ℓ-adic analogue also holds.

The same statements hold if Sp_{2n} is replaced by a quasi-split special unitary group for a CM quadratic extension $E \supset F$ (assuming the twisted weighted fundamental lemma is valid).

Proof. As in the proof of Theorem 4.9, [Fra98, Thm. 3 and 4] imply that

$$H^i_{(2)}(Y_G,\mathbb{C})\cong H^i(\mathfrak{g},\widetilde{K}^\circ_\infty;L^2([G]))\cong H^i(\mathfrak{g},\widetilde{K}^\circ_\infty;L^2_{\mathrm{disc}}([G]))\oplus H^i(\mathfrak{g},\widetilde{K}^\circ_\infty;L^2_{\mathrm{cont}}([G])),$$
$$H^i_{(2)}(Y_{G_{\mathrm{der}}},\mathbb{C})\cong H^i(\mathfrak{g}_{\mathrm{der}},K^\circ_\infty;L^2([G_{\mathrm{der}}]))\cong H^i(\mathfrak{g}_{\mathrm{der}},K^\circ_\infty;L^2_{\mathrm{disc}}([G_{\mathrm{der}}]))\oplus H^i(\mathfrak{g}_{\mathrm{der}},K^\circ_\infty;L^2_{\mathrm{cont}}([G_{\mathrm{der}}])).$$

By Remark 4.10,

$$H^i(\mathfrak{g}_{\mathrm{der}},K^\circ_\infty;L^2_{\mathrm{cont}}([G_{\mathrm{der}}]))=H^i(\mathfrak{g},\widetilde{K}^\circ_\infty;L^2_{\mathrm{cont}}([G]))=0.$$

Note also that for $*\in\{\emptyset,\mathrm{disc}\}$

$$L^2_*([G])=\widehat{\bigoplus}_\omega L^2_*([G],\omega),$$

where ω run through unitary characters of the compact abelian group $[Z_G]=Z_G(\mathbb{A})/Z_G(\mathbb{Q})A_G(\mathbb{R})^\circ$ and $L^2_*([G],\omega)$ is the ω-isotypic part. If π_p^0 appears as a subquotient of $H^i_{(2)}(Y_G,\mathbb{C})$, then there exists a cohomological discrete automorphic representation $\pi\subset L^2_{\mathrm{disc}}([G],\omega)$ with $\pi_p\cong\pi_p^0$ for some ω that is trivial on $Z_G(\mathbb{R})^\circ$. As in [LS19, §5], the space $L^2([G],\omega)$ is the induction of

$$L^2(G(\mathbb{Q})\backslash Z_G(\mathbb{A})G(\mathbb{Q})G_{\mathrm{der}}(\mathbb{A}),\omega)\cong L^2(G_{\mathrm{der}}(\mathbb{Q})\backslash G_{\mathrm{der}}(\mathbb{A}),\omega_1)$$

from $Z_G(\mathbb{A})G(\mathbb{Q})G_{\mathrm{der}}(\mathbb{A})$ to $G(\mathbb{A})$, where ω_1 is the restriction of ω to $[Z_{G_{\mathrm{der}}}]$. Moreover, the same holds for the discrete spectrum as $Z_G(\mathbb{A})G(\mathbb{Q})G_{\mathrm{der}}(\mathbb{A})\backslash G(\mathbb{A})$ is an abelian compact group [LS19, Prop. 3.3.3]. Therefore, $\pi|_{G_{\mathrm{der}}(\mathbb{A})}$, which is the Hilbert direct sum of irreducible representations, contains a discrete automorphic representation π' with $\pi'_p\cong\pi^0_{p,j}$ for some j [LS19, Prop. 2.3.3 and Lem. 5.1.1]. Furthermore, π'_∞ is a summand of $\pi_\infty|_{G_{\mathrm{der}}(\mathbb{R})}$ and in particular cohomological by [NP21, Prop. 4]. We have the Künneth formula

$$H^\bullet(\mathfrak{g},\widetilde{K}^\circ_\infty,\pi_\infty)\cong H^\bullet(\mathfrak{g}_{\mathrm{der}},(G_{\mathrm{der}}(\mathbb{R})\cap K_\infty)^\circ,\pi_\infty)\otimes H^\bullet(\mathfrak{a}',\mathbb{C}),$$

once we choose a splitting

$$\mathfrak{g}=\mathfrak{g}_{\mathrm{der}}\oplus(\mathrm{Lie}\,A_G(\mathbb{R}))_{\mathbb{C}}\oplus(\mathrm{Lie}\,(Z_G(\mathbb{R})\cap K))_{\mathbb{C}}\oplus\mathfrak{a}'$$

for some $\mathfrak{a}'\subset\mathfrak{g}$, and $H^{i'}(\mathfrak{g}_{\mathrm{der}},(G_{\mathrm{der}}(\mathbb{R})\cap K_\infty)^\circ,\pi'_\infty)$ is nonzero for some i'.

Therefore, if the case $G=G_{\mathrm{der}}$ is verified, we have

$$q^\flat(G_{\mathrm{der}})-\max_{\mathcal{N}\le\mathcal{N}(\pi^0_{p,j})}a^{(2)}_G(\mathcal{N},0)\le i'\le q^\flat(G_{\mathrm{der}})+\max_{\mathcal{N}\le\mathcal{N}(\pi^0_{p,j})}a^{(2)}_G(\mathcal{N},0).$$

As $q^\flat(G)=q^\flat(G_{\mathrm{der}})+\dim\mathfrak{a}'$, the case of G follows.

It remains to verify the case $G=G_{\mathrm{der}}$. If G is the Weil restriction of Sp_{2n}, $n\ge2$, it follows from Corollary 4.12. Now suppose $G=\mathrm{Res}^F_{\mathbb{Q}}\mathrm{SL}_2=\mathrm{Res}^F_{\mathbb{Q}}\mathrm{Sp}_2$ or a quasi-split special unitary group. We already know the case $G=\mathrm{Res}^F_{\mathbb{Q}}\mathrm{GL}_2$ or a quasi-split unitary group by Corollary 4.12. As in the previous argument, there exists a cohomological discrete automorphic representation π of $G_{\mathrm{der}}(\mathbb{A})$ with $\pi_p\cong\pi_p^0$ and central character ω^1 that is trivial on $Z_{G_{\mathrm{der}}}(\mathbb{R})$. Arguing similarly as in the proof of [LS19, Thm. 5.2.2], we can find a discrete automorphic representation $\widetilde{\pi}$ of $G(\mathbb{A})$ lifting π that is trivial on $Z_G(\mathbb{R})$. Then the Künneth formula as above implies that $\widetilde{\pi}$ is cohomological, and in turn, the case of SL_2 and quasi-split special unitary groups. ∎

5. Torsion and ℓ-adic coefficients

5.1. **Assignment of nilpotent conjugacy classes: torsion and ℓ-adic coefficients.** In the cases coefficients are torsion, even *formulating* a similar conjectural statement is a nontrivial task for several reasons. One reason is that the Aubert–Zelevinsky involution may not preserve irreducible representations, and another is that there is no precise form of the mod ℓ local Langlands correspondence even in a conjectural form. In fact, even the case of $\overline{\mathbb{Q}}_\ell$-coefficients is subtle.

For instance, upon choosing an isomorphism $\iota\colon \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$, we can translate Question 4.3 into the ℓ-adic setting. However, it depends on ι a priori. More precisely, if $\mathfrak{m}_\mathfrak{p}$ arises from a $\overline{\mathbb{Q}}_\ell$-valued character, then the invariant $\mathcal{N}_{\mathfrak{m}_\mathfrak{p}}$ relies on the local Langlands correspondence with $\overline{\mathbb{Q}}_\ell$-coefficients, which depends on ι. The following axiom says it is actually independent of ι. [4]

(A2^{++}) **(A2^{+})** holds, and moreover $\phi_\pi|_{\mathrm{SL}_2^D}$ for each $\pi \in \mathrm{Irr}(G(F_\mathfrak{p}))$ is invariant under any field automorphism $\sigma\colon \mathbb{C} \xrightarrow{\cong} \mathbb{C}$ in the following sense: $\phi_\pi|_{\mathrm{SL}_2^D}$ is conjugate to $\phi_{\pi^\sigma}|_{\mathrm{SL}_2^D}$. If $\sigma(q^{1/2}) = q^{1/2}$, then ϕ_π^σ is conjugate to ϕ_{π^σ}.

In fact, more canonically, it is expected that $({}^C\phi_\pi)^\sigma$ is conjugate to ${}^C\phi_{\pi^\sigma}$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$; see [Ima24, Conj. 2.4], for instance. In particular, as is well-known, **(A2^{++})** is true for general linear groups [ST14, (3.2)], [Ima24, Corollary 2.12]. We will discuss the case of quasi-split classical groups in the next subsection.

