COHOMOLOGY OF IGUSA CURVES – A SURVEY

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Abstract. We illustrate the strategy to compute the ℓ-adic cohomology of Igusa varieties in the setup of ordinary modular curves, with updates on the literature towards a genrealization.

1. Introduction

Igusa curves were introduced by Igusa [Igu68] to understand the mod p geometry of modular curves when the level is divisible by p, for each prime p. As a generalization, we have Igusa varieties in the setup of Shimura varieties of Hodge type thanks to [HT01, Man05, Ham17, Zha, HK19] Igusa varieties shed light on the mod p and p-adic geometry of Shimura varieties via the so-called product structure. Moreover they play a vital role in the applications to the Langlands correspondence, vanishing results on the cohomology of Shimura varieties, and p-adic automorphic forms. We refer to [KS, §1] for a detailed introduction to Igusa varieties and further references. For an application to (an extension of) the Kottwitz conjecture on the cohomology of Rapoport–Zink spaces, see [Shi12, BM].

A fundamental problem on Igusa varieties is to compute their ℓ-adic cohomology (with or without compact support) for primes $\ell \neq p$. To this end, the Langlands–Kottwitz (LK) method for Shimura varieties has been adapted to Igusa varieties in [HT01, Shi09, MC21], at least when the level structure at p is hyperspecial. (In [HT01], one can go a little further.) While there are excellent exposotions[1] on the LK method for modular curves (with good reduction mod p) by Clozel [Clo93, §3] and Scholze [Sch11, §5], in addition to Langlands's original papers [Lan73, Lan76], there is no counterpart for Igusa curves. The goal of this article is to spell out the LK method for Igusa curves in a somewhat informal style, thereby to serve as a friendly entry point for the subject.

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Notation and Conventions. When R is a commutative ring with unity, we often use R to mean Spec$R$ when there is no danger of confusion. For example, a scheme X over R means a scheme over Spec$R$, and $X \times_R S$ means $X \times_{\mathrm{Spec}R} \mathrm{Spec}S$ when a ring homomorphism $R \to S$ is given. By (Set) (resp. (Sch/R)) we denote the category of sets (resp. schemes over R). We also write X_S for $X \times_R S$ if $R \to S$ is clear from the context. Similarly if X is a scheme, we write (Sch/X) for the

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[1] There are also several valuable surveys on the LK method for more general Shimura varieties with different emphases, such as [BR94, Clo93, GN09, Zhu20].

category of schemes over X. Write $\widehat{\mathbb{Z}}^p := \varprojlim_{(N,p)=1} \mathbb{Z}/N\mathbb{Z}$ and $\mathbb{A}^{\infty,p} := \widehat{\mathbb{Z}}^p \otimes_{\mathbb{Z}} \mathbb{Q}$ for the ring of adèles away from ∞, p. By $C_c^\infty(X)$, we mean the space of smooth compactly supported functions on a locally compact group X (with values in $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$).

2. Modular curves

Let $N \in \mathbb{Z}_{\geq 3}$. Consider the moduli functor

$$Y_N : (\mathrm{Sch}/\mathbb{Z}[1/N]) \longrightarrow (\mathrm{Set})$$

sending S to the set of isomorphism classes of pairs (E, α), where E is an elliptic curve over S, and $\alpha : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N]$ is an isomorphism of group schemes over S.

Theorem 2.1 (Igusa, Deligne–Rapoport)**.** *The functor Y_N is represented by a smooth affine curve over $\mathbb{Z}[1/N]$.*

We keep writing Y_N for the curve it represents. Denote by $\mathcal{E} \to Y_N$ the universal elliptic curve. Consider the following inverse limit

$$Y_{\mathbb{C}} := \varprojlim_{N \geq 3} Y_N \times_{\mathbb{Z}[1/N]} \mathbb{C},$$

which exists in the category of $\mathbb{C}$-schemes as the transition maps are finite étale. We have

$$\pi_0(Y_{\mathbb{C}}) := \varprojlim_{N \geq 3} \pi_0(Y_{N,\mathbb{C}}) = \varprojlim_{N \geq 3} (\mathbb{Z}/N\mathbb{Z})^\times = \widehat{\mathbb{Z}}^\times. \tag{2.1}$$

From now on, fix a prime p once and for all. We restrict the level N to integers coprime to p. Recall that an elliptic curve E over a field k of characteristic p is said to be **supersingular** if $\#E[p](\overline{k}) = 1$. Otherwise E is said to be **ordinary**, in which case $\#E[p](\overline{k}) = p$. Accordingly we have a partition of the topological space

$$Y_{N,\mathbb{F}_p} = Y^{\mathrm{ord}}_{N,\mathbb{F}_p} \coprod Y^{\mathrm{ss}}_{N,\mathbb{F}_p}, \tag{2.2}$$

where $Y^{\mathrm{ord}}_{N,\mathbb{F}_p}$ (resp. $Y^{\mathrm{ss}}_{N,\mathbb{F}_p}$) is the subset of $x \in Y_{N,\mathbb{F}_p}$ such that the fiber $\mathcal{E}_x$ is an ordinary (resp. supersingular) elliptic curve. Thus we can view $Y^{\mathrm{ord}}_{N,\mathbb{F}_p}$ as an open subscheme of $Y_{N,\mathbb{F}_p}$ (and $Y^{\mathrm{ss}}_{N,\mathbb{F}_p}$ as a closed 0-dimensional subscheme). As $N \in \mathbb{Z}_{\geq 3}$ varies over prime-to-p integers, the transition maps are finite étale and compatible with the partition (2.2).

In this survey we will concentrate on the ordinary case though there is a parallel story in the supersingular case.

Remark 2.2. The stratification (2.2) admits a vast generalization to general Shimura varieties. The reader is referred to excellent articles such as [Man20, HR17].

