GEOMETRIC REDUCTIVE DUAL PAIRS AND A MOD $p$ THETA CORRESPONDENCE

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Abstract. This paper aims to recast the theory of theta correspondence in the geometric context of abelian schemes and nondegenerate line bundles, which replace the role of symplectic vector spaces, continuing our previous work. We start by formulating the notion of reductive dual pairs which fit well with our notion of Weil representations and metaplectic group functors constructed in the earlier work. When specialized to the case of base scheme $\text{Spec} \mathbb{F}_p$, our work provides an experimental framework for a mod $p$ theta correspondence for $p$-adic reductive groups. In the case of type II pairs, two results are obtained on the structure of mod $p$ Weil representations. First, the Weil representation turns out to have few quotients, which tells us that the naive analogue of the classical theta correspondence is hopeless and would demand a new approach. Second, a weak analogue of the unramified version of Howe’s conjecture still holds.

1. Introduction

1.1. Basic ingredients. In §1.1 of [Shi], we mentioned that the following are needed to formulate the classical theta correspondence. That paper studied (i), (ii) and (iii) in the context of abelian schemes and nondegenerate line bundles. Once (iii) is obtained from (i) and (ii) and realized on explicit models, it is often harmless to forget (i) and (ii).

(i) a $p$-adic Heisenberg group arising from a symplectic vector space $(V, \langle \cdot, \cdot \rangle)$ over $\mathbb{Q}_p$,

(ii) Stone Von Neumann theorem and Schur’s lemma for representations of the Heisenberg group,

(iii) the Weil representations of the $p$-adic metaplectic group $\text{Mp}(V, \langle \cdot, \cdot \rangle)$, and

(iv) reductive dual pairs in $\text{Sp}(V, \langle \cdot, \cdot \rangle)$.

Our first goal is to give a definition of (iv), in a general setting, which goes well with our earlier construction of (i), (ii) and (iii). This will get us ready for exploring the “theta correspondence” in a broader context, in particular for representations of $p$-adic reductive groups on $\mathbb{F}_p$-vector spaces.

In fact when formulating the theta correspondence, one also needs a lemma amounting to the classical fact that two elements in the metaplectic group commute if and only if their images in the symplectic group commute. In the study of type II correspondence (§§5.4-5.5) the lemma is easy to prove. Although we conjecture that the lemma is true in general, we have not been able to confirm it. (See §5.3 for detail.)

1.2. Mod $l$ and mod $p$ theta correspondences. There have been a few attempts (e.g. Minguez ([Min08b]) to extend the local theta correspondence for $p$-adic groups to $\mathbb{F}_l$-vector spaces (rather than $\mathbb{C}$-vector spaces) when $l \neq p$. In fact, the basic objects like (i)-(iv) above all carry over from $\mathbb{C}$ to $\mathbb{F}_l$ rather easily, basically by replacing $\mathbb{C}$ with $\mathbb{F}_l$ everywhere. Nevertheless, finding the correct formulation of mod $l$ theta correspondence may be a hard task. Minguez observed that already in the type II case, the naive analogue of Howe’s conjecture (§5.1) on the bijectivity of the local theta correspondence does not hold (if $l$ is not a “banal” prime). Still, one has a good starting point for investigating a suitable mod $l$ theta correspondence. By contrast, perhaps there has not been much prospect for a mod $p$ theta correspondence for $p$-adic groups as (i)-(iv) were missing to our knowledge (before [Shi]).

1.3. Reductive dual pairs. We are naturally led to build a new definition of reductive dual pairs (marked as (iv) above) which fits well into the geometric picture and coincides with the usual notion in the classical case ($S = \text{Spec} \mathbb{C}$). Recall that the usual way to construct a reductive dual pair (of type I) is as follows: a

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Hermitian space \((W_1, \langle \cdot, \cdot \rangle_1)\) and a skew-Hermitian space \((W_2, \langle \cdot, \cdot \rangle_2)\), say over \(\mathbb{Q}_p\), give rise to a symplectic space \((W_1 \otimes_{\mathbb{Q}_p} W_2, \langle \cdot, \cdot \rangle)\). Then the natural embedding
\[
\text{Aut}(W_1, \langle \cdot, \cdot \rangle_1) \times \text{Aut}(W_2, \langle \cdot, \cdot \rangle_2) \hookrightarrow \text{Sp}(W_1 \otimes W_2, \langle \cdot, \cdot \rangle)
\]
realizes \((\text{Aut}(W_1, \langle \cdot, \cdot \rangle_1), \text{Aut}(W_2, \langle \cdot, \cdot \rangle_2))\) as a reductive dual pair in the symplectic group. Our idea is basically to consider \((A_2, L_2)\) instead of \((W_2, \langle \cdot, \cdot \rangle_2)\), where \(A_2\) is an abelian scheme and \(L_2\) is a nondegenerate line bundle over \(A_2\). Or more precisely, \((W_2, \langle \cdot, \cdot \rangle_2)\) is replaced with \((V_p A_2, \hat{\epsilon}^{L_2})\), consisting of the rational \(p\)-adic Tate module of \(A\) and the \(L_2\)-Weil pairing. Once we make sense of \((A, L) := (W_1 \otimes A_2, \langle \cdot, \cdot \rangle_1 \otimes L_2)\) as an abelian scheme with a line bundle, we will be able to think of \(\text{Aut}(W_1, \langle \cdot, \cdot \rangle_1)\) and \(\text{Aut}(V_p A_2, \hat{\epsilon}^{L_2})\) as a reductive dual pair in \(\text{Sp}(V_p A, \hat{\epsilon}_p^L)\).