In the rest of this subsection, we assume **(A2^{++})** holds. We want to use it as a crutch to attach a nilpotent conjugacy class to mod ℓ representations of $G(F_\mathfrak{p})$. To get around the lack of mod ℓ local Langlands in a precise form, we proceed in the following ad hoc manner. Let $\pi_\mathfrak{p}$ be an irreducible smooth $\overline{\mathbb{F}}_\ell$-representation of $G(F_\mathfrak{p})$ with ℓ prime to $\mathfrak{p}$. According to [DHKM, Prop. 4.9], there exists an irreducible integral smooth $\overline{\mathbb{Q}}_\ell$-representation $\widetilde{\pi}_\mathfrak{p}$ whose mod ℓ reduction contains $\pi_\mathfrak{p}$ as a subquotient. We then work with the nilpotent conjugacy class $\mathcal{N}(\widetilde{\pi}_\mathfrak{p})$ in characteristic 0. However $\widetilde{\pi}_\mathfrak{p}$ may not be unique, and we will consider all possible such lifts.

Remark 5.2. If the Fargues–Scholze semisimple L-parameter of $\pi_\mathfrak{p}$ is of weakly Langlands–Shahidi type in the sense of [HL, Def. 6.2] and the semisimplification of the ℓ-adic L-parameters agree with those of Fargues–Scholze, then the mod ℓ reduction of $\phi_{\widetilde{\pi}_\mathfrak{p}}$ must have the trivial monodromy. Therefore, such a $\pi_\mathfrak{p}$ is A-generic in the sense that $\mathcal{N}(\widetilde{\pi}_\mathfrak{p})$ are all trivial.

Example 5.3. Suppose $G(F_\mathfrak{p})$ is a general linear group. In this case, the mod ℓ local Langlands correspondence is constructed by Vignéras [Vig01]; let us denote the resulting Weil–Deligne parameter of $\pi_\mathfrak{p}$ by $\phi^{\mathrm{V}}_{\pi_\mathfrak{p}}$ with the conjugacy class of nilpotent operator $\mathcal{N}(\phi^{\mathrm{V}}_{\pi_\mathfrak{p}})$. It is also known that a suitable modification of Aubert–Zelevinsky involution $\pi_\mathfrak{p} \mapsto \widehat{\pi}_\mathfrak{p}$ preserving irreducible representations exists. Moreover, the local Langlands correspondence is compatible with reduction modulo ℓ up to Aubert–Zelevinsky involution in the following sense [Vig01, 1.8.5]: any $\pi_\mathfrak{p}$ has the form $J_\ell(\widetilde{\pi}_\mathfrak{p})$ for some $\widetilde{\pi}_\mathfrak{p}$, where $J_\ell(-)$ is the "most generic" subquotient of the mod ℓ reduction [Vig01, 1.8.4] (see also [Vig98, V.9.2]), and $\phi^{\mathrm{V}}_{\widehat{\pi}_\mathfrak{p}}$ is equivalent to the mod ℓ reduction of $\phi_{\widehat{\widetilde{\pi}}_\mathfrak{p}}$ when $\pi_\mathfrak{p} \cong J_\ell(\widetilde{\pi}_\mathfrak{p})$.

If $\pi_\mathfrak{p}$ is a subquotient of the mod ℓ reduction of some irreducible $\widetilde{\pi}_\mathfrak{p}$ but not isomorphic to $J := J_\ell(\widetilde{\pi}_\mathfrak{p})$, then the nilpotent orbit $\mathcal{N}(\phi^{\mathrm{V}}_{\widehat{\pi}_\mathfrak{p}})$ is larger in the closure ordering than $\mathcal{N}(\phi^{\mathrm{V}}_{\widehat{J}})$, which is the mod ℓ reduction of $\mathcal{N}(\phi_{\widehat{\widetilde{\pi}}_\mathfrak{p}})$ as above. Indeed, this follows from the displayed equality in the

[4] Compare with the recent elegant result of Scholze that the map from the spectral Bernstein center to the Bernstein center, and hence Fargues–Scholze semisimple L-parameters, is independent of ℓ [Sch, Cor. 6.2].

proof of [MS14, Lem. 9.41], for instance.[5] Therefore, including such $\widetilde{\pi}_\mathfrak{p}$ in consideration does not change the nilpotent conjugacy class as long as we look at the maximal one.

We conclude that it is reasonable to define $\mathcal{N}(\pi_\mathfrak{p})$ to be the nilpotent conjugacy class $\mathcal{N}(\phi^{\mathrm{V}}_{\widehat{\pi}_\mathfrak{p}})$ in characteristic ℓ. The L-parameter ϕ is of weakly Langlands–Shahidi type if and only if there is no possible nontrivial monodromy. In particular, there is a unique irreducible representation $\pi_\mathfrak{p}$ whose semisimplified L-parameter is ϕ and it is A-generic in the sense that $\mathcal{N}(\pi_\mathfrak{p})$ is trivial.

Remark 5.4. One may also try to work with maximal ideals of (local) unramified Hecke algebras. Note that two non-isomorphic unramified irreducible representations can have the same mod ℓ Satake parameter, and the relation with the above consideration is more subtle than the non-torsion case. A related issue is that even in the case of general linear groups, there could be multiple "maximal" nilpotent operators for a given representation of the Weil group.

Let us also recall that if $G(F_\mathfrak{p})$ is a general linear group, the generic condition in [Kosb] means there is no possible nontrivial monodromy. In particular, there is a unique corresponding unramified representation and any its irreducible lift is generic (and its mod ℓ reduction is automatically irreducible).

5.5. **Invariance of L-parameters for quasi-split classical groups.** For simplicity of notation, let F denote a nonarchimedan local field of characteristic 0 for the rest of §5. Assume G is GL_N, split symplectic, or qusai-split orthogonal/unitary over F. We are going to use results from Section 2.15.2 to verify **(A2^{++})** in this case. In particular, when G is a classical group, the twisted weighted fundamental lemma is assumed to be able to assign local L-parameters ϕ_π to $\pi \in \mathrm{Irr}(G(F)$.

Let $\sigma \in \mathrm{Aut}(\mathbb{C})$. Suppose $\sigma(q^{1/2}) = -q^{1/2}$. We let $\eta = \eta_\sigma$ denote the unramified character sending any Frobenius lift to $-1 \in Z(\widehat{G}) \subset \widehat{G}$ if $G = \mathrm{GL}_N$, odd orthogonal, or unitary. Note that $-1 = 2\rho(-1) \in Z(\widehat{G})^{W_F}$ if $G = \mathrm{GL}_{\mathrm{even}}$, odd orthogonal, or even unitary and $2\rho(-1)$ is trivial otherwise. If $\sigma(q^{1/2}) = q^{1/2}$, we let η be trivial.

Theorem 5.6. *Let $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $\pi \in \mathrm{Irr}(G(F))$.*

(i) If $\sigma(q^{1/2}) = q^{1/2}$, ϕ_{π^σ} is conjugate to ϕ_π^σ.

(ii) Suppose $\sigma(q^{1/2}) = -q^{1/2}$. If G is $\mathrm{GL}_{\mathrm{odd}}$, symplectic, even orthogonal, or odd unitary, then ϕ_{π^σ} is conjugate to ϕ_π^σ. If G is $\mathrm{GL}_{\mathrm{even}}$, odd orthogonal, or even unitary, ϕ_{π^σ} is conjugate to $\eta\phi_\pi^\sigma$.

In other words, C-parameters ${}^C\phi_{\pi^\sigma}$ and $({}^C\phi_\pi)^\sigma$ agree. Moreover, if π_{st} denotes the standard representation with unique irreducible quotient π, then π_{st}^σ is the standard representation with unique irreducible quotient π^σ.

Corollary 5.7. $\mathcal{N}(\pi^\sigma) = \mathcal{N}(\pi)$.

Proof. Observe that $(\widehat{\pi})^\sigma = \widehat{\pi^\sigma}$ (since the Aubert involution can be defined only via the unnormalized version of parabolic induction and Jacquet modules) and that the twist by the central character has no effect. □

5.7.1. *General linear groups.* As already mentioned, $\mathrm{Aut}(\mathbb{C})$-invariance of C-parameters for general linear groups is well-known [ST14, (3.2)], [Ima24, Cor. 2.12]. Therefore, the nontrivial claim is that $\mathrm{Aut}(\mathbb{C})$ respects standard modules. Any standard module has the form

$$\pi_1|-|^{s_1} \times \cdots \times \pi_r|-|^{s_r} := \text{n-ind}(\pi_1|-|^{s_1} \boxtimes \cdots \boxtimes \pi_r|-|^{s_r}),$$

[5] We pass from $Z(\mathfrak{m})$ to $L(\mathfrak{m})$ as we are taking the Zelevinsky involution. Note also that the inequality there has to be reversed as the partition $\mu_\mathfrak{m}$ [MS14, Def. 9.14] is the dual to the partition we are interested in via the order-preserving correspondence between nilpotent orbits and partitions.

where $\pi_1, \dots, \pi_r$ are (unitary) tempered representations and $s_1 > \cdots > s_r$ are real numbers, and each π_i has the form

$$\pi_i = \Delta_{i1} \times \cdots \times \Delta_{ij_i} := \text{n-ind}(\Delta_{i1} \boxtimes \cdots \boxtimes \Delta_{ij_i}),$$

where Δ_{**} is a discrete series corresponding to a segment; see [Zel80, §3, §9].