3. Igusa curves

We keep fixing a prime p and let $N \geq 3$ be an integer coprime to p. We still write $\mathcal{E}$ for the universal elliptic curve over $Y^{\mathrm{ord}}_{N,\mathbb{F}_p}$. For each integer $m \geq 1$, we have the slope filtration $0 \to \mathcal{E}[p^m]^\circ \to \mathcal{E}[p^m] \to \mathcal{E}[p^m]^{\text{ét}} \to 0$ such that $\mathcal{E}[p^m]^{\text{ét}}$ is the maximal étale quotient. We introduce the Igusa functor of level Np^m as

$$\mathrm{Ig}^{\mathrm{ord}}_{N,m} : (\mathrm{Sch}/Y^{\mathrm{ord}}_{N,\mathbb{F}_p}) \longrightarrow (\mathrm{Set}), \qquad S \mapsto \{(j_m^{\text{ét}}, j_m^\circ)\}, \tag{3.1}$$

where $j_m^{\text{ét}} : \mathbb{Z}/p^m\mathbb{Z} \xrightarrow{\sim} \mathcal{E}[p^m]^{\text{ét}}_S$ and $j_m^\circ : \mu_{p^m} \xrightarrow{\sim} \mathcal{E}[p^m]^\circ_S$ are isomorphisms of groups schemes over S. A fundamental theorem by Igusa (reproduced by Katz–Mazur) is the following.

Theorem 3.1. *The functor* $\mathrm{Ig}^{\mathrm{ord}}_{N,m}$ *is represented by a scheme, which is an étale* $\mathrm{GL}_1(\mathbb{Z}/p^m\mathbb{Z}) \times \mathrm{GL}_1(\mathbb{Z}/p^m\mathbb{Z})$*-torsor over* $Y^{\mathrm{ord}}_{N,\mathbb{F}_p}$*.*

As $N \in \mathbb{Z}_{\geq 3}$ and $m \in \mathbb{Z}_{\geq 0}$ vary, $\{\mathrm{Ig}^{\mathrm{ord}}_{N,m}\}$ forms a projective system with finite étale transition maps, equippd with a prime-to-p Hecke action of $\mathrm{GL}_2(\mathbb{A}^{\infty,p})$ defined in the same way for modular curves. Define a $\mathbb{Q}_p$-group $J := \mathrm{GL}_1 \times \mathrm{GL}_1$, so that $J(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$ is the automorphism group of $\mathbb{Q}_p/\mathbb{Z}_p \times \mu_{p^\infty}$ in the isogeny category of p-divisible groups. The obvious action of $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ on $\mathrm{Ig}^{\mathrm{ord}}_{N,m}$ by translating $(j^{\text{ét}}_m, j^{\circ}_m)$ induces an action on

$$H^i_c(\mathrm{Ig}^{\mathrm{ord}}_\infty, \overline{\mathbb{Q}}_\ell) := \varinjlim_{\substack{N\geq 3,\ (N,p)=1,\\ m\geq 1}} H^i_c(\mathfrak{Ig}^{\mathrm{ord}}_{N,m,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell), \qquad i \geq 0, \tag{3.2}$$

which uniquely extends to an action of $J(\mathbb{Q}_p)$. (Mantovan [Man05] proved that the action extends. The same also follows from Caraiani–Scholze's approach [CS17, CS].) The $J(\mathbb{Q}_p)$-action turns out to commute with the $\mathrm{GL}_2(\mathbb{A}^{\infty,p})$-action. Finite-dimensionality of cohomology for each N and m tells us that $H^i_c(\mathrm{Ig}^{\mathrm{ord}}_\infty, \overline{\mathbb{Q}}_\ell)$ is an admissible representation of $\mathrm{GL}_2(\mathbb{A}^{\infty,p}) \times J(\mathbb{Q}_p)$. Now we can state the goal of this article:

Goal: Compute (3.2) as a representation of $\mathrm{GL}_2(\mathbb{A}^{\infty,p}) \times J(\mathbb{Q}_p)$.

Caraiani and Scholze [CS17, CS] defined another version of Igusa varieties directly at level Np^∞. In our case, their definition specializes to the following functor, where $(\mathrm{Perf}/Y^{\mathrm{ord}}_{N,\mathbb{F}_p})$ means the category of perfect schemes over $Y^{\mathrm{ord}}_{N,\mathbb{F}_p}$:

$$\mathfrak{Ig}^{\mathrm{ord}}_{N,\infty} : (\mathrm{Perf}/Y^{\mathrm{ord}}_{N,\mathbb{F}_p}) \longrightarrow (\mathrm{Set}),$$

sending S to the set of isomorphisms $\mathbb{Q}_p/\mathbb{Z}_p \times \mu_{p^\infty} \xrightarrow{\sim} \mathcal{E}[p^\infty]_S$ between p-divisible groups over S. This can be compared with the scheme

$$\mathrm{Ig}^{\mathrm{ord}}_{N,\infty} := \varprojlim_{m\geq 1} \mathrm{Ig}^{\mathrm{ord}}_{N,m},$$

where the "level-decreasing" transition maps are finite étale. As $\mathrm{Ig}^{\mathrm{ord}}_{N,\infty}$ form a projective system with finite étale transition maps as N varies, we can take the limit scheme $\mathrm{Ig}^{\mathrm{ord}}_\infty$.

Theorem 3.2 (Caraiani–Scholze)**.** *The functor* $\mathfrak{Ig}^{\mathrm{ord}}_{N,\infty}$ *is represented by a perfect scheme over* $Y^{\mathrm{ord}}_{N,\mathbb{F}_p}$ *and canonically isomorphic to the perfection of* $\mathrm{Ig}^{\mathrm{ord}}_{N,\infty}$*.*

Since perfection does not affect topological information such as étale cohomology or the set of $\overline{\mathbb{F}}_p$-points, we can use either $\mathrm{Ig}^{\mathrm{ord}}_{N,\infty}$ or $\mathfrak{Ig}^{\mathrm{ord}}_{N,\infty}$. Since the former is built out of finite-level Igusa curves, it is useful for applying a fixed point formula. On the other hand, $\mathfrak{Ig}^{\mathrm{ord}}_{N,\infty}$ is a little more convenient for defining group actions and describing the $\overline{\mathbb{F}}_p$-points.