From the representation theoretic viewpoint, it may not be always desirable that \(\text{Sp}(V_p A, \hat{\epsilon}_p^L)\) (which is viewed as a sheaf on the base scheme \(S\) of \(A\)) varies when \(A\) moves in a family of abelian varieties, though this may be an interesting phenomenon (especially in mixed characteristic). Thus we introduce a level structure which is an isomorphism from \(V_p A\) onto a fixed object with compatible symplectic structure. In particular, it induces an isomorphism from \(\text{Sp}(V_p A, \hat{\epsilon}_p^L)\) onto a fixed symplectic group. When \(A\) is over \(\mathbb{C}\), a level structure provides a way to recover the classical notion of (i)-(iv) from our construction. When \(A\) is over \(\bar{\mathbb{F}}_p\), a level structure is an isomorphism from \(V_p A\) onto \(V_p \Sigma\) for a fixed \(p\)-divisible group \(\Sigma\). The group \(\text{Sp}(V_p A, \hat{\epsilon}_p^L)\) is now different from the usual symplectic group. (For instance, if \(A\) is ordinary then \(\text{Sp}(V_p A, \hat{\epsilon}_p^L)\) is isomorphic to a general linear group.)

In fact level structures alone are usually insufficient. To consider all possible reductive dual pairs, we need to also consider an endomorphism structure for \(A\) as well. In other words, the formalism of reductive dual pairs naturally requires a PEL structure\(^2\) on \(A\), modulo the fact that we prefer a nondegenerate line bundle to a polarization. Often the geometric literature restricts attention to ample line bundles, but it is worth pointing out that we do need to treat nondegenerate line bundles in order to deal with all reductive dual pairs. The reason is that \(L\) may not be ample even if \(L_2\) is ample, in the above notation. As an aside, we recall that Howe asked in [How79, Rem 5.(c)] whether the reductive dual pairs have something to do with the data defining PEL Shimura varieties. Our paper provides a partial answer to his question by pulling them close to each other.

This completes a geometric construction of the basic objects which are needed to consider the theta correspondence, which recovers the classical objects if we work over \(\text{Spec } \mathbb{C}\). As a consequence, we broaden the scope of the theta correspondence. When the base scheme is \(\text{Spec } \mathbb{F}_p\), this gets us ready for exploring a mod \(p\) theta correspondence. (See §1.4 below.)

1.4. A mod \(p\) theta correspondence for \(p\)-adic groups. There would be two approaches to the theta correspondence for \(\mathbb{F}_p\)-representations of \(p\)-adic groups. The first way is to realize the classical theta correspondence on \(\overline{\mathbb{Q}}_p\)-representations (whenever possible), prove that certain \(\mathbb{Z}_p\)-structure is preserved, and then take modulo \(p\). However, this may not be the most natural approach. In the context of the \(p\)-adic Langlands program, it is more natural to consider the mod \(p\) of a unitary Banach representation than the mod \(p\) of a classical admissible representation of a \(p\)-adic group. Unfortunately we do not have a \(p\)-adic Banach version of the Weil representation on a \(p\)-adic vector space, and this approach cannot be taken.

The second approach is based on the mod \(p\) Weil representation constructed in this article, which is natural from the geometric viewpoint. Although we have the notion of reductive dual pairs, the difficulty here is that it is not even clear how to formulate useful and plausible conjectures. The mod \(p\) Weil representations have few quotients, and the naïve analogue of Howe’s conjecture breaks down completely. Nevertheless, we also prove a positive result, namely that a weak analogue of the unramified theta correspondence can be shown for certain type II pairs. See §§5.4-5.5 for more detail.

It is worth recalling from [Shi] that the mod \(p\) versions of the Heisenberg group and the Heisenberg/Weil representation can be constructed from a \(p\)-divisible group with a symplectic pairing, without any use of \((A, L)\). When the \(p\)-divisible group is ordinary, the Schrodinger model exists and provides a shortcut to the Weil representation and the theta correspondence (Example 4.2), with few prerequisites. We use the ordinary Schrodinger model in §5.

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\(^2\) Polarization, E=Endomorphism, L=Level structure
1.5. Further developments and speculations. The geometric interpretation of the reductive dual pairs may be of some use in the classical theta correspondence. We mentioned some idea in [Shi, §1.5]. As for a conjectural mod \( p \) theta correspondence for \( p \)-adic groups, it is imperative to come up with a sensible statement replacing Howe’s conjecture. To do so, a first step would be to collect many computational examples. We restricted ourselves to type II pairs and ordinary \( p \)-divisible groups in §5 but the story would be even more exciting for type I pairs and supersingular \( p \)-divisible groups. Eventually the proposed mod \( p \) correspondence for \( p \)-adic groups should be compatible with the global theta correspondence for mod \( p \) automorphic forms (whenever it becomes available) and interact reasonably with the mod \( p \) Langlands program. Still not many tools are available in the mod \( p \) representations theory of \( p \)-adic groups. It would be nice if our work eventually adds another weapon in constructing and classifying representations.

1.6. Organization. The organization of the paper is simple. Section 2 recollects basic materials on Hermitian and skew-Hermitian pairings (for which [MVW87] is a comprehensive reference) and adapt them to rational Tate modules. In §3 and §4 we define geometric analogues of classical reductive dual pairs of type I and II and give examples. The last section 5 reviews Howe’s conjecture in two versions, tests the validity of their analogues in the setting of type II mod \( p \) theta correspondence, and ends with a brief speculation.

1.7. Acknowledgments. As this paper is a natural continuation of [Shi], I express my gratitude again to all the people in the acknowledgments of that paper. Part of this work was completed during my stay at the Institute for Advanced Study. I thank for its generous support.