On the other hand, recall that for any sequence of segments $\Delta_1, \dots, \Delta_m$, the representation $\Delta_1 \times \cdots \times \Delta_m$ has a unique irreducible quotient if Δ_i does not precede Δ_j for any $i < j$, and $\Delta_1 \times \cdots \times \Delta_m$ is irreducible if Δ_i and Δ_j are not linked for any $i < j$. If Δ_i and Δ_j are linked and Δ_i does not precede Δ_j, then $\Delta_i \times \Delta_j$ is a standard representation with two irreducible constituents.

Lemma 5.8. *If Δ_i does not precede Δ_j for any $i < j$, then $\Delta_1 \times \cdots \times \Delta_m$ is isomorphic to a standard representation.*

Proof. Suppose Δ_i and Δ_{i+1} are not linked, then the full normalized parabolic induction does not change by swapping Δ_i and Δ_{i+1} as $\Delta_i \times \Delta_{i+1} \cong \Delta_{i+1} \times \Delta_i$. Therefore, it is possible to arrange the sequence so that $\Delta_i = \rho|-|^{s_i}$ with ρ_i unitary and $s_1 \geq \cdots \geq s_r$. Its induction is a standard representation. □

When we take the normalized induction using $\sigma(q^{1/2})$ in place of $q^{1/2}$ (so the square root of the modulus character $\delta_P^{1/2}$ is replaced with $\sigma(\delta_P^{1/2})$, where P is the standard parabolic from which one is inducing), we write $\pi_1 \times_\sigma \cdots \times_\sigma \pi_r$ instead; if $\sigma(q^{1/2}) = q^{1/2}$, this is simply $\pi_1 \times \cdots \times \pi_r$.

Lemma 5.9. *The representation*

$$(\pi_1|-|^{s_1})^\sigma \times_\sigma \cdots \times_\sigma (\pi_r|-|^{s_r})^\sigma = (\Delta_{11}|-|^{s_1})^\sigma \times_\sigma \cdots \times_\sigma (\Delta_{1j_1}|-|^{s_1})^\sigma \times_\sigma \cdots \times_\sigma (\Delta_{r1}|-|^{s_r})^\sigma \times_\sigma \cdots \times_\sigma (\Delta_{rj_r}|-|^{s_r})^\sigma$$

is isomorphic to a standard representation.

Proof. Suppose Δ_{ij} is a representation of $\mathrm{GL}_{n_{ij}}$ with $N = \sum n_{ij}$. Then,

$$(\pi_1|-|^{s_1})^\sigma \times_\sigma \cdots \times_\sigma (\pi_r|-|^{s_r})^\sigma = \eta^{N-n_{11}}(\Delta_{11}|-|^{s_1})^\sigma \times \cdots \times \eta^{N-n_{ij}}(\Delta_{ij}|-|^{s_i})^\sigma \times \cdots \times \eta^{N-n_{rj_r}}(\Delta_{rj_r}|-|^{s_r})^\sigma.$$

The L-parameter of $\eta^{N-n_{ij}}(\Delta_{ij}|-|^{s_i})^\sigma$ is $\eta^{N-1}\phi^\sigma_{\Delta_{ij}|-|^{s_i}}$. In particular, $(\pi_1|-|^{s_1})^\sigma \times_\sigma \cdots \times_\sigma (\pi_r|-|^{s_r})^\sigma$ is fully induced from an essentially discrete series. Moreover, $\eta^{N-n_{i_1j_1}}(\Delta_{i_1j_1}|-|^{s_{i_1}})^\sigma$ does not precede $\eta^{N-n_{i_2j_2}}(\Delta_{i_2j_2}|-|^{s_{i_2}})^\sigma$ for any $(i_1, j_1) < (i_2, j_2)$ in the lexicographic order. Therefore $(\pi_1|-|^{s_1})^\sigma \times_\sigma \cdots \times_\sigma (\pi_r|-|^{s_r})^\sigma$ is standard by Lemma 5.8. □

5.9.1. *Irreducibility criterion.* From now on, let G be a split symplectic, a quasi-split orthogonal, or a quasi-split unitary group; in the even orthogonal case, we will actually work with full (disconnected) orthogonal groups following the convention of [AGI$^+$], for example. We need the following well-known irreducibility criterion for certain parabolically induced representations. It is also a special case of a general form of the standard module conjecture, which is known for many groups.

Lemma 5.10. *Assume that the twisted weighted fundamental lemma is true (to access results in* [Art13, Mok15]*, cf. §2.15.2). Suppose G is not a unitary group. Let π be a discrete series of a general linear group corresponding to a segment $[\rho, \dots, \rho|-|^{r-1}], r \in \mathbb{Z}_{>0}$. Suppose an L-parameter $\phi\colon W_F \times \mathrm{SL}_2 \to \widehat{G} \hookrightarrow \mathrm{GL}_N$ is the direct sum of irreducible self-dual representations. For $\tau \in \Pi_\phi$ and $s \in \mathbb{R}$, the representation*

$$\pi|-|^s \rtimes \tau := \text{n-ind}(\pi|-|^s \boxtimes \tau)$$

is irreducible in one of the following cases:

(i) $\rho|-|^{(r-1)/2}$ *is not self-dual.*

(ii) $2s \notin \mathbb{Z}$.

In particular, $\pi|-|^s \rtimes \tau \cong \pi^\vee|-|^{-s} \rtimes \tau$ *in these cases.*

An analogous claim holds for quasi-split unitary groups G, with "conjugate self-dual" in place of "self-dual".

Proof. Assume $s \neq 0$. By repeatedly applying [Mui05, Lem. 2.1, Lem. 2.4], which also makes sense for unitary groups, we reduce the proof to the two cases: (a) where τ is a discrete series of a smaller group of the same type as G, and (b) the case with τ replaced by a self-dual (or conjugate-self-dual) discrete series of a general linear group. Case (a) is [Mui04, Thm. 2.2] or [Tad13, App. A] whose basic assumption (BA), as in Mœglin–Tadić's classification of discrete series, is verified in [Mœg14, §3, §7], [Xu17, Prop. 3.2, Cor. 9.1, Thm. 11.1], conditionally on [Art13] (see also [Tad13, Rem. 2.2]). Case (b) is clear once interpreted using segments.

If $s = 0$, the claim follows from the structure of tempered L-packets, or the classification of tempered representations [Tad13, Thm. 5.3]. □

Proof of Theorem 5.6. First suppose that both ϕ_π and ϕ_π^σ are tempered. Then, [ST14, Prop. 5.2] implies the claim (note that the L-homomorphism in *loc. cit.* involves a half-integral twist in the case of even orthogonal groups as in [ST14, §4.2]). [6]

To simplify the notation, assume G is not a unitary group. To discuss a general π, we recall that any L-parameter $\phi\colon W_F \times \mathrm{SL}_2^D \to \widehat{G} \hookrightarrow \mathrm{GL}_N$ can be written as

$$\phi = \phi' \oplus \phi_0 \oplus (\phi')^\vee,$$

where ϕ_0 is the direct sum of irreducible self-dual representations of the type specified by G and ϕ' is the direct sum of irreducible representations that are either non-self-dual or self-dual of different type. Note that ϕ' may not be unique. The summand ϕ_0 gives rise to an absolutely tempered L-parameter of a smaller group of the same type, and it is known that $\#\Pi_\phi = \#\Pi_{\phi_0}$.

Assume $\phi' = \phi_1|-|^{s_1} \oplus \cdots \oplus \phi_r|-|^{s_r}$ with ϕ_i irreducible tempered and $s_1 \geq s_2 \geq \cdots \geq s_r \geq 0$. Let π_i denote the essentially discrete series representation corresponding to $\phi_i|-|^{s_i}$. Then $\pi_1 \times \cdots \times \pi_r$ is a standard representation of a general linear group, and in fact

$$\pi_1 \times \cdots \times \pi_r \rtimes \tau, \quad \tau \in \Pi_{\phi_0},$$

is a standard representation of G. Their unique irreducible quotients have the L-parameter ϕ and these quotients are non-isomorphic to each other by looking at Jacquet modules. Therefore, these quotients exhaust the L-packet Π_ϕ.