Remark 3.3. We can define $\mathrm{Ig}^{\mathrm{ss}}_{N,\infty}$ and $\mathfrak{Ig}^{\mathrm{ss}}_{N,\infty}$ in analogy with the ordinary case. Then $\mathrm{Ig}^{\mathrm{ss}}_{N,\infty}$ is already perfect and $\mathrm{Ig}^{\mathrm{ss}}_{N,\infty} = \mathfrak{Ig}^{\mathrm{ss}}_{N,\infty}$.

4. $\overline{\mathbb{F}}_p$-points of Igusa curves

In order to achieve the aforementioned goal via a fixed-point formula, we need to describe the set of $\overline{\mathbb{F}}_p$-points of Igusa curves

$$\mathfrak{Ig}^{\mathrm{ord}}_{\infty}(\overline{\mathbb{F}}_p) = \varprojlim_{(N,p)=1,\ N\geq 3} \mathfrak{Ig}^{\mathrm{ord}}_{\infty}(\overline{\mathbb{F}}_p) = \varprojlim_{(N,p)=1,\ N\geq 3} \mathrm{Ig}^{\mathrm{ord}}_{\infty}(\overline{\mathbb{F}}_p)$$

with the $\mathrm{GL}_2(\mathbb{A}^{\infty,p}) \times J(\mathbb{Q}_p)$-action.

Let us set up some more notation. Write $\breve{\mathbb{Z}}_p := W(\overline{\mathbb{F}}_p)$ and $\breve{\mathbb{Q}}_p := W(\overline{\mathbb{F}}_p)[1/p]$. Denote by Ell^0 the set of isogeny classes of elliptic curves over $\overline{\mathbb{F}}_p$. Those of ordinary elliptic curves define a subset $\mathrm{Ell}^{0,\mathrm{ord}}$. We identify $\mathrm{Ell}^{0,\mathrm{ord}}$ with a set of representatives by fixing a representative in each isogeny class.

When E is an elliptic curve over $\overline{\mathbb{F}}_p$, define

- $I(E) := (\mathrm{End}_{\overline{\mathbb{F}}_p}(E) \otimes_{\mathbb{Z}} \mathbb{Q})^{\times}$,
- $T^p(E) := \varprojlim_{(N,p)=1} E[N](\overline{\mathbb{F}}_p)$,
- $\breve{T}_p(E)$ to be the covariant Dieudonné module of $E[p^{\infty}]$.

As we are concerned with the ordinary case, $I(E) = F^{\times}$ for an imaginary quadratic field F. (As an algebraic group over $\mathbb{Q}$, $I(E) = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$.) A standard fact is that T^pE is a free $\widehat{\mathbb{Z}}^p$-module of rank 2, and $\breve{T}_pE$ is free of rank 2 over $\breve{\mathbb{Z}}_p$ (which is the $\breve{\mathbb{Z}}_p$-linear dual of $H^1_{\mathrm{cris}}(E/\breve{\mathbb{Z}}_p)$). At p, we have the extra structure of semi-linear maps F^{-1}, V^{-1} on $\breve{T}_pE$ such that $F^{-1}V^{-1} = V^{-1}F^{-1} = p$. (The minus sign comes from the covariant convention.) It is useful to think of T^pE as a $\widehat{\mathbb{Z}}^p$-lattice in the free $\mathbb{A}^{\infty,p}$-module $V^pE := T^pE \otimes_{\mathbb{Z}} \mathbb{Q}$ of rank 2. Similarly $\breve{T}_pE$ is an F^{-1}, V^{-1}-invariant lattice in $\breve{V}_pE := \breve{T}_pE \otimes_{\mathbb{Z}} \mathbb{Q}$. (We have linear extensions of F^{-1}, V^{-1} to self-bijections on $\breve{V}_pE$.)

Now we start our analysis of $\overline{\mathbb{F}}_p$-points from (3.1).

$$\begin{aligned}
\mathfrak{Ig}^{\mathrm{ord}}_{\infty}(\overline{\mathbb{F}}_p) &= \left\{ \begin{array}{l} E : \text{elliptic curve}/\overline{\mathbb{F}}_p, \\ \alpha : (\widehat{\mathbb{Z}}^p)^2 \xrightarrow{\sim} T^pE, \\ j : \mathbb{Q}_p/\mathbb{Z}_p \times \mu_{p^\infty} \xrightarrow{\sim} E[p^\infty] \end{array} \right\} / \simeq \\
&= \coprod_{E_0 \in \mathrm{Ell}^{0,\mathrm{ord}}} \left\{ \begin{array}{l} (E,\alpha,j) \text{ as above,} \\ \text{s.t. } \exists \text{ an isogeny } f : E \to E_0 \end{array} \right\} / \simeq \\
&= \coprod_{E_0 \in \mathrm{Ell}^{0,\mathrm{ord}}} I(E_0)\backslash \left\{ \begin{array}{l} (L^p, \phi^p, L_p, \phi_p) : \\ L^p \subset V^pE_0 \text{ is a } \widehat{\mathbb{Z}}^p\text{-lattice, } \phi^p : (\widehat{\mathbb{Z}}^p)^2 \xrightarrow{\sim} L^p, \\ L_p \subset \breve{V}_pE_0 \text{ is an } F^{-1}, V^{-1}\text{-invariant } \breve{\mathbb{Z}}_p\text{-lattice,} \\ \phi_p : \breve{\mathbb{Z}}_p^2 \xrightarrow{\sim} L_p \text{ carries } (1, p^{-1})\sigma \text{ on } \breve{\mathbb{Z}}_p^2 \text{ to } F \text{ on } L_p. \end{array} \right\}.
\end{aligned}$$

In the last expression, ϕ^p and ϕ_p are respectively $\widehat{\mathbb{Z}}^p$-linear and $\breve{\mathbb{Z}}_p$-linear. Each equality above is natural and equivariant with respect to the natural action of $G(\mathbb{A}^{\infty,p}) \times J(\mathbb{Q}_p)$. To see the last equality, one starts from (E,α,j) and chooses an isogeny $f : E \to E_0$. Then take $L^p = f(T^pE)$ and $L_p = f(\breve{T}_pE)$. We leave it as an exercise to give ϕ^p and ϕ_p from (E,α,j) and to show that the left quotient by $I(E_0)$ cancels out the choice of f (so that the quotient set is independent of the choice).