1.8. Notation and Convention. When \( S \) is a scheme, denote by \((\text{Sch}/S)\) the category of \( S \)-schemes. Most of the time we work with schemes and ind schemes over \( S \) (often with group structure), in which case \( \times \) always means a fiber product over \( S \). Underlined notation such as \( \text{Hom}, \text{Aut} \) and \( \text{End} \) is used to denote a sheaf or a functor whereas \( \text{Hom}, \text{Aut} \) and \( \text{End} \) denote the corresponding set, group and a ring. When \( X \) is an abelian variety or a \( p \)-divisible group, we write \( \text{End}^0(X) \) for \( \text{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \) (equivalently the endomorphism ring in the category in which isogenies are inverted).

The same notation as in the previous paper [Shi] will also be used here. For the reader’s convenience, we recall some of the notation with reference points in that paper.

- \( A \) is an abelian scheme over \( S \) and \( L \) is a nondegenerate line bundle.
- \( \lambda_L : A \to A^\vee \) is the morphism given by \( x \mapsto T^*_x L \otimes L^{-1} \) on points. (§2)
- \( TA \) (resp. \( VA \)) are (resp. ind-) group scheme versions of the (resp. rational) Tate module for \( A \). (§3.1)
- \( \hat{\epsilon}_{L, \text{Weil}} : VA \times VA \to VG_m \) is the \( L \)-Weil pairing. (§3.4)
- \( \hat{\epsilon}_L : VA \times VA \to G_m \) is the commutator pairing of the Heisenberg group, satisfying \( \hat{\epsilon}_L = \hat{\epsilon}_{L, \text{Weil}} \) via the natural map \( : VG_m \to G_m \). (§3.4)
- \( \text{Mp}(VA, \hat{\epsilon}_L) \) and \( \text{Sp}(VA, \hat{\epsilon}_L) \) are metaplectic and symplectic groups, defined as group functors on \((\text{Sch}/S)\). (§5.1)

2. Hermitian and skew-Hermitian pairings

This section contains preliminaries to be used later as a reference. It is recommended that the reader skip §2 and come back as needed. Throughout this section,

- \( F_0 \) is a field,
- \( D \) is a finite dimensional division algebra over \( F_0 \), so that \( E := Z(D) \supset F_0 \),
- \(*\) is an involution on \( D \) acting as the identity on \( F_0 \),
- \( F := E^{*} = 1 \), so that \( F_0 \subset F \subset E \) and \( 1 \leq |E : F| \leq 2 \).

If \( E = F \) (resp. \( E \supset F \)) then \(*\) is an involution of the first (resp. second) kind.

2.1. Basic definitions and properties.

**Definition 2.1.** Let \( \epsilon \in \{ \pm 1 \} \). Let \( V \) (resp. \( W \)) be a left (resp. right) \( D \)-module. We say that a nondegenerate bilinear map \( \langle \cdot , \cdot \rangle : V \times V \to D \) a left \((D, \epsilon)\)-linear \( \epsilon \)-Hermitian pairing if

\[
\langle v', v \rangle = \epsilon \cdot \langle v, v' \rangle^* \quad \langle dv, d'v' \rangle = d \langle v, v' \rangle (d')^*, \quad d, d' \in D, \ v, v' \in D.
\] (2.1)
Similarly, a nondegenerate bilinear map \( \langle \cdot, \cdot \rangle : W \times W \to D \) is a right \((D, \ast)\)-linear \(\epsilon\)-Hermitian pairing if
\[
\langle w', w \rangle = \epsilon \cdot \langle w, w' \rangle^*, \quad (dw, d'w') = d^* \langle w, w' \rangle d', \quad d, d' \in D, \; w, w' \in D.
\]
Often we say Hermitian (resp. skew-Hermitian) in place of \(\epsilon\)-Hermitian if \(\epsilon = 1\) (resp. \(\epsilon = -1\)).

With the above notation, define an \(F\)-group \(\text{Aut}(V, \langle \cdot, \cdot \rangle)\) by the rule
\[
\text{Aut}(V, \langle \cdot, \cdot \rangle)(R) = \{ g \in \text{End}_R(V \otimes_F R) | \langle gv, gw' \rangle = \langle v, w' \rangle, \; \forall v, w' \in V \}
\]
for each \(F\)-algebra \(R\). An \(F\)-group \(\text{Aut}(W, \langle \cdot, \cdot \rangle)\) is defined similarly. When the context makes it clear whether \((W, \langle \cdot, \cdot \rangle)\) is unitary, symplectic or orthogonal, we will often write \(U, \text{Sp}, \text{O}\) in place of \(\text{Aut}\).

**Definition 2.2.** Let \(\epsilon, V, W\) be as above. A nondegenerate \(F_0\)-bilinear map \(\langle \cdot, \cdot \rangle : V \times V \to F_0\) is a left \((D, \ast)\)-\(\epsilon\)-Hermitian \(F_0\)-valued pairing on \(V\) if
\[
\langle v, v' \rangle = \epsilon \cdot (v, v')^*, \quad (dv, d'v') = (v, d^*v'), \quad d, d' \in D, \; v, v' \in D.
\]
A right \((D, \ast)\)-\(\epsilon\)-Hermitian \(F_0\)-valued pairing on \(W\) is defined similarly. When \(F_0\) is clear from the context, we will omit the reference to \(F_0\). (Note that different brackets are used in (2.1) and (2.2).)

Define an \(F_0\)-group
\[
\text{Aut}(V, \langle \cdot, \cdot \rangle)(R) = \{ g \in \text{End}_R(V \otimes_{F_0} R) | \langle gv, gw' \rangle = \langle v, w' \rangle, \; \forall v, w' \in V \}
\]
for \(F_0\)-algebras \(R\). Set \(\text{tr}_{D/F_0} := \text{tr}_{F/F_0} \circ \text{tr}_{D/E}\) where \(\text{tr}_{D/E}\) is the reduced trace. Let \(\text{Res}_{F/F_0}\) denote the Weil restriction of scalars. The following lemma is obvious.