Let $1 \leq i \leq r$. Suppose either ϕ_i is not self-dual or $2s_i \notin \mathbb{Z}$. If $(\phi_i|-|^{s_i})^\sigma$ is a twist of a tempered L-parameter by a negative power of $|-|$, let π_i' denote the dual of π_i^σ, which corresponds to $(\phi_i(|-|^{s_i})^{\sigma,\vee}$. Otherwise, we let $\pi_i' := \pi_i^\sigma$. If ϕ_i is self-dual and $2s_i \in \mathbb{Z}$, then

$$(\phi_i|-|^{s_i})^\sigma = \phi_i^\sigma(|-|_\sigma^{1/2})^{2s_i}$$

with ϕ_i^σ being tempered. Here, $|-|_\sigma^{1/2}$ is defined using $\sigma(q^{1/2})$. We set $\pi_i' := \pi_i^\sigma$ in this case as well. Note that if we write $\phi_{\pi_i^\sigma} = \phi_i'|-|^{s_i'}$ with ϕ_i' tempered, then the following holds: ϕ_i is self-dual and $2s_i \in \mathbb{Z}$, if and only if $\phi_i', \eta\phi_i'$ are self-dual and $2s_i' \in \mathbb{Z}$. (Of course, ϕ_i' is self-dual if and only if $\eta\phi_i'$ is self-dual, since $\eta^2 = 1$.)

[6] In fact there is a gap in the proof of [ST14, Prop. 5.2] when ϕ_π^σ is not tempered (which does happen in general), in which case the endoscopic character identity therein is not guaranteed unless, for example, ϕ_π^σ is shown to be a standard module. Our proof here can be viewed as filling in the gap in the case at hand.

By Lemma 5.10, the full normalized parabolic induction does not change by replacing π_i^σ by π_i':

$$\pi_1^\sigma \times_\sigma \cdots \times_\sigma \pi_r^\sigma \rtimes_\sigma \tau^\sigma \cong \pi_1' \times_\sigma \cdots \times_\sigma \pi_r' \rtimes_\sigma \tau^\sigma.$$

By Lemma 5.9, $\pi_1^\sigma \times_\sigma \cdots \times_\sigma \pi_r^\sigma$ is isomorphic to a standard representation. The L-parameter of its unique quotient is the direct sum of $\eta^{N'-1}(\phi_i|-|^{s_i})^\sigma$ with $N' := \dim \phi'$. By the first paragraph of the proof, τ^σ itself is tempered and has the L-parameter ϕ_0^σ for symplectic and even orthogonal cases, and $\eta\phi_0^\sigma$ for the odd orthogonal case.

Finally, observe that if G is symplectic or even orthogonal, then

$$\pi_1^\sigma \times_\sigma \cdots \times_\sigma \pi_r^\sigma \rtimes_\sigma \tau^\sigma \cong \eta^{N'+1}(\pi_1^\sigma \times_\sigma \cdots \times_\sigma \pi_r^\sigma) \rtimes \tau^\sigma$$

and if G is odd orthogonal,

$$\pi_1^\sigma \times_\sigma \cdots \times_\sigma \pi_r^\sigma \rtimes_\sigma \tau^\sigma \cong \eta^{N'}(\pi_1^\sigma \times_\sigma \cdots \times_\sigma \pi_r^\sigma) \rtimes \tau^\sigma.$$

We conclude that these are standard and the L-parameters of their irreducible quotients are $\phi^\sigma, \eta\phi^\sigma$ respectively.

The case of unitary groups is similar. □

6. Vanishing range for the cohomology of Shimura varieties and moduli spaces of local shtukas

6.1. **Global Shimura varieties.** Let (G, X) be a Shimura datum, or a little more generally, consider a connected reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of a group homomorphism $h\colon \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}\mathbf{G}_m \to G_{\mathbb{R}}$ satisfying

(SV1) The cocharacter $\mu_h(z) := h_{\mathbb{C}}(z, 1)$ is minuscule, and

(SV2) $\mathrm{Ad}(h(i))$ is a Cartan involution of $G_{\mathbb{R}}^{\mathrm{ad}}$.

The pair (G, X) need not be a Shimura datum in that we are not imposing (2.1.1.3) but only (2.1.1.1)–(2.1.1.2) of [Del79]; the latter corresponds to our (SV1)–(SV2).

As before, $A_{G_{\mathbb{R}}}$ denotes the maximal $\mathbb{R}$-split subtorus of $Z_{G,\mathbb{R}}$. The centralizer $\mathrm{Cent}_{G_{\mathbb{R}}}(h)$ is connected (it becomes a Levi subgroup of G over $\mathbb{C}$) and $\mathrm{Cent}(h)/Z_{G,\mathbb{R}}$ is anisotropic by (SV2); we drop the subscript $G_{\mathbb{R}}$ when the context is clear. The map $\pi_0(A_{G_{\mathbb{R}}}(\mathbb{R})) \to \pi_0(\mathrm{Cent}(h)(\mathbb{R}))$ is surjective by Matsumoto's theorem, cf. [Tim22].

Let K_h° denote the unique[7] maximal compact subgroup of $\mathrm{Cent}(h)(\mathbb{R})^\circ$, equivalently, the identity component of the maximal compact subgroup of $\mathrm{Cent}(h)(\mathbb{R})$. It is a maximal connected compact subgroup of $G(\mathbb{R})$ and the image of K_h° in $G_{\mathrm{ad}}(\mathbb{R})$ is a maximal compact subgroup of $G_{\mathrm{ad}}(\mathbb{R})^\circ$.

Lemma 6.2.

$$\mathrm{Cent}(h)(\mathbb{R}) = A_{G_{\mathbb{R}}}(\mathbb{R})K_h^\circ.$$

In particular, the conjugacy class X of h identifies with the quotient of $G(\mathbb{R})/A_{G_{\mathbb{R}}}(\mathbb{R})^\circ K_h^\circ$ by the action of the finite abelian group $\pi_0(A_{G_{\mathbb{R}}}(\mathbb{R})) = A_{G_{\mathbb{R}}}(\mathbb{R})/A_{G_{\mathbb{R}}}(\mathbb{R})^\circ$. Moreover, all the connected components of both spaces are isomorphic.

Proof. We first show that $Z_G(\mathbb{R})$ is contained in $A_{G_{\mathbb{R}}}(\mathbb{R})K_h^\circ$. As Z_G is contained in any maximal torus T of $\mathrm{Cent}(h)$ and $A_{G_{\mathbb{R}}}(\mathbb{R}) \to \pi_0(T(\mathbb{R}))$ is surjective, given $z \in Z_G(\mathbb{R})$, we can find $a \in A_{G_{\mathbb{R}}}(\mathbb{R})$ such that a, z are in the same connected component of $T(\mathbb{R})$. Thus, we may assume $z \in T(\mathbb{R})^\circ$. Recall that $T(\mathbb{R})^\circ$ is the product of $A_{G_{\mathbb{R}}}(\mathbb{R})^\circ$ and the maximal connected compact subgroup K_T° of $T(\mathbb{R})$. So, we may further assume that $z \in K_T^\circ$. But K_T° is contained in K_h°.

The claim now follows as the quotient $\mathrm{Cent}(h)(\mathbb{R})/Z_G(\mathbb{R})$ is connected and compact. □

[7]The surjection $Z_{\mathrm{Cent}(h)} \times \mathrm{Cent}(h)_{\mathrm{der}} \to \mathrm{Cent}(h)$ induces a surjection $Z_{\mathrm{Cent}(h)}(\mathbb{R})^\circ \times \mathrm{Cent}(h)_{\mathrm{der}}(\mathbb{R})^\circ \to \mathrm{Cent}(h)(\mathbb{R})^\circ$ and $\mathrm{Cent}(h)_{\mathrm{der}}(\mathbb{R})^\circ$ is compact.

For neat open compact subgroups $K \subset G(\mathbb{A}^\infty)$, set

$$\mathrm{Sh}_K := \mathrm{Sh}_K(G,X) := G(\mathbb{Q})\backslash G(\mathbb{A})/(K \times A_{G_\mathbb{R}}(\mathbb{R})K_h^\circ) \cong G(\mathbb{Q})\backslash(X \times G(\mathbb{A}^\infty))/K.$$

In the case of Shimura data, this is the Shimura variety. We use the same notation in general, that is, when [Del79, (2.1.1.3)] is not assumed. It has the structure of a complex variety, and we let $d := \dim \mathrm{Sh}_K$ denote the *complex* dimension.