To proceed, we give more convenient parametrizations of (L^p, ϕ^p) and (L_p, ϕ_p). We describe the right $\mathrm{GL}_2(\mathbb{A}^{\infty,p})$-set (which is a torsor for the action)

$$X^p(E_0) := \{(L^p, \phi^p) \text{ as above}\} = \{(\mathbb{A}^{\infty,p})^2 \overset{\sim}{\to} V^p E_0\},$$

where $\mathrm{GL}_2(\mathbb{A}^{\infty,p})$ acts on the last set through its natural action on $(\mathbb{A}^{\infty,p})^2$. To obtain the inverse map, notice that an isomorphism $(\mathbb{A}^{\infty,p})^2 \overset{\sim}{\to} V^p E_0$ determines (L^p, ϕ^p) by restriction to $(\widehat{\mathbb{Z}}^p)^2$. The above identification is also equivariant for the left action of $I(E_0)$, which naturally acts on $V^p E_0$.

Similarly we have a bijection of right $J(\mathbb{Q}_p)$-sets (which are $J(\mathbb{Q}_p)$-torsors)

$$\breve{X}_p(E_0) := \{(L_p, \phi_p) \text{ as above}\} = \{(\breve{\mathbb{Q}}_p)^2 \overset{\sim}{\to} \breve{V}_p E_0, \text{ s.t. } (1, p^{-1})\sigma \leftrightarrow F\},$$

where $J(\mathbb{Q}_p)$ acts as automorphisms of the isocrystal $((\breve{\mathbb{Q}}_p)^2, (1, p^{-1})\sigma)$ associated with the p-divisible group $\mathbb{Q}_p/\mathbb{Z}_p \times \mu_{p^\infty}$. The above equality is equivariant for the left action of $I(E_0)$, which acts as automorphisms of the isocrystal $(\breve{V}_p E_0, F)$.

The progress so far may be summarized as follows. As right $\mathrm{GL}_2(\mathbb{A}^{\infty,p}) \times J(\mathbb{Q}_p)$-sets,

$$\mathfrak{Ig}^{\mathrm{ord}}_\infty(\overline{\mathbb{F}}_p) = \coprod_{E_0 \in \mathrm{Ell}^{0,\mathrm{ord}}} I(E_0) \backslash \left(X^p(E_0) \times X_p(E_0)\right). \tag{4.1}$$

By choosing a base point, we can identify the right $\mathrm{GL}_2(\mathbb{A}^{\infty,p}) \times J(\mathbb{Q}_p)$-torsor $X^p(E_0) \times X_p(E_0)$ with $\mathrm{GL}_2(\mathbb{A}^{\infty,p}) \times J(\mathbb{Q}_p)$ equipped with an embedding of groups

$$I(E_0) \hookrightarrow \mathrm{GL}_2(\mathbb{A}^{\infty,p}) \times J(\mathbb{Q}_p),$$

well defined up to $\mathrm{GL}_2(\mathbb{A}^{\infty,p}) \times J(\mathbb{Q}_p)$-conjugacy. On the other hand, a special case of Honda–Tate theory over $\overline{\mathbb{F}}_p$ (cf. [HT01, V.2] or [Shi09, §8]) tells us that $\mathrm{Ell}^{0,\mathrm{ord}}$ is in bijection with the set $\mathrm{IQF}(p)(p)$ of imaginary quadratic fields (up to isomorphism) in whic p splits, where we assign $\mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ (which is an imaginary quadratic field since E is ordinary) to each $E \in \mathrm{Ell}^{0,\mathrm{ord}}$. Thus we can rewrite (4.1) as follows.

Proposition 4.1. *As right* $\mathrm{GL}_2(\mathbb{A}^{\infty,p}) \times J(\mathbb{Q}_p)$*-sets,*

$$\mathfrak{Ig}^{\mathrm{ord}}_\infty(\overline{\mathbb{F}}_p) = \coprod_{F \in \mathrm{IQF}(p)} F^\times \backslash \left(\mathrm{GL}_2(\mathbb{A}^{\infty,p}) \times J(\mathbb{Q}_p)\right),$$

where the quotient is taken with respect to the embedding of $I(E_0) = F^\times$ *above.*

Remark 4.2. Mack-Crane [MC21] recently obtained the analogue for Igusa varieties in the setup of Hodge-type Shimura varieties with hyperspecial level at p, generalizing [Shi09] on the PEL case.

5. FROM $\overline{\mathbb{F}}_p$-POINTS TO THE TRACE FORMULA

Before we go from Proposition 4.1 to compute the ℓ-adic cohomology, we need some preparation. Let $N \geq 3$ and $m \geq 1$. Define

$$\begin{aligned} K^p = K^p(N) &:= \ker(\mathrm{GL}_2(\widehat{\mathbb{Z}}^p) \to \mathrm{GL}_2(\widehat{\mathbb{Z}}^p/N\widehat{\mathbb{Z}}^p)) \subset \mathrm{GL}_2(\mathbb{A}^{\infty,p}), \\ K_p = K_{p,m} &:= (1 + p^m\mathbb{Z}_p) \times (1 + p^m\mathbb{Z}_p) \subset J(\mathbb{Q}_p). \end{aligned}$$

Then $\mathrm{Ig}^{\mathrm{ord}}_{N,m} = \mathrm{Ig}^{\mathrm{ord}}_\infty / K^p \times K_p$. Let

$$g^p \in \mathrm{GL}_2(\mathbb{A}^{\infty,p}), \qquad g_p = (g_{p,1}, g_{p,2}) \in J(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times.$$

We say that g_p is *acceptable* if the additive p-adic valuations satisfy the inequality $v_p(g_{p,1}) > v_p(g_{p,2})$.