**Lemma 2.3.** If \(\langle \cdot, \cdot \rangle : V \times V \to D\) (resp. \(\langle \cdot, \cdot \rangle : W \times W \to D\)) is a left (resp. right) \((D, \ast)\)-linear \(\epsilon\)-Hermitian pairing then \(\text{tr}_{D/F_0} \circ \langle \cdot, \cdot \rangle\) is a left (resp. right) \((D, \ast)\)-\(\epsilon\)-Hermitian \(F_0\)-valued pairing. Further, there is a canonical isomorphism
\[
\text{Res}_{F/F_0}(\text{Aut}(V, \langle \cdot, \cdot \rangle)) \simeq \text{Aut}(V, \text{tr}_{D/F_0} \circ \langle \cdot, \cdot \rangle).
\]

**Remark 2.4.** In fact the lemma is true if \(\text{tr}_{D/F_0}\) is replaced by any nondegenerate \(F_0\)-linear pairing \(t_{D/F_0} : D \times D \to F_0\).

**Lemma 2.5.** There is a natural bijection between the two sets consisting of
(i) left \((D, \ast)\)-linear \(\epsilon\)-Hermitian pairings \(\langle \cdot, \cdot \rangle\) on \(V\) and
(ii) left \((D, \ast)\)-\(\epsilon\)-Hermitian \(F_0\)-valued pairings \(\langle \cdot, \cdot \rangle\) on \(V\), respectively,
induced by \(\langle \cdot, \cdot \rangle \mapsto \text{tr}_{D/F_0} \circ \langle \cdot, \cdot \rangle\). Moreover the same map induces a bijection on the sets of isomorphism classes of (i) and (ii).

**Proof.** The first bijectivity is deduced from the following claim: for any given \(\langle \cdot, \cdot \rangle\), there exists a unique \(\langle \cdot, \cdot \rangle\) such that
\[
\langle v, v' \rangle = \text{tr}_{D/F_0}(\langle v, v' \rangle), \quad v, v' \in V.
\]
To prove the claim, for each pair \(v, v' \in V\) define \(\langle v, v' \rangle\) to be the unique element \(\delta \in D\) such that \(\text{tr}_{D/F_0}(\delta \delta') = (dv, dv')\) for all \(d \in D\). (Such a \(\delta\) uniquely exists since the trace pairing is nondegenerate.) Then it is a routine check that \(\langle \cdot, \cdot \rangle\) is \((D, \ast)\)-linear and \(\epsilon\)-Hermitian. To verify the uniqueness of \(\langle \cdot, \cdot \rangle\), it is enough to note that \(\text{tr}_{D/F_0} \circ \langle \cdot, \cdot \rangle = \text{tr}_{D/F_0} \circ \langle \cdot, \cdot \rangle'\) implies \(\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle'\) as the trace pairing is nondegenerate.

Let us show the bijectivity on the level of isomorphism classes. Let \(\gamma \in \text{Aut}_D(V)\). What need to be proved is the equivalence that
\[
\text{tr}_{D/F_0}(\gamma v, \gamma v') = \text{tr}_{D/F_0}(v, v'), \quad \forall v, v' \in V \quad \Leftrightarrow \quad \langle \gamma v, \gamma v' \rangle = \langle v, v' \rangle \; \forall v, v' \in V.
\]
The implication \(\Leftarrow\) is obvious. To see \(\Rightarrow\), substitute \(dv\) for \(v\) to obtain
\[
\text{tr}_{D/F_0}(d(\gamma v, \gamma v')) = \text{tr}_{D/F_0}(dv, dv'), \quad \forall v, v' \in V, \; \forall d \in D.
\]
The latter implies \(\langle \gamma v, \gamma v' \rangle = \langle v, v' \rangle\) as the trace pairing is nondegenerate. \(\square\)

So far \(F_0\) has been any field. Let us put ourselves in the adelic situation in the case of \(F_0 = \mathbb{Q}\).

**Lemma 2.6.** There is a bijection between the two sets consisting of
(i) left \((D \otimes \mathbb{Q} \mathcal{A}_\infty, \ast)\)-linear \(\epsilon\)-Hermitian pairings \(\langle \cdot, \cdot \rangle\) on \(V \otimes \mathbb{Q} \mathcal{A}_\infty\) and
(ii) left \((D \otimes \overline{\mathbb{Q}} \cdot A^\infty, \star)_\epsilon\)-Hermitian \(A^\infty\)-valued pairings \((\cdot, \cdot)\) on \(V \otimes \overline{\mathbb{Q}} A^\infty\), respectively, induced by \((\cdot, \cdot) \mapsto \text{tr}_{D/k} \circ (\cdot, \cdot)\). The same map induces a bijection on the sets of isomorphism classes.