Choose a prime p. The colimit

$$H^i_\star(\mathrm{Sh}_{K_p},\mathbb{C}) := \varinjlim_{K^p} H^i_\star(\mathrm{Sh}_{K_pK^p},\mathbb{C}), \qquad \star \in \{(2),\emptyset,c\}$$

over open compact subgroups $K^p \subset G(\mathbb{A}^{p,\infty})$ is a $\mathbb{T}_p$-module with a commuting $G(\mathbb{A}^{p,\infty})$-action. Similarly, we define a $G(\mathbb{A}^\infty)$-module $H^i_\star(\mathrm{Sh},\mathbb{C})$ by taking colimit over $K_p \subset G(\mathbb{Q}_p)$ as well as K^p. Recall also that, by the Zucker conjecture proven by Looijenga and Saper–Stern, [Loo88, SS90]. $H^i_{(2)}(\mathrm{Sh}_K,\mathbb{C})$ is isomorphic to the intersection cohomology of the Baily–Borel–Satake minimal compactification Sh_K^*

$$\mathrm{IH}^i(\mathrm{Sh}_K,\mathbb{C}) := H^i(\mathrm{Sh}_K^*, j_{!*}\mathbb{C}),$$

where $j\colon \mathrm{Sh}_K \hookrightarrow \mathrm{Sh}_K^*$ denotes the open immersion into the compactification.

Lemma 6.3.

$$H^i_{(2)}(\mathrm{Sh}_K,\mathbb{C}) \cong \bigoplus_\pi H^i(\mathfrak{g}, A_{G_\mathbb{R}}(\mathbb{R})^\circ K_h^\circ, \pi_\infty) \otimes (\pi^\infty)^K,$$

where $\pi \subset L^2_{\mathrm{disc}}([G])$ such that $A_{G_\mathbb{R}}(\mathbb{R})$ acts trivially on π_∞. It is also isomorphic to

$$\bigoplus_\pi H^i(\mathfrak{g}, \mathrm{Cent}(h)(\mathbb{R}), \pi_\infty) \otimes (\pi^\infty)^K,$$

where there is no assumption on $\pi \subset L^2_{\mathrm{disc}}([G])$.

Proof. The first part follows from the second: use that if $H^i(\mathfrak{g}, \mathrm{Cent}(h)(\mathbb{R}), \pi_\infty)$ is nonzero then the central character of π_∞ is trivial on $A_{G_\mathbb{R}}(\mathbb{R}) \subset \mathrm{Cent}(h)(\mathbb{R})$ and, in which case,

$$H^i(\mathfrak{g}, \mathrm{Cent}(h)(\mathbb{R}), \pi_\infty) \cong H^i(\mathfrak{g}, A_{G_\mathbb{R}}(\mathbb{R})^\circ K_h^\circ, \pi_\infty)$$

by the definition of relative Lie algebra cohomology.

For the second claim, by slight variants of [Fra98, Thm. 3 and 4], there is an identification

$$H^i_{(2)}(\mathrm{Sh}_K,\mathbb{C}) \cong H^i(\mathfrak{g}, \mathrm{Cent}(h)(\mathbb{R}), L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/A_{G_\mathbb{R}}(\mathbb{R})^\circ K)).$$

(We are replacing $A_G(\mathbb{R})^\circ$ by $A_{G_\mathbb{R}}(\mathbb{R})^\circ$; this is harmless in the argument. Note also that Franke [Fra98, p.184] allows any open subgroup of a maximal compact subgroup K_∞, and in particular we take it to be the unique maximal compact subgroup of $A_{G_\mathbb{R}}(\mathbb{R})K_h^\circ$, which together with $A_{G_\mathbb{R}}(\mathbb{R})^\circ$ generates $\mathrm{Cent}(h)(\mathbb{R})$.) By [BC83], cf. Remark 4.10, this is isomorphic to

$$H^i(\mathfrak{g}, \mathrm{Cent}(h)(\mathbb{R}), L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})/A_{G_\mathbb{R}}(\mathbb{R})^\circ K)).$$

As in the first paragraph of the proof, this is isomorphic to

$$H^i(\mathfrak{g}, \mathrm{Cent}(h)(\mathbb{R}), L^2_{\mathrm{disc}}([G])^K) = H^i(\mathfrak{g}, \mathrm{Cent}(h)(\mathbb{R}), L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})/A_G(\mathbb{R})^\circ K))$$

as the discrete spectrum is the Hilbert direct sum of irreducible representation with finite multiplicities. ∎

Remark 6.4. The underlying manifolds of Shimura varieties are not exactly locally symmetric spaces but they can be related as follows. The pullback $H^i_{(2)}(\mathrm{Sh}_K, \mathbb{C}) \to H^i_{(2)}(Y_{G,K}, \mathbb{C})$ along the natural proper maps $Y_{G,K} \to \mathrm{Sh}_K$ are compatible with natural maps

$$H^i(\mathfrak{g}, \mathrm{Cent}(h)(\mathbb{R}), \pi_\infty) \otimes (\pi^\infty)^K \to H^i(\mathfrak{g}, A_G(\mathbb{R})^\circ K_h^\circ, \pi_\infty) \otimes (\pi^\infty)^K.$$

In fact, if $A_G(\mathbb{R})$ acts trivially on π_∞, we have the Künneth formula (cf. proof of Lemma 2.7),

$$H^\bullet(\mathfrak{g}, A_G(\mathbb{R})^\circ K_h^\circ, \pi_\infty) \cong H^\bullet(\mathfrak{g}^0, \mathrm{Cent}(h), \pi_\infty) \otimes \wedge^\bullet \mathfrak{a}'^*,$$

where $\mathfrak{a}'$ is the canonical complement of $(\mathrm{Lie}\, A_{G_\mathbb{R}})_\mathbb{C}$ in $(\mathrm{Lie}\, Z_G)_\mathbb{C}$ and $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{a}'$ is the induced decomposition. In particular, after passing to the colimit over K, if π_p appears in $H^i_{(2)}(\mathrm{Sh}, \mathbb{C})$ then π_p appears in $H^j_{(2)}(Y_G, \mathbb{C})$ for $j \in [i, i + \dim A_{G_\mathbb{R}} - \dim A_G]$. The invariants for Y_G and Sh are related as follows.

$$q^\flat(G_\infty) - a_G^{(2)}(\mathcal{N}_{\psi_\infty}, 0) = d - \overline{a}_G^{(2)}(\mathcal{N}_{\psi_\infty}, 0),$$
$$q^\flat(G_\infty) + a_G^{(2)}(\mathcal{N}_{\psi_\infty}, 0) = d + \overline{a}_G^{(2)}(\mathcal{N}_{\psi_\infty}, 0) + \dim A_{G_\mathbb{R}} - \dim A_G.$$

We would like to reformulate Conjectures 4.2 and 4.4 for Shimura varieties, using the following result of Arthur. The conjugacy class of μ_h, which depends only on (G, X), determines a unique element of $X^*(\widehat{T})$ dominant for $\widehat{B}$, which we denote by μ. Let $r_{-\mu}$ denote an irreducible representation of $\widehat{G}$ with highest weight conjugate to the character corresponding to $-\mu$.[8]

Proposition 6.5 (cf. [Art89, Prop. 9.1])**.** *For each cohomological Arthur parameter ψ_∞ of G_∞, the largest weight of $r_{-\mu} \circ \psi_\infty$ as a representation of $\mathbf{G}_m^A$ is equal to $\overline{a}_G^{(2)}(\mathcal{N}_{\psi_\infty}, 0)$. In other words,*

$$\overline{a}_G^{(2)}(\mathcal{N}_{\psi_\infty}, 0) = \langle \psi_\infty|_{\mathbf{G}_m^A}, \mu \rangle$$

if the representative ψ_∞ is chosen so that $\psi_\infty|_{\mathbf{G}_m^A}$ is dominant. More generally, for any $\mathcal{N} \in \mathit{Nilp}_{\widehat{G}}$ and the corresponding map $f \colon \mathrm{SL}_2 \to \widehat{G}$ (see §3.1) with $f|_{\mathbf{G}_m}$ dominant,

$$\overline{a}_G^{(2)}(\mathcal{N}, 0) = \langle f|_{\mathbf{G}_m}, \mu \rangle.$$

Proof. By definition, $\overline{a}_G^{(2)}(\mathcal{N}_{\psi_\infty}, 0) = \overline{q}(L)$ for a θ-stable Levi L such that $\psi_\infty|_{\mathrm{SL}_2^A}$ is principal in $\widehat{L}$. Then $\psi_\infty|_{\mathbf{G}_m^A} = 2\rho_L$ [NP21, Prop. 1]. As μ is dominant, **(SV1)** and **(SV2)** together imply that $\overline{q}(L) = \langle 2\rho_L, \mu \rangle$ as in Example 2.5. The second part is similar. ☐

From now on, a representative of any L-parameter ϕ is chosen such that $\phi|_{\mathbf{G}_m^D}$ is dominant.