Write $\mathbf{1}_{K^pg^pK^p}$ and $\mathbf{1}_{K_pg_pK_p}$ for the characteristic functions on the corresponding double cosets, viewed as elements of Hecke algebras for $\mathrm{GL}_2(\mathbb{A}^{\infty,p})$ and $J(\mathbb{Q}_p)$, respectively. Let $[K^pg^pK^p]$ and $[K_pg_pK_p]$ denote the double coset operators on the set of $\overline{\mathbb{F}}_p$-points or cohomology of $\mathrm{Ig}^{\mathrm{ord}}_{N,m}$. Denote by H_c the alternating sum $\sum_{i\geq 0}(-1)^iH^i_c$ in the Grothendieck group of representations. To achieve the goal stated in §3, we compute

$$\mathrm{tr}\left(\mathbf{1}_{K^pg^pK^p}\times\mathbf{1}_{K_pg_pK_p}\mid H_c(\mathrm{Ig}^{\mathrm{ord}}_{\infty},\overline{\mathbb{Q}}_\ell)\right),$$

which is equal to (the volume of $K^p\times K_p$ times)

$$\mathrm{tr}\left([K^pg^pK^p]\times[K_pg_pK_p]\mid H_c(\mathrm{Ig}^{\mathrm{ord}}_{N,m,\overline{\mathbb{F}}_p},\overline{\mathbb{Q}}_\ell)\right). \tag{5.1}$$

Let $\mathrm{Fix}(A|B)$ denote the set of fixed points of an operator A acting on a mathematical object B. We apply the fixed-point formula for non-proper varieties à la Fujiwara and Varshavsky[2] to obtain the following.

$$\begin{aligned}(5.1)\quad &= \quad \#\mathrm{Fix}\left([K^pg^pK^p]\times[K_pg_pK_p]\mid \mathrm{Ig}^{\mathrm{ord}}_{N,m}(\overline{\mathbb{F}}_p)\right)\\ &\overset{\text{Prop. 4.1}}{=} \quad \#\mathrm{Fix}\left([K^pg^pK^p]\times[K_pg_pK_p]\mid \sum_{F\in\mathrm{IQF}(p)}F^\times\backslash\left(\mathrm{GL}_2(\mathbb{A}^{\infty,p})\times J(\mathbb{Q}_p)\right)/K^p\times K_p\right).\end{aligned}$$

The details are omitted, but this is turned into the following via the combinatorial lemma of [Mil92, §5]:

$$=\sum_{F\in\mathrm{IQF}(p)}\sum_{a\in F^\times}\#\left(F^\times\backslash(Y^p(a)\times\breve{Y}_p(a))\right), \tag{5.2}$$

where

$$\begin{aligned}Y^p(a) &:= \{y^p\in\mathrm{GL}_2(\mathbb{A}^{\infty,p})/K^p : y^pg^p=ay^p \text{ in } \mathrm{GL}_2(\mathbb{A}^{\infty,p})/K^p\}\\ &= \left\{y^p\in\mathrm{GL}_2(\mathbb{A}^{\infty,p})/K^p : (y^p)^{-1}ay^p\in K^pg^pK^p\right\},\\ \breve{Y}_p(a) &:= \{y_p\in J(\mathbb{Q}_p)/K_p : y_pg_p=ay_p \text{ in } J(\mathbb{Q}_p)/K_p\}\\ &= \left\{y_p\in J(\mathbb{Q}_p)/K_p : y_p^{-1}ay_p\in K_pg_pK_p\right\}.\end{aligned}$$

Of course $y_p^{-1}ay_p=a$ in our setup since $J(\mathbb{Q}_p)$ is abelian, but we chose to write $y_p^{-1}ay_p$ since this is the correct expression for general Igusa varieties where J is not a torus. Thus we can rewrite (5.2) as

$$=\sum_{F\in\mathrm{IQF}(p)}\sum_{a\in F^\times}\int_{F^\times\backslash\mathrm{GL}_2(\mathbb{A}^{\infty,p})\times J(\mathbb{Q}_p)}\mathbf{1}_{K^pg^pK^p}((y^p)^{-1}ay^p)\times\mathbf{1}_{K_pg_pK_p}(y_p^{-1}ay_p)d(y^p,y_p).$$

Since the integrand depends only on the $F^\times_{\mathbb{A}^\infty}$-coset of (y^p,y_p), we can rewrite $\int_{F^\times\backslash\mathrm{GL}_2(\mathbb{A}^{\infty,p})\times J(\mathbb{Q}_p)}$ as $\mathrm{vol}(F^\times\backslash F^\times_{\mathbb{A}^\infty})\int_{F^\times\backslash\mathrm{GL}_2(\mathbb{A}^{\infty,p})\times J(\mathbb{Q}_p)}$, and then express the integral as an orbital integral at a:

$$=\sum_{F\in\mathrm{IQF}(p)}\sum_{a\in F^\times}\mathrm{vol}(F^\times\backslash F^\times_{\mathbb{A}^\infty})O_a^{\mathrm{GL}_2(\mathbb{A}^{\infty,p})}(\mathbf{1}_{K^pg^pK^p})O_a^{J(\mathbb{Q}_p)}(\mathbf{1}_{K_pg_pK_p}).$$

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[2] To apply this formula, we need to twist the double coset operator by a sufficiently high power of Frobenius. In this article we will gloss over this point, but this turns out to be harmless for computing the cohomology. See [HT01, V.1] or [Shi09, §6] for details.