Proof. The method of proof is the same as for Lemma 2.5. \(\square\)

2.2. Hermitian pairings on \(V A\). Let \(F_0 = \overline{\mathbb{Q}}\) and \((D, \star)\) be as before. Let \(A\) be an abelian scheme over \(S\) equipped with a \(D\)-action via a \(\overline{\mathbb{Q}}\)-algebra morphism \(i : D \hookrightarrow \text{End}_{\overline{\mathbb{Q}}}(A)\). Let \(D \otimes \overline{\mathbb{Q}} V G_m\) denote the ind-group scheme over \(\overline{\mathbb{Q}}\) isomorphic as a group functor to the ind-scheme \(\prod_{i \in I} V G_m\) equipped with group law

\[
\prod_{j \in J} V G_m(T) \times \prod_{k \in K} V G_m(T) \to \prod_{i \in I} V G_m(T)
\]

\[
((\gamma_j)_{j \in J}, (\gamma_k)_{k \in K}) \mapsto \left( \sum_{j,k} (\gamma_j \gamma_k)^{a_{ijk}} \right)_{i \in I}.
\]

Note that \(V G_m(T)\) is a \(\mathbb{Q}\)-module (even an \(A^\infty\)-module), thus \((\gamma_j \gamma_k)^{a_{ijk}}\) makes sense. There is an obvious trace map \(\text{tr}_{D/k} : D \otimes \overline{\mathbb{Q}} V G_m \to V G_m\) given by

\[
(D \otimes \overline{\mathbb{Q}} V G_m)(T) = D \otimes \overline{\mathbb{Q}} V G_m(T) \xrightarrow{\text{tr}_{D/k} \otimes \text{id}} \overline{\mathbb{Q}} \otimes \overline{\mathbb{Q}} V G_m(T) = V G_m(T).
\]

Let

\[
\tilde{e} : V A \times V A \to D \otimes \overline{\mathbb{Q}} V G_m \quad \text{and} \quad e : V A \times V A \to V G_m
\]

be morphisms of \(S\)-group schemes which are bilinear in both arguments and nondegenerate in the obvious sense. We say that \(\tilde{e}\) is a \((D, \star)\)-linear \(\epsilon\)-Hermitian pairing if (2.1) holds on scheme-valued points of \(V A\) (with \((\cdot, \cdot)\) replaced by \(\tilde{e}\)). Similarly \(e\) is said to be a \((D, \star)\)-\(\epsilon\)-Hermitian pairing if (2.2) holds for \(e\). (In our convention the \(D\)-action on an abelian scheme is always a left action, so the word “left” will be omitted.)

**Lemma 2.7.** There is a bijection between the two sets consisting of isomorphism classes of

(i) \((D, \star)\)-linear \(\epsilon\)-Hermitian pairings \(\tilde{e} : V A \times V A \to D \otimes \overline{\mathbb{Q}} V G_m\) and

(ii) \((D, \star)\)-\(\epsilon\)-Hermitian pairings \(e : V A \times V A \to V G_m\), respectively, induced by \(\tilde{e} \mapsto \text{tr}_{D/k} \circ \tilde{e}\).

Proof. The proof is omitted as it is an adaptation of the proof of Lemma 2.5 to the scheme-theoretic setting and the essential idea is unchanged. \(\square\)

2.3. Hermitian pairings on \(V pA\). Now consider the \(p\)-adic case where \(F_0 = \mathbb{Q}_p\). Thus \(D\) is a finite dimensional division \(\mathbb{Q}_p\)-algebra. We define \(D \otimes \mathbb{Q}_p V p G_m\) and the trace map \(\text{tr}_{D/p} : D \otimes \mathbb{Q}_p V p G_m \to V p G_m\) analogously as before.

**Lemma 2.8.** There is a bijection between the two sets consisting of isomorphism classes of

(i) \((D, \star)\)-linear \(\epsilon\)-Hermitian pairings \(\tilde{e} : V pA \times V pA \to D \otimes \mathbb{Q}_p V p G_m\) and

(ii) \((D, \star)\)-\(\epsilon\)-Hermitian pairings \(e : V pA \times V pA \to V p G_m\), respectively, induced by \(\tilde{e} \mapsto \text{tr}_{D/p} \circ \tilde{e}\).

Proof. Essentially the same as the proof of Lemma 2.7. \(\square\)

3. Reductive dual pairs of Type I

To achieve our goal of formulating a candidate theta correspondence, we need to bring classical reductive dual pairs into our context. This will take up sections 3 and 4. Rather than pursuing an abstract definition, we give an explicit construction of reductive dual pairs. When the base \(S\) is \(\text{Spec} \mathbb{C}\) we essentially recover the classical dual pairs. In sections 3 and 4 we always assume that \(W_1, W_2\) are nonzero \(D\)-modules and that \(A_2\) has positive dimension unless specified otherwise.
3.1. Classical reductive dual pairs of type I. As the $p$-adic case is similar, we only recall the number field case. Let
- $F_0$ be a finite extension field of $\mathbb{Q}$,
- $D, *, E, F$ be as at the start of §2,
- $W$ be a finite dimensional $F_0$-vector space of even dimension,
- $\langle \cdot, \cdot \rangle : W \times W \to F_0$ be a non-degenerate alternating $F_0$-linear pairing.

Recall ([How79, §5], [MVW87, 1.19-1.20]) that a type I reductive dual pair arises from the following data.
- a right $D$-module $W_1$ and a left $D$-module $W_2$,
- a right $(D, \ast)$-linear Hermitian pairing $\langle \cdot, \cdot \rangle_1 : W_1 \times W_1 \to D$,
- a left $(D, \ast)$-linear skew-Hermitian pairing $\langle \cdot, \cdot \rangle_2 : W_2 \times W_2 \to D$

such that there is an $F_0$-vector space isomorphism $W_1 \otimes_D W_2 \cong W$ under which we have
$$\langle w_1 \otimes w_2, w'_1 \otimes w'_2 \rangle = \text{tr}_{D/F_0}(\langle w_1, w'_1 \rangle_1^* \langle w_2, w'_2 \rangle_2) \quad (3.1)$$
for all $w_1, w'_1 \in W_1$ and $w_2, w'_2 \in W_2$. For $i = 1, 2$, define $G_i := \text{Aut}_D(W_i, \langle \cdot, \cdot \rangle_i)$. The groups $G_1$ and $G_2$ form a reductive dual pair in $\text{Sp}_w$ for all $w$.