Lemma 6.6. *For any $\mathcal{N}_1, \mathcal{N}_2 \in \mathit{Nilp}_{\widehat{G}}$ with $\mathcal{N}_1 \leq \mathcal{N}_2$, let $f_1, f_2 \colon \mathrm{SL}_2 \to \widehat{G}$ denote the corresponding morphisms (§3.1) and choose them such that $f_1|_{\mathbf{G}_m}, f_2|_{\mathbf{G}_m}$ are dominant. Then*

$$\langle f_1|_{\mathbf{G}_m}, \mu \rangle \leq \langle f_2|_{\mathbf{G}_m}, \mu \rangle$$

Proof. Let V_μ denote the underlying vector space of the representation r_μ with highest weight μ. We first reduce the claim to the case of $\mathrm{GL}(V_\mu)$ with the standard representation. It is clear that r_μ induces a map $\mathit{Nilp}_{\widehat{G}} \to \mathit{Nilp}_{\mathrm{GL}(V_\mu)}$ [9] and that

$$r_\mu(\mathcal{N}_1) \leq r_\mu(\mathcal{N}_2).$$

[8] For our purpose, the sign is not essential but this sign is more natural if we consider the Galois action on the cohomology of Shimura varieties as in the Kottwitz conjecture.

[9] Since r_μ and $r_{-\mu}$ induce the same map on nilpotent conjugacy classes, we work with r_μ for convenience.

The weight decomposition of V_μ determines a maximal torus $T_\mu \subset \mathrm{GL}(V_\mu)$ and it is also possible to extend it to a Borel pair (T_μ, B_μ) of $\mathrm{GL}(V_\mu)$ such that $f_1|_{\mathbf{G}_m}, f_2|_{\mathbf{G}_m}$ are dominant. Note also that μ is induced by the cocharacter of T_μ corresponding to the highest weight. These are enough to reduce to the case of $\mathrm{GL}(V_\mu)$.

In the case $\widehat{G} = \mathrm{GL}(V_\mu)$, it is well-known that nilpotent orbits correspond to partitions of $\dim V_\mu$ and the order $\mathcal{N}_1 \leq \mathcal{N}_2$ corresponds to the dominance order of partitions. This implies that $f_1|_{\mathbf{G}_m} \leq f_2|_{\mathbf{G}_m}$ as cocharacter of $X_*(T_\mu)$ with dominance order. As μ is dominant, the claim immediately follows. □

Conjecture 4.4 implies the following via Proposition 6.5 in light of Remark 6.4:

Conjecture 6.7. *Let π_p be an irreducible smooth representation of $G(\mathbb{Q}_p)$. If π_p is a subquotient, equivalently a summand, of*

$$H^i_{(2)}(\mathrm{Sh}, \mathbb{C}) \cong \mathrm{IH}^i(\mathrm{Sh}, \mathbb{C}),$$

then $d - \langle \phi_{\widehat{\pi}_p}|_{\mathbf{G}_m^D}, \mu\rangle \leq i \leq d + \langle \phi_{\widehat{\pi}_p}|_{\mathbf{G}_m^D}, \mu\rangle$.

Given this reformulation, we expect the following along the line of Question 4.3.

Conjecture 6.8. *(i) If π_p is a subquotient of $H^i(\mathrm{Sh}, \mathbb{C})$, then $i \geq d - \langle \phi_{\widehat{\pi}_p}|_{\mathbf{G}_m^D}, \mu\rangle$.*
(ii) If π_p is a subquotient of $H^i_c(\mathrm{Sh}, \mathbb{C})$, then $i \leq d + \langle \phi_{\widehat{\pi}_p}|_{\mathbf{G}_m^D}, \mu\rangle$.

Remark 6.9. Assuming **(A2^{++})**, one can formulate the ℓ-adic version of the above statements; L^2-cohomology with ℓ-adic coefficients itself does not make sense, but the intersection cohomology does. Statements (i) and (ii) are related via the Poincaré duality and the behavior of L-parameters under taking contragredients, which is expected to coincide with composing with the Chevalley involution.

Corollary 4.12 and Theorem 5.6 also imply that

Corollary 6.10. *Conjecture 6.7 and its ℓ-adic analogue hold true for (G, X) if G is*

- *an inner form of GL_2 that is split at p, or*
- *a quasi-split classical group,*

with the twisted weighted fundamental lemma assumed in the last case. (See §2.15.2.)

Similarly, Corollary 4.13 implies the following.

Corollary 6.11. *Suppose $G_{\mathrm{der}} \simeq \mathrm{Res}^F_{\mathbb{Q}} \mathrm{Sp}_{2n}$ for a totally real field F and $n \in \mathbb{Z}_{\geq 1}$; if $n \geq 2$, assume that the twisted weighted fundamental lemma is valid. If $\pi_p \in \mathrm{Irr}(G_p)$ appears as a summand of $H^i_{(2)}(\mathrm{Sh}_K, \mathbb{C})$ as a $G(\mathbb{Q}_p)$-module for $i \in \mathbb{Z}_{\geq 0}$ then*

$$d - \max_{\pi_{p,j}} \langle \phi_{\widehat{\pi}_{p,j}}|_{\mathbf{G}_m^D}, \mu\rangle \leq i \leq d + \max_{\pi_{p,j}} \langle \phi_{\widehat{\pi}_{p,j}}|_{\mathbf{G}_m^D}, \mu\rangle,$$

where $\pi_{p,j}$ are irreducible summands of the restriction of π_p to $G_{\mathrm{der}}(\mathbb{Q}_p)$ as in §3.5.

The same statement holds if Sp_{2n} is replaced by quasi-split special unitary groups associated with CM extensions $E \supset F$ (assuming the twisted weighted fundamental lemma is valid).

Example 6.12. (i) The Hilbert modular varieties are associated with groups $G = \mathrm{Res}^F_{\mathbb{Q}} GL_2$ for totally real fields F, or the subgroups H as in Example 3.6. The representation $r_{-\mu}$ is the tensor product of standard representations of GL_2. If π_v is unramified with Satake parameter α_v, β_v satisfying $\alpha_v/\beta_v \notin \{q_v, q_v^{-1}\}$ for a place v dividing p, then our argument implies that π_v appears only in the middle degree. This case is unconditional.

(ii) More generally, the Hilbert–Siegel varieties are associated with groups $G = G(\mathrm{Res}^F_{\mathbb{Q}}\mathrm{Sp}_{2n})$ or $\mathrm{Res}^F_{\mathbb{Q}}\mathrm{GSp}_{2n}$, F totally real. As we saw in Example 3.7, $\widehat{\pi}_{p,j}$ are in the same L-packet. The representation $r_{-\mu}$ restricts to the tensor product of the spin representations of $\mathrm{Spin}_{2n+1} \subset \mathrm{GSpin}_{2n+1} = \widehat{\mathrm{GSp}}_{2n}$.

(iii) The even quasi-split unitary Shimura varieties are associated with $\mathrm{Res}^F_{\mathbb{Q}}U^*_{2n,E/F}$ or its similitude group for CM quadratic extensions E over totally real F. Its signature is (n,n) at all real places of F, and $r_{-\mu}$ is the tensor product of the n-th exterior powers of the standard representations of GL_{2n}.

Remark 6.13. For Hilbert modular varieties and unitary Shimura varieties, the statement like "non-CAP cuspidal automorphic representations appear only in the middle degree" has been known under several assumptions (see [RW24, Conj. 7.5] for a more general conjecture made precise along this line). Such automorphic representations are generic, hence A-generic, (at least) at split places, and it is a special case of Conjectures 6.7. The case of GSp_4 is unconditionally proven in [RW24, Prop. 8.2] as well. The generic case of Conjectures 6.7, 6.8 without global condition was also known in some special situations [Clo93, HT01].

6.14. **A torsion analogue and known results.** Specializing to the A-generic case, i.e., the case where $\phi_{\widehat{\pi}_p}|_{\mathbf{G}_m^D}$ is trivial, Conjecture 6.7 implies that π_p appears only in the middle degree.

Torsion analogues of such specializations to the (A-)generic case have been studied extensively these years [CS17, CS24, Kosb, dSS, HL, DvHKZ]. The papers [Boy19], [CT23, Theorem B, Theorem 7.5.2] contain results beyond the generic case as well.

Let us formulate a general conjecture in the torsion case. We only consider the constant coefficients, which would also imply the case of non-constant coefficients with the same vanishing range, cf. [CS17, Rem. 1.7.1]. For sharper estimate, let us only refer to [LS12, LS13].