Now we reparametrize the pairs (F, a) in the sum. We will view $J(\mathbb{Q}_p) = \mathrm{GL}_1(\mathbb{Q}_p) \times \mathrm{GL}_1(\mathbb{Q}_p)$ as the diagonal subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$ below. Recall that an element $\gamma_0 \in \mathrm{GL}_2(\mathbb{Q})$ is said to be $\mathbb{R}$-*elliptic* if γ_0 is either central or has imaginary eigenvalues. We use the symbol $\sim$ to designate the conjugacy relation. Then

Lemma 5.1. *There is a natural bijection between the following two sets:*

(i) $\{(F, a) : F \in \mathrm{IQF}(p),\ a \in F^\times,\ \text{s.t. } a \text{ is conjugate to an acceptable element of } J(\mathbb{Q}_p)\}$,

(ii) $\mathscr{C} := \{(\gamma_0, \delta) : \gamma_0 \in \mathrm{GL}_2(\mathbb{Q})/\sim\ \mathbb{R}\text{-elliptic},\ \delta \in J(\mathbb{Q}_p)/\sim\ \text{acceptable}\}$,

given as follows. For each pair (F, a)*, write* $a = (a_1, a_2) \in F^\times_{\mathbb{Q}_p} \cong \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$*; then* $v_p(a_1) \neq v_p(a_2)$ *by the condition on* a*. The pair is sent to* (a, δ)*, where* $\delta = (a_1, a_2)$ *if* $v_p(a_1) > v_p(a_2)$*, and* $\delta = (a_2, a_1)$ *otherwise.*

Proof. Left as an exercise. ☐

We apply the lemma to the preceding formula to obtain

$$(5.1) = \sum_{(\gamma_0,\delta)\in\mathscr{C}} \mathrm{vol}(F^\times \backslash F^\times_{\mathbb{A}^\infty}) O_{\gamma_0}^{\mathrm{GL}_2(\mathbb{A}^{\infty,p})}(\mathbf{1}_{K^p g^p K^p}) O_\delta^{J(\mathbb{Q}_p)}(\mathbf{1}_{K_p g_p K_p}).$$

As we can take finite linear combinations of test functions of the form $\mathbf{1}_{K^p g^p K^p}$ (resp. $\mathbf{1}_{K_p g_p K_p}$), we arrive at the following.

Theorem 5.2. *Let* $f^p \in C_c^\infty(\mathrm{GL}_2(\mathbb{A}^{\infty,p}))$*,* $f'_p \in C_c^\infty(J(\mathbb{Q}_p))$*, and assume that* f'_p *is supported on acceptable elements. Then*

$$\mathrm{tr}\left(f^p \times f'_p \mid H_c(\mathrm{Ig}^{\mathrm{ord}}_\infty, \overline{\mathbb{Q}}_\ell)\right) = \sum_{(\gamma_0,\delta)\in\mathscr{C}} \mathrm{vol}(F^\times \backslash F^\times_{\mathbb{A}^\infty}) O_{\gamma_0}^{\mathrm{GL}_2(\mathbb{A}^{\infty,p})}(f^p) O_\delta^{J(\mathbb{Q}_p)}(f'_p).$$

6. ℓ-adic cohomology of Igusa curves

To achieve the goal stated in §3, the final step is to extract spectral information about the cohomology from the trace formula above. We need some input from the theory of automorphic forms. There are two key ingredients, local and global, which we state without proofs but include references. We fix an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ to identify $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ coefficients of representations.

(Local) There exists a "transfer" from $f'_p \in C_c^\infty(J(\mathbb{Q}_p))$, supported on acceptable elements, to $f_p \in C_c^\infty(\mathrm{GL}_2(\mathbb{Q}_p))$ such that for every semisimple element $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$,

$$O_\gamma^{\mathrm{GL}_2(\mathbb{Q}_p)}(f_p) = \begin{cases} O_\delta^{J(\mathbb{Q}_p)}(f'_p), & \text{if } \exists\, \text{acceptable } \delta \in J(\mathbb{Q}_p) \text{ s.t. } \delta \sim \gamma, \\ 0, & \text{otherwise.} \end{cases} \tag{6.1}$$

Moreover a character identity is satisfied by f'_p and f_p:

$$\mathrm{tr}\,\pi_p(f_p) = \mathrm{tr}\,(J_{N^{\mathrm{op}}}(\pi_p) \otimes \delta^{1/2}_{P(\mathbb{Q}_p)})(f'_p), \quad \forall \pi_p : \text{irred. adm. representation of } \mathrm{GL}_2(\mathbb{Q}_p), \tag{6.2}$$

where P is the upper triangular Borel subgroup of GL_2, N^{op} is the unipotent radical of the opposite parabolic of P, $\delta_{P(\mathbb{Q}_p)}$ is the modulus character on $P(\mathbb{Q}_p)$, and $J_{N^{\mathrm{op}}}$ is the normalized Jacquet module relative to N^{op} from representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to those of $J(\mathbb{Q}_p)$. We remark that [Shi10, Lem. 3.9] proves this local fact in a more general setup. (See the proof of [KS, Lem. 3.1.2] for a small correction to the statement and proof of [Shi10, Lem. 3.9].)