3.2. Setup for a geometric analogue. Consider the following data.
- $S$ is a locally noetherian scheme,
- a finite dimensional division algebra $D$ with involution $\ast$ whose center is finite over $\mathbb{Q}$,
- $F_0$ is a field contained in $D^{\ast} = 1$,
- a pair $(W_1, \langle \cdot, \cdot \rangle_1)$ where
  - $W_1$ is a finite right $D$-module,
  - $\langle \cdot, \cdot \rangle_1 : W_1 \times W_1 \to D$ is a $(D, \ast)$-linear Hermitian pairing and
- a triple $(A_2, L_2, \iota_2)$ where
  - $A_2$ is an abelian variety over $S$,
  - $L_2$ is a nondegenerate symmetric line bundle over $A_2$ of index $\iota_2(L_2)$,
  - $\iota_2 : D \hookrightarrow \text{End}^0(A_2)$ such that $\iota_2(d) = \iota_2(d^\ast)$ for all $d \in D$, where $\iota_L : \text{End}^0(A_2)$ is the map $\phi \mapsto \iota_L^{-1} \phi \iota_L$.

The aim of §3.2 is to construct $(A, L)$ with respect to which $(W_1, \langle \cdot, \cdot \rangle_1)$ and $(A_2, L_2, \iota_2)$ define a reductive dual pair in a suitable sense, in analogy with §3.1.

Remark 3.1. Note that the triple $(A_2, L_2, \iota_2)$ is almost the datum in a moduli problem for abelian varieties (cf. [Mumf76, §9]). However we assume neither that $\ast$ is positive nor that $L_2$ is ample.

Lemma 3.2. $\hat{\epsilon}^{L_2, \text{Weil}} : VA_2 \times VA_2 \to V G_m$ is a $(D \otimes \mathbb{A}^\infty, \ast)$-skew-Hermitian pairing.

Proof. It is a standard fact that $\hat{\epsilon}^{L_2, \text{Weil}}$ is $\mathbb{A}^\infty$-linear and alternating. So it is enough to verify that
$$\hat{\epsilon}^{L_2, \text{Weil}}(\iota_2(d)v, v') = \hat{\epsilon}^{L_2, \text{Weil}}(v, \iota_2(d^\ast)v'), \quad \forall d \in D, \ v, v' \in VA_2(T).$$

Let $e : VA \times VA^\vee \to VG_m$ denote the canonical pairing. Then $\hat{\epsilon}^{L_2, \text{Weil}}(v, v') = e(v, \lambda_{L_2}v')$. Thus
$$\hat{\epsilon}^{L_2, \text{Weil}}(v, \iota_2(d^\ast)v') = e(v, \lambda_{L_2}(\lambda_{L_2}^{-1}(\iota_2(d))(\lambda_{L_2}v')) = e(v, \iota_2(d)^\ast \lambda_{L_2}v') = e(\iota_2(d)v, \lambda_{L_2}v') = \hat{\epsilon}^{L_2, \text{Weil}}(\iota_2(d)v, v').$$

Corollary 3.3. There exists a $(D \otimes \mathbb{A}^\infty, \ast)$-linear skew-Hermitian pairing
$$\hat{\epsilon}^{L_2, \text{Weil}} : VA_2 \times VA_2 \to D \otimes \mathbb{Q} VG_m$$

such that $\hat{\epsilon}^{L_2, \text{Weil}} = \text{tr}_{D/\mathbb{Q}} \circ \hat{\epsilon}^{L_2, \text{Weil}}$.

Proof. Immediate from the last lemma and Lemma 2.7. 

In order to proceed we make the following hypothesis on $(W_1, \langle \cdot, \cdot \rangle_1)$ from now on, which seems to be harmless in practice. (For instance the case of standard orthogonal pairing or hermitian pairing of signature $(p, q)$ can be captured, and $L$ has a simple form in this case as in (3.3) below.)
**Hypothesis 3.4.** There are a $D$-basis $\{\epsilon_i\}_{i \in I}$ of $W_1$ and $\{\alpha_{i,j} \in \mathbb{Q}|i, j \in I\}$, $\{\beta_{i,j} \in D|i, j \in I\}$ such that $W_1 = \oplus_{i \in I} \epsilon_i \cdot D$ and $\langle \epsilon_i, \epsilon_j \rangle_1 = \alpha_{i,j} \delta_{i,j}$.

Since $\langle \epsilon_i, \epsilon_j \rangle_1 = \langle \epsilon_j, \epsilon_i \rangle^*_1$, we may and will arrange that $\alpha_{i,j} = \alpha_{j,i}$ and $\beta_{i,j} = \beta_{j,i}$. By scaling $\{\epsilon_i\}_{i \in I}$ we may and will assume that $\alpha_{i,j} \in \mathbb{Z}$ and $\beta_{i,j} \in \mathcal{O}_D$.

We are ready to construct $(A, L)$. Set

$$
A := \prod_{i \in I} A_2
$$

$$
L := \bigotimes_{i \in I} \left( p_i^* t_2(\beta_{i,i})^* t_2(\beta_{i,i})^* L_2^{-\alpha_{i,i}} \right) \otimes \bigotimes_{i < j} (p_i, p_j)^* t_2(\beta_{i,j})^* L_2^{\alpha_{i,j}}
$$

(3.2)

We define $(\cdot, \cdot)^*_1 \in \mathbb{L}^{\text{Well}} : VA \times VA = (\prod_{i \in I} V A_2) \times (\prod_{i \in I} V A_2) \rightarrow D \otimes V G_m$ by

$$(v_i)_{i \in I}, (v'_j)_{j \in I} \mapsto \prod_{i,j \in I} \langle \epsilon_i, \epsilon_j \rangle_1^* \cdot \mathbb{L}^{\text{Well}}(v_i, v'_j)$$

where $v_i, v'_j \in VA_2(T)$ for each $S$-scheme $T$. In fact there is no need for $*$ in the above formula as $\langle \epsilon_i, \epsilon_j \rangle_1^* = \langle \epsilon_i, \epsilon_j \rangle_1$. Nevertheless we keep $*$ in order to remember the correct $D$-linear action. (cf. (3.1))

**Lemma 3.6.** $\mathbb{L}^{\text{Well}} = \text{tr}_{D/\mathbb{Q}}(\langle \cdot, \cdot \rangle_1 \otimes \mathbb{L}^{\text{Well}})$.