Conjecture 6.15. *Let $\ell \neq p$ and assume that **(A2^{++})** holds.*[10] *Let π_p be an irreducible smooth $\overline{\mathbb{F}}_\ell$-representation of $G(\mathbb{Q}_p)$.*

(i) If π_p is a subquotient of $H^i(\mathrm{Sh}, \overline{\mathbb{F}}_\ell)$, then

$$i \geq d - \max_{\widetilde{\pi}_p} \langle \phi_{\widehat{\widetilde{\pi}}_p}|_{\mathbf{G}_m^D}, \mu\rangle,$$

where $\widetilde{\pi}_p$ run through irreducible integral smooth $\overline{\mathbb{Q}}_\ell$-representations whose mod ℓ reduction contain π_p as subquotients.

(ii) If π_p is a subquotient of $H^i_c(\mathrm{Sh}, \overline{\mathbb{F}}_\ell)$, then $i \leq d + \max_{\widetilde{\pi}_p} \langle \phi_{\widehat{\widetilde{\pi}}_p}|_{\mathbf{G}_m^D}, \mu\rangle$.

Example 6.16. Suppose Sh_K is a unitary Shimura variety of PEL type. Assume $G_{\mathbb{Q}_p}$ is the product of general linear groups over $\mathbb{Q}_p$. If π_p is an unramified irreducible principal series whose L-parameter is of weakly Langlands–Shahidi type, then (a version of) Conjecture 6.15 is shown in [Kosb, dSS]; the argument works also for, not necessarily unramified, irreducible principal series. Later works [HL, DvHKZ] allow more general groups and p, but the representation π is assumed to be of Langlands–Shahidi type.

Example 6.17. Another extreme is the case π_v is a supercuspidal representation of a general linear group for some finite place $v|p$. Any lift of such a π_v, in the sense that π_v appears as a subquotient of the reduction, is supercuspidal and π_v is A-generic. In fact, we expect that π_v appears only in

[10] In fact, the conjecture (for a given class of π_p) makes sense once we have a notion of L-parameters for a suitable class of $\overline{\mathbb{Q}}_\ell$-representations of $G(\mathbb{Q}_p)$. Results in literature will be understood in this way.

the middle degree. Such results were obtained in [Fuj06, Shi15, IM20] for certain unitary Shimura varieties of PEL type.

Example 6.18. Assume Sh_K is of Harris–Taylor type with a CM field $F \supset F^+$ as in [HT01, Boy09], and let p be any prime such that every finite place v of F^+ above p splits over F. The result of Boyer [Boy09] can be understood as a finite level version of Conjecture 6.15 for any irreducible unramified representation π of $G(\mathbb{Q}_p)$ after taking (derived) K_p-invariant of the conjecture for a hyperspecial subgroup K_p and unipotent L-parameters. Note that $r_{-\mu}$ in this case is essentially the standard representation of a general linear group.

Example 6.19. Assume (G, X) is attached to a central quaternion algebra B over a totally real field F in the sense that $G = \mathrm{Res}^F_{\mathbb{Q}} B^\times$, and let p be any odd prime that completely splits in F and splits B. Conjecture 6.15 for unipotent L-parameters implies the result of Caraiani–Tamiozzo [CT23, Theorem 7.1.6, Theorem 7.5.2 (2)] by taking K_p-invariants as in the preceding example.

6.20. **Local analogues.** Let G be a connected reductive group over a nonarchimedean local field F whose residue characteristic is not ℓ, b an element of the Kottwitz set $B(G)$, and $\{\mu\}$ the conjugacy class of a cocharacter of $G_{\overline{F}}$. This datum determines the moduli space of G-shtukas $\mathrm{Sht}(G, b, \mu)_K$ for open compact subgroups $K \subset G(F)$, defined over the completion of the maximal unramified extension of the reflex field. Using the geometric Satake correspondence, Fargues–Scholze define the compactly supported cohomology of the "IC sheaf" with $\overline{\mathbb{Q}}_\ell$-coefficients

$$R\Gamma_c(\mathrm{Sht}(G, b, \mu)_K, \mathrm{IC}_\mu) \in D(G_b(F)),$$

where $D(G_b(F))$ is the derived ∞-category of smooth $\mathbb{Z}_\ell$-representations of the group $G_b(F)$ attached to b; we will ignore the action of the Weil group of the reflex field. This is denoted as $f_{K\natural}\mathcal{S}'_W$ in [FS, Prop. IX.3.2]. If μ is minuscule, $\mathrm{Sht}(G, b, \mu)_{K,C}$ is a smooth rigid-analytic variety over C of dimension $d = \langle 2\rho_G, \mu\rangle$ for an algebraically closed nonarchimedean extension C of $\widehat{\overline{F}}$, and

$$R\Gamma_c(\mathrm{Sht}(G, b, \mu)_K, \mathrm{IC}_\mu) \cong R\Gamma_c(\mathrm{Sht}(G, b, \mu)_{K,C}, \overline{\mathbb{Q}}_\ell)[d].$$

By passing to the colimit, we may also consider

$$R\Gamma_c(\mathrm{Sht}(G, b, \mu)_\infty, \mathrm{IC}_\mu) := \varinjlim_K R\Gamma_c(\mathrm{Sht}(G, b, \mu)_K, \mathrm{IC}_\mu) \in D(G_b(F) \times G(F)).$$

Before formulating the local conjectures, we need to discuss axioms we assume. If the valuation ring of F has mixed characteristic and if **(A2^{++})**, or rather a stronger version mentioned there, holds, then the ℓ-adic C-parameter is canonically defined. Our previous axioms are global in nature, and we prefer to make a different axiom including the case of equal characteristic.

(ℓ-LLC) Fix $q^{1/2} \in \overline{\mathbb{Q}}_\ell$. For each $b \in B(G)$ and each irreducible smooth representation π_b of $G_b(F)$, an L-parameter $\phi_{\pi_b} \colon W_F \times \mathrm{SL}_2 \to \widehat{G} \rtimes W_F$ defined over $\overline{\mathbb{Q}}_\ell$ is attached.

We do not attempt to characterize the parametrization $\pi_b \mapsto \phi_{\pi_b}$. In fact, it is expected that a canonical C-parameter ${}^C\phi_{\pi_b} \colon W_F \times \mathrm{SL}_2 \to {}^CG$ is attached to π_b independently of the choice of $q^{1/2}$, which should determine the L-parameter ϕ_{π_b} upon choosing $q^{1/2}$, cf. §2.3.

We will write ν_b for the slope morphism of b, regarded as an element of $X^*(\widehat{T})^+_{\mathbb{Q}}$. The following is our main local conjecture. The torsion case, at least with minuscule μ, can be formulated in a way similar to Conjecture 6.15.

Conjecture 6.21. *Assume **(ℓ-LLC)**. Let π, π_b be irreducible smooth representations of $G(F), G_b(F)$ respectively.*

(i) *For any integer*

$$i \notin [\langle 2\rho_G, \nu_b\rangle, \langle 2\rho_G, \nu_b\rangle + \langle \phi_{\widehat{\pi}}|_{\mathbf{G}_m^D}, \mu - \nu_b\rangle],$$

π is not a subquotient of $H^i(R\Gamma_c(\mathrm{Sht}(G,b,\mu)_\infty, \mathrm{IC}_\mu))$.

(ii) *For any integer*

$$i \notin [\langle 2\rho_G, \nu_b\rangle, \langle 2\rho_G, \nu_b\rangle + \langle \phi_{\widehat{\pi}_b}|_{\mathbf{G}_m^D} + 2\rho_G - 2\rho_{G_b}, \mu - \nu_b\rangle],$$

π_b is not a subquotient of $H^i(R\Gamma_c(\mathrm{Sht}(G,b,\mu)_K, \mathrm{IC}_\mu))$.

Remark 6.22. The lower bound $\langle 2\rho_G, \nu_b\rangle$ is independently conjectured by David Hansen. In the case μ is minuscule and b is basic, the lower bound follows from the conjecture of Hansen and Scholze that local Shimura varieties are Stein [Hanb, Conj. 1.10]. The lower bound for a general b follows from the categorical local Langlands conjecture of Fargues–Scholze [FS, Conj. I.10.2] together with a conjecture for generalized coherent Springer sheaves [Hana, Prop. 3.2.2]. We found the conjectural upper bounds for basic b as local analogues of the global conjecture, and for general b based on some computations inspired by the categorical local Langlands.

Remark 6.23. As in [Kosb], there would be another constraint for a representation π of $G(F)$ to contribute to $R\Gamma_c(\mathrm{Sht}(G,b,\mu)_\infty, \mathrm{IC}_\mu)$ if G_b is not quasi-split. A minimal requirement is that $\phi_\pi^{\mathrm{ss}} = C \circ \phi_{\pi_b}^{\mathrm{ss}}$ for some irreducible smooth representation π_b of $G_b(F)$ and the Chevalley involution $C\colon {}^LG \to {}^LG$, and this indeed holds when the semisimplified L-parameters agree with those of Fargues–Scholze.