(Global) There exists an Euler–Poincaré function $f_\infty \in C_c^\infty(\mathrm{GL}_2(\mathbb{R})/\mathbb{R}_{>0}^\times)$, which encodes the "weight 2 condition" for classical modular forms in the following sense: for each irreducible unitary representation π_∞ of $\mathrm{GL}_2(\mathbb{R})$ whose central character is trivial on $\mathbb{R}_{>0}^\times$,

$$\operatorname{tr}\pi_\infty(f_\infty) = \begin{cases} -1, & \text{if } \pi_\infty \text{ is the weight 2 discrete series representation,} \\ 1, & \text{if } \pi_\infty = \chi\circ\det \text{ for } \chi = \mathbf{1} \text{ or } \chi = \mathrm{sgn}, \\ 0, & \text{otherwise.} \end{cases} \tag{6.3}$$

Here we have written sgn for the sign character on $\mathbb{R}^\times$. Moreover, a simiple trace formula of the following form holds:

$$\begin{aligned} &\operatorname{tr}\left(f^p f_p f_\infty | L^2_{\mathrm{disc}}(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/\mathbb{R}_{>0}^\times)\right) \qquad (6.4)\\ = &\sum_{\substack{\gamma\in\mathrm{GL}_2(\mathbb{Q})/\sim \\ \mathbb{R}\text{-elliptic}}} (\text{volume})\cdot O_\gamma^{\mathrm{GL}_2(\mathbb{A}^\infty)}(f^p f_p) \ + \ (\text{error terms}). \end{aligned}$$

When γ is noncentral so that it generates an imaginary quadratic field F over $\mathbb{Q}$, the volume term is precisely $\mathrm{vol}(F^\times\backslash F^\times_{\mathbb{A}^\infty})$ that we saw before. (The existence of f_∞ and the simple trace formula are respectively due to Clozel–Delorme [CD85] and Arthur [Art89] in quite a general setup.)

Write $\mathcal{A}_{\mathbf{1}}(\mathrm{GL}_2)$ for the set of 1-dimensional automorphic representations π of $\mathrm{GL}_2(\mathbb{A})$ such that $\pi_\infty \in \{\mathbf{1}, \mathrm{sgn}\circ\det\}$. By $\mathcal{A}_{\mathrm{wt}\,2}(\mathrm{GL}_2)$ we denote the set of cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbb{A})$ arising from weight 2 modular forms. By $(\ )^{\mathrm{ss}}$, we mean the semisimplification of a representation with respect to the given action.

Theorem 6.1. *As* $\mathrm{GL}_2(\mathbb{A}^{\infty,p})\times J(\mathbb{Q}_p)$*-representations, we have:*

$$\begin{aligned} H^2_c(\mathrm{Ig}^{\mathrm{ord}}_\infty, \overline{\mathbb{Q}}_\ell) &= \bigoplus_{\pi\in\mathcal{A}_{\mathbf{1}}(\mathrm{GL}_2)} \pi^{\infty,p}\otimes(J_{N^{\mathrm{op}}}(\pi_p)\otimes\delta^{1/2}_{P(\mathbb{Q}_p)}), \\ H^1_c(\mathrm{Ig}^{\mathrm{ord}}_\infty, \overline{\mathbb{Q}}_\ell)^{\mathrm{ss}} &= \bigoplus_{\pi\in\mathcal{A}_{\mathrm{wt}\,2}(\mathrm{GL}_2)} \pi^{\infty,p}\otimes(J_{N^{\mathrm{op}}}(\pi_p)\otimes\delta^{1/2}_{P(\mathbb{Q}_p)}) + (\text{error terms}), \\ H^0_c(\mathrm{Ig}^{\mathrm{ord}}_\infty, \overline{\mathbb{Q}}_\ell) &= 0. \end{aligned}$$

Remark 6.2. One can think of $\delta^{1/2}_{P(\mathbb{Q}_p)}$ as "raising weight by 1". So when π_p is tempered (thus so is $J_{N^{\mathrm{op}}}(\pi_p)$), we expect $J_{N^{\mathrm{op}}}(\pi_p)\otimes\delta^{1/2}_{P(\mathbb{Q}_p)}$ to appear in H^1_c (except that H^i_c need not be pure of weight i due to non-properness). When $\dim\pi_p = 1$ (nontempered), $J_{N^{\mathrm{op}}}(\pi_p)$ is not unitary but has "weight 1", so $J_{N^{\mathrm{op}}}(\pi_p)\otimes\delta^{1/2}_{P(\mathbb{Q}_p)}$ is expected to contribute to H^2_c. To make the use of weight precise, the point is that $\mathrm{Ig}^{\mathrm{ord}}_\infty$ is defined over $\mathbb{F}_p$ (not just $\overline{\mathbb{F}}_p$) and the geometric Frobenius action coincides with the action of $(1,p)\in J(\mathbb{Q}_p)$. Notice that indeed $\delta^{1/2}_{P(\mathbb{Q}_p)}((1,p)) = p$.

Remark 6.3. The error terms in H^1_c arise from spectral interpretation of the geometric error terms (on the proper Levi subgroup $\mathrm{GL}_1\times\mathrm{GL}_1$ of GL_2) in (6.4) as can be seen in the proof below. We invite the reader to explicitly describe the error terms in H^1_c.

Remark 6.4. The top-degree cohomology with compact support classsifies the set of irreducible components; in our setup, this coincides with the set of connected components by (formal) smoothnes. We leave it to the reader to notice the following: the description of $H^2_c(\mathrm{Ig}^{\mathrm{ord}}_\infty, \overline{\mathbb{Q}}_\ell)$ implies that $\pi_0(\mathrm{Ig}^{\mathrm{ord}}_\infty)$ is in $\mathrm{GL}_2(\mathbb{A}^{\infty,p})\times J(\mathbb{Q}_p)$-equivariant bijection with $\pi_0(Y_{\mathbb{C}}) = \widehat{\mathbb{Z}}^\times$, cf. (2.1).