**Proof.** Let $v = (v_i)_{i \in I}$, $v' = (v'_j)_{j \in I}$ be in $VA(T)$. Then

$$
\mathbb{L}^{\text{Well}}(v, v') = \prod_{i \in I} \left( \mathbb{L}^{\text{Well}}(t_2(\beta_{i,i})(v_i), t_2(\beta_{i,i})(v'_j))_{a_{i,j}} \prod_{j \neq i} \mathbb{L}^{\text{Well}}(t_2(\beta_{i,j})(v_i), t_2(\beta_{i,j})(v'_j))_{a_{i,j}} \right)
$$

$$
\times \prod_{i < j} \mathbb{L}^{\text{Well}}(t_2(\beta_{i,j})(v_i + v_j), t_2(\beta_{i,j})(v'_j))_{a_{i,j}}
$$

$$
= \prod_{i,j \in I} \text{tr}_{D/\mathbb{Q}} \left( \mathbb{L}^{\text{Well}}(t_2(\beta_{i,j})(v_i), t_2(\beta_{i,j})(v'_j))_{a_{i,j}} \right)
$$

$$
= \prod_{i,j \in I} \text{tr}_{D/\mathbb{Q}} \left( \langle \alpha_{i,j}, \beta_{i,j}^*, \beta_{i,j} \rangle \cdot \mathbb{L}^{\text{Well}}(v_i, v'_j) \right)
$$

$$
= \text{tr}_{D/\mathbb{Q}} \left( \prod_{i,j \in I} \langle \epsilon_i, \epsilon_j \rangle_1 \cdot \mathbb{L}^{\text{Well}}(v_i, v'_j) \right)
$$

$\Box$
Remark 3.7. The definition of $L$ certainly depends on the choice of basis $\{e_i\}_{i \in I}$, and possibly $\alpha_{i,j}$ and $\beta_{i,j}$ as well. The isogeny class of $(A, L, \iota)$ should be independent of such choices but we do not check it here.

Remark 3.8. It would have been more natural to have defined $A$ as the $S$-group scheme representing $T \mapsto \mathcal{O}_D \otimes_{\mathbb{Z}} A_2(T)$. We did not take this approach because we have not found a natural definition of $L$ to go with such a definition of $A$.

**Lemma 3.9.** $L$ is symmetric and nondegenerate.

**Proof.** We have $(-1)^s L \simeq L$ because the multiplication by $-1$ commutes with $m$ and $p_i$'s. To see $L$ is nondegenerate, it suffices to show that $L_s$ is nondegenerate for each $s \in S$, so the proof is reduced to the case where $S = \text{Spec} \, k$. We may even assume that $k = k$. It is enough to verify that $\hat{\mathcal{L}}_L,\text{Weil}$ is a nondegenerate pairing for a prime $l$, which we choose to be prime to $\text{char}(k)$. Note that $\hat{\mathcal{L}}_L$ is a symplectic form on the constant group scheme $V$, which can be thought of as a symplectic $\mathbb{Q}_l$-vector space. Since $\langle \cdot, \cdot \rangle_1$ and $\hat{\mathcal{L}}_L,\text{Weil}$ are nondegenerate by the initial assumption, Lemma 3.6 implies that $\hat{\mathcal{L}},\text{Weil}$, in particular $\hat{\mathcal{L}},\text{Weil}$, is nondegenerate.

**Remark 3.10.** Even if $L_2$ is ample, $L$ may not be ample. To see this, suppose that $S = \text{Spec} \, \mathbb{C}$, that $D = F_0 = \mathbb{Q}$, and that $(W_1, \langle \cdot, \cdot \rangle_1)$ is an orthogonal space. In the notation of [Mum74, §2], write $L_2 = L(H_2, \alpha_2)$ and $L = L(H, \alpha)$ for nondegenerate hermitian forms $H_2$ and $H$. They are related via $H = \langle \cdot, \cdot \rangle_1 \otimes H_2$. If $L_2$ is ample then $H_2$ is positive definite. However, unless $\langle \cdot, \cdot \rangle_1$ is positive definite (for instance unless $I_2$ is empty in Example 3.5), $H$ is not positive definite and thus $L$ is not ample. This remark explains one major reason why we did not restrict to ample line bundles in developing the theory of Heisenberg groups and representations in [Shi].

Keeping Remark 3.8 in mind, it is clear how to define an action of $\alpha \in \text{End}_D(W_1)$ as a $\mathbb{Q}$-isogeny on $A$. For each $i \in I$, write $\alpha(e_i) = \sum_{j \in I} \epsilon_j \cdot d_{ji}$ for some $d_{ji} \in D$. Then $\alpha$ acts on $A = \prod_{i \in I} A_2$ by

\[
(x_i)_{i \in I} \mapsto \left( \prod_{j \in I} \iota_2(d_{ji})(x_j) \right)_{i \in I}.
\]

(3.4)

On the other hand, each $\beta \in \text{End}(VA_2)$ acts on $VA = \prod_{i \in I} VA_2$ by diagonal action. This induces a map of ring-valued functors on $(\text{Sch}/S)$

\[
\text{End}_D(W_1) \times \text{End}(VA_2) \to \text{End}(VA).
\]

(As usual, $\text{End}_D(W_1)$ is viewed as the constant ring scheme over $S$ associated with $\text{End}_D(W_1)$.) By restricting to the automorphisms preserving pairings, we obtain the following lemma, whose proof is not difficult.