Example 6.24. Assume b is basic and ϕ_π, ϕ_{π_b} are supercuspidal. It has been conjectured that such π, π_b appear only in degree 0. Known results include [Mie10], [Ito13, IM, Mie], [Hanb]. It is a special case of Conjecture 6.21 as $\widehat{\pi} = \pi$, $\widehat{\pi}_b = \pi_b$ and the L-parameters are trivial on $\mathbf{G}_m^D$ in this case.

Example 6.25. In the case of GSp_4 and $\mathrm{GU}_{1,2}$, Ito and Mieda [Ito13, Mie] have observed that supercuspidal representations whose L-parameter are not supercuspidal appear outside the middle degree. Their computation (or the vanishing result [IM]) is consistent with Conjecture 6.21.

Example 6.26. In the Lubin–Tate case (and the Drinfeld case by the Faltings isomorphism), Boyer gives a complete description of cohomology [Boy09, Théorème 2.3.5] (in the case of mixed characteristic). His result is consistent with Conjecture 6.21.

Example 6.27. Suppose that π, π_b are the trivial representations of $G(F), G_b(F)$. It is expected that $\phi_{\widehat{\pi}}|_{\mathbf{G}_m^D} = 2\rho_G$ and $\phi_{\widehat{\pi}_b}|_{\mathbf{G}_m^D} = 2\rho_{G_b}$. Hence, the upper bound is $\langle 2\rho_G, \mu\rangle$ in both cases. If μ is minuscule, this upper bound follows from the ℓ-cohomological dimension of $\mathrm{Sht}(G,b,\mu)_K$.

6.28. **A relation between the local and global conjectures.** The local conjecture implies the conjecture for global Shimura varieties, with the help of Mantovan's formula. Choose an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Fix a Shimura datum (G, X) with dominant cocharacter μ as before and put $d := \langle 2\rho_G, \mu\rangle$, which equals the complex dimension of the associated Shimura variety (at any finite level). For each $b \in B(G_{\mathbb{Q}_p}, \mu^{-1})$, we have the Igusa variety Ig^b of dimension $d_b := \langle 2\rho_G, \nu_b\rangle$. To simplify the discussion, we assume the following version of Mantovan's formula based on the work of Hamann–Li [HL].

Hypothesis 6.29. For each sufficiently small open compact subgroup $K^p \subset G(\mathbb{A}^{\infty,p})$, the compactly supported cohomology $R\Gamma_c(\mathrm{Sh}_{K^p}, \overline{\mathbb{Q}}_\ell)[d]$ admits a finite filtration in the derived ∞-category $D(G(\mathbb{Q}_p))$ whose graded pieces are, for $b \in B(G_{\mathbb{Q}_p}, \mu^{-1})$,

$$R\Gamma_c(\mathrm{Sht}(G_{\mathbb{Q}_p}, b, \mu)_\infty, \overline{\mathbb{Q}}_\ell)[d] \otimes^L_{C_c(G_b(\mathbb{Q}_p))} R\Gamma_{c-\partial}(\mathrm{Ig}^b, \overline{\mathbb{Q}}_\ell)[2d_b],$$

where $R\Gamma_{c-\partial}$ denotes the partially compactly supported cohomology defined using a partial compactification $g_b \colon \mathrm{Ig}^b \hookrightarrow \mathrm{Ig}^{b*}$ into an affine scheme Ig^{b*} that is the perfection of the limit of open immersions $\mathrm{Ig}^b_m \hookrightarrow \mathrm{Ig}^{b*}_m$ at finite levels. More precisely, we define

$$R\Gamma_{c-\partial}(\mathrm{Ig}^b, \overline{\mathbb{Q}}_\ell) := (R \varprojlim_m R\Gamma(\mathrm{Ig}^{b*}, g_{b!}\mathbb{Z}/\ell^m\mathbb{Z})) \otimes^L_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell,$$

where the derived limit is taken in $D(G_b(\mathbb{Q}_p))$.

Remark 6.30. With $\overline{\mathbb{F}}_\ell$-coefficients, Hypothesis 6.29 is proven in [HL, Thm. 1.12] for the PEL case of type A and C under a mild assumption, and in [DvHKZ, Thm 8.5.7, proof of Thm. 8.6.2, Prop. 4.1.6] for the compact case of Hodge type under a mild assumption. (In general, the existence of a partial compactification $g_b : \mathrm{Ig}^b \hookrightarrow \mathrm{Ig}^{b*}$ as above is a hypothesis in itself.) In fact, the argument also lifts to the case with coefficients in $\mathbb{Z}/\ell^m\mathbb{Z}$ for any $m \geq 1$ in a compatible way. Using the notation of [FS], we can write each graded piece as $i_1^* T_\mu i_{b!} R\Gamma(\mathrm{Ig}^{b*}, g_{b!}\mathbb{Z}/\ell^m\mathbb{Z})$, up to shift and twist, so [FS, IX.2.2] implies that $R \varprojlim_m R\Gamma_c(\mathrm{Sh}_{K^p}, \mathbb{Z}/\ell^m\mathbb{Z})[d]$ admits a filtration with graded pieces

$$R \varprojlim_m i_1^* T_\mu i_{b!} R\Gamma(\mathrm{Ig}^{b*}, g_{b!}\mathbb{Z}/\ell^m\mathbb{Z}) \cong i_1^* T_\mu i_{b!} R \varprojlim_m R\Gamma(\mathrm{Ig}^{b*}, g_{b!}\mathbb{Z}/\ell^m\mathbb{Z}),$$

up to shift and twist. Observe that $R \varprojlim_m R\Gamma_c(\mathrm{Sh}_{K^p}, \mathbb{Z}/\ell^m\mathbb{Z})$ is $R\Gamma_c(\mathrm{Sh}_{K^p}, \mathbb{Z}_\ell)$ as the derived limit is taken in $D(G(\mathbb{Q}_p))$. Inverting ℓ and extending the scalars, we have verified Hypothesis 6.29 in these situations.

Proposition 6.31. *Let $(G, X), \mu$ be as above. Conjecture 6.21 (i) for all $b \in B(G_{\mathbb{Q}_p}, \mu^{-1})$ and Hypothesis 6.29 imply the ℓ-adic analogue of Conjecture 6.8 (ii).*

Proof. By the Artin vanishing and limit arguments,

$$R\Gamma_{c-\partial}(\mathrm{Ig}^b, \overline{\mathbb{Q}}_\ell)[2d_b]$$

lives in $D^{[-2d_b, -d_b]}(G_b(\mathbb{Q}_p))$. Therefore, Conjecture 6.21 (i) implies that, for any irreducible smooth representation π and an integer $i > \langle \phi_{\widehat{\pi}}|_{\mathbf{G}_m^D}, \mu - \nu_b \rangle$, π is not a subquotient of

$$H^i(R\Gamma_c(\mathrm{Sht}(G, b, \mu)_{(G_{\mathbb{Q}_p}, b, \mu), \infty}, \overline{\mathbb{Q}}_\ell)[d] \otimes^L_{C_c(G_b(\mathbb{Q}_p))} R\Gamma_{c-\partial}(\mathrm{Ig}^b, \overline{\mathbb{Q}}_\ell)[2d_b]).$$

Then, Hypothesis 6.29 implies that π is not a subquotient of

$$H_c^{d+i}(\mathrm{Sh}_{K^p}, \overline{\mathbb{Q}}_\ell)$$

for any $i > \langle \phi_{\widehat{\pi}}|_{\mathbf{G}_m^D}, \mu \rangle$. □

Example 6.32. Consider the setting of [Kosb]. In particular, G is a unitary similitude group for a CM quadratic extension E of a totally real field F, and p splits completely in E.

(i) In the compact case, any generic unramified representation π_p of $G(\mathbb{Q}_p)$ appears only in the middle degree cohomology $H^d(\mathrm{Sh}, \overline{\mathbb{Q}}_\ell)$. This is a special case of the ℓ-adic analogue of Conjecture 6.7 and follows from the above discussion, Poincaré duality, and the $\overline{\mathbb{Q}}_\ell$-version of [Kosb, Thm. 1.1], which can be shown by a similar (and simpler) argument, cf. Remark 6.23. This case is related to purity and was known to some extent [CS17, Rem. 1.8], e.g. [HT01, Cor. VI.2.7].

(ii) In the quasi-split (hence non-compact) case, any generic unramified π_p appears only in $H_c^{\leq d}(\mathrm{Sh}, \overline{\mathbb{Q}}_\ell)$, and dually only in $H^{\geq d}(\mathrm{Sh}, \overline{\mathbb{Q}}_\ell)$. This is a special case of the ℓ-adic analogue of Conjecture 6.8 and holds by the same reasoning as (i). Recall that Corollary 6.11 says π_p appears only in $\mathrm{IH}^d(\mathrm{Sh}, \overline{\mathbb{Q}}_\ell)$, conditionally on twisted weighted fundamental lemma.

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