Proof. Since $\mathrm{Ig}^{\mathrm{ord}}_{N,m}$ is affine of dimension 1, we have $H^i_c(\mathrm{Ig}^{\mathrm{ord}}_\infty, \overline{\mathbb{Q}}_\ell) = 0$ unless $i \in \{1,2\}$. To understand

$$H^2_c(\mathrm{Ig}^{\mathrm{ord}}_\infty, \overline{\mathbb{Q}}_\ell) - H^1_c(\mathrm{Ig}^{\mathrm{ord}}_\infty, \overline{\mathbb{Q}}_\ell)$$

in the Grothendieck group of admissible $\mathrm{GL}_2(\mathbb{A}^{\infty,p}) \times J(\mathbb{Q}_p)$-representations, we rewrite the formula in Theorem 5.2 in terms of automorphic representations.

$$\begin{aligned}
& \mathrm{tr}\,\Big(f^p \times f'_p \mid H_c(\mathrm{Ig}^{\mathrm{ord}}_\infty, \overline{\mathbb{Q}}_\ell)\Big) \\
= \quad & \sum_{(\gamma_0,\delta)\in\mathscr{C}} \mathrm{vol}(F^\times \backslash F^\times_{\mathbb{A}^\infty}) O^{\mathrm{GL}_2(\mathbb{A}^{\infty,p})}_{\gamma_0}(f^p) O^{J(\mathbb{Q}_p)}_{\delta}(f'_p) \\
\overset{(6.1)}{=\!=} \quad & \sum_{\gamma_0 \in \mathrm{GL}_2(\mathbb{Q})/\sim} \mathrm{vol}(F^\times \backslash F^\times_{\mathbb{A}^\infty}) O^{\mathrm{GL}_2(\mathbb{A}^{\infty,p})}_{\gamma_0}(f^p) O^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\delta}(f_p) \\
\overset{(6.4)}{=\!=} \quad & \mathrm{tr}\,\big(f^p f_p f_\infty \mid L^2_{\mathrm{disc}}(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{R}^\times_{>0})\big) \ + \ (\text{error terms}) \\
\overset{(6.2)}{=\!=} \quad & \mathrm{tr}\,\Big(f^p f'_p f_\infty \mid J_{N^{\mathrm{op}}}\big(L^2_{\mathrm{disc}}(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{R}^\times_{>0})\big) \otimes \delta^{1/2}_{P(\mathbb{Q}_p)}\Big) \ + \ (\text{error terms}) \\
\overset{(6.3)}{=\!=} \quad & \mathrm{tr}\,\left(f^p f'_p \mid \sum_{\pi \in \mathcal{A}_{\mathbf{1}}(\mathrm{GL}_2)} \pi^{\infty,p} \otimes (J_{N^{\mathrm{op}}}(\pi_p) \otimes \delta^{1/2}_{P(\mathbb{Q}_p)})\right) \\
& - \mathrm{tr}\,\left(f^p f'_p \mid \sum_{\pi \in \mathcal{A}_{\mathrm{wt}\,2}(\mathrm{GL}_2)} \pi^{\infty,p} \otimes (J_{N^{\mathrm{op}}}(\pi_p) \otimes \delta^{1/2}_{P(\mathbb{Q}_p)})\right) \ + \ (\text{error terms})
\end{aligned}$$

At this point, we can remove the assumption that f'_p is supported on acceptable elements. Indeed, [Shi09, Lem. 6.4] tells us that the trace identity between the first and last expressions in the displayed formula above holds for all f^p and all f'_p. Therefore, we have the identity in the Grothendieck group

$$\begin{aligned}
& H^2_c(\mathrm{Ig}^{\mathrm{ord}}_\infty, \overline{\mathbb{Q}}_\ell) - H^1_c(\mathrm{Ig}^{\mathrm{ord}}_\infty, \overline{\mathbb{Q}}_\ell) \qquad (6.5) \\
= & \sum_{\pi \in \mathcal{A}_{\mathbf{1}}(\mathrm{GL}_2)} \pi^{\infty,p} \otimes (J_{N^{\mathrm{op}}}(\pi_p) \otimes \delta^{1/2}_{P(\mathbb{Q}_p)}) - \sum_{\pi \in \mathcal{A}_{\mathrm{wt}\,2}(\mathrm{GL}_2)} \pi^{\infty,p} \otimes (J_{N^{\mathrm{op}}}(\pi_p) \otimes \delta^{1/2}_{P(\mathbb{Q}_p)}) \ + \ (\text{error terms}).
\end{aligned}$$

To separate H^2_c from H^1_c, the basic idea is that there is no cancellation between H^2_c and H^1_c since the geometric Frobenius action (encoded by $(1,p) \in J(\mathbb{Q}_p)$; see Remark 6.2) has weight=2 in H^2_c and weight≤ 1 in H^1_c. There are at least a couple of ways to proceed.

(i) Show that the geometric Frobenius action has weight=2 in the first summation and <2 apart from it. (This method generalizes to study the top-degree H_c, or dually H^0, of Igusa varieties in the Hodge-type setting, as carried out in [KS].)

(ii) Verify that everything in the error terms has the negative sign. (This is harder to generalize to higher dimensions when there are many cohomological degrees.)

Either way, we obtain

$$H^2_c(\mathrm{Ig}^{\mathrm{ord}}_\infty, \overline{\mathbb{Q}}_\ell) = \sum_{\pi \in \mathcal{A}_{\mathbf{1}}(\mathrm{GL}_2)} \pi^{\infty,p} \otimes (J_{N^{\mathrm{op}}}(\pi_p) \otimes \delta^{1/2}_{P(\mathbb{Q}_p)})$$

in the Grothendieck group. Since distinct 1-dimensional representations of $\mathrm{GL}_2(\mathbb{A}^{\infty,p}) \times J(\mathbb{Q}_p)$ have no extensions between each other, we obtain the formula for H^2_c in the theorem. From this, we can compute H^1_c up to semisimplification from (6.5). □

Finally we remark that endoscopy complicates the whole computation when considering general Igusa varieties, just like endoscopy intervenes in the computation of cohomology of Shimura varieties. See [Shi20] for an illustrative account of endoscopic calculation for Igusa varieties associated with certain unitary similitude groups. Endoscopy does not show up in the present artcile only because we are restricting to the GL_2-case.

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