**Lemma 3.11.** The map (3.5) induces a map of group functors on $(\text{Sch}/S)$

\[
\text{Aut}_D(W_1, \langle \cdot, \cdot \rangle_1) \times \text{Aut}(VA_2, \hat{\mathcal{L}}, \iota, \iota_2) \to \text{Sp}(VA, \mathcal{L}, \iota) \to \text{Sp}(VA, \mathcal{L})
\]

which is an injection on $\text{Aut}_D(W_1, \langle \cdot, \cdot \rangle_1) \times \{1\}$ and $\{1\} \times \text{Aut}(VA_2, \hat{\mathcal{L}}, \iota, \iota_2)$.

**Proof.** The injectivity is straightforward. In the rest of the proof we check that the image lands in $\text{Sp}(VA, \mathcal{L}, \iota)$ (not just in $\text{End}(VA)$). Let $\beta \in \text{Aut}(VA_2, \hat{\mathcal{L}}, \iota, \iota_2)$, $v, v' \in VA$ and write $v = (v_i)_{i \in I}$ and $v' = (v'_j)_{j \in I}$. Then

\[
\hat{\mathcal{L}},\text{Weil}(\beta(v), \beta(v')) = \text{tr}_{D/\mathbb{Q}} \left( \prod_{i,j} \langle \epsilon_i, \epsilon_j \rangle_1 \cdot \hat{\mathcal{L}},\text{Weil}((\beta(v_i))_{i \in I}, (\beta(v'_j))_{j \in I}) \right)
\]

\[
= \text{tr}_{D/\mathbb{Q}} \left( \prod_{i,j} \langle \epsilon_i, \epsilon_j \rangle_1 \cdot \hat{\mathcal{L}},\text{Weil}((v_i)_{i \in I}, (v'_j)_{j \in I}) \right) = \hat{\mathcal{L}},\text{Weil}(v, v').
\]
Now consider $\alpha \in \text{End}_D(W_1)$ and write $\alpha(\epsilon_i) = \sum_{j \in I} \epsilon_j \cdot d_{ji}$ as above. Let $k, l \in I$. It is enough to consider “basis elements” $v, v'$ with $v_i = 0$ and $v'_i = 0$ for all $i \neq k$ and $j \neq l$. Then $\tilde{c}^L, \text{Weil}(\alpha(v), \alpha(v'))$ equals

$$\text{tr}_{D/Q} \left( \prod_{i,j} (\epsilon_i, \epsilon_j) \cdot \tilde{c}^L,\text{Weil}(d_{ik} v_k, d_{jl} v'_l) \right) = \text{tr}_{D/Q} \left( \prod_{i,j} (\epsilon_i, \epsilon_j) \cdot d_{ik} \tilde{c}^L,\text{Weil}(v_k, v'_l) \right)$$

$$= \text{tr}_{D/Q} \left( \prod_{i,j} (\epsilon_i, \epsilon_j) \cdot \tilde{c}^L,\text{Weil}(v_k, v'_l) \right) = \text{tr}_{D/Q} \left( \langle \alpha(\epsilon_k), \alpha(\epsilon_l) \rangle \cdot \tilde{c}^L,\text{Weil}(v_k, v'_l) \right)$$

$$= \text{tr}_{D/Q} \left( \langle \epsilon_k, \epsilon_l \rangle \cdot \tilde{c}^L,\text{Weil}(v_k, v'_l) \right) = \tilde{c}^L,\text{Weil}(v, v').$$

3.3. Level structure. We wish to view Lemma 3.11 as presenting the analogue of a classical reductive dual pair. This is more than an analogy when there is a level structure, as treated in [Shi, §6] from which we import notation. To begin with, assume that $S$ is a $\mathbb{Q}$-scheme. A nontrivial morphism of group schemes $\psi : \mathbb{A}^\infty \to \mathbb{G}_m$ over $S$ (where $\mathbb{A}^\infty$ is a constant group scheme) plays the role of additive character. Let $\langle \cdot, \cdot \rangle_{2,\psi}$ denote the pairing $W_2 \otimes \mathbb{A}^\infty \times W_2 \otimes \mathbb{A}^\infty \to \mathbb{G}_m$ obtained from $\langle \cdot, \cdot \rangle_2$ and $\psi$ in the obvious manner. Similarly $\langle \cdot, \cdot \rangle : V \otimes \mathbb{A}^\infty \times V \otimes \mathbb{A}^\infty \to \mathbb{G}_m$ is defined. Let

$$\alpha_2 : (W_2 \otimes \mathbb{A}^\infty)_S \simeq VA_2$$

be a $D \otimes \mathbb{A}^\infty$-linear isomorphism (of ind-group schemes over $S$) compatible with $(D \otimes \mathbb{A}^\infty, \ast)$-skew Hermitian pairings $\langle \cdot, \cdot \rangle_{2,\psi}$ and $\tilde{c}^L$. As we have $VA_2 \simeq W_1 \otimes_D VA_2$ by construction, the isomorphism $\alpha_2$ induces

$$\alpha : VA \simeq (W_1 \otimes_D W_2) \otimes \mathbb{A}^\infty$$

matching $\tilde{c}^L$ and $\langle \cdot, \cdot \rangle_\psi$. Then $\alpha_2$ and $\alpha$ induce

\[
\begin{align*}
\text{Aut}(VA_2, \tilde{c}^L, \iota_2) & \simeq \text{Aut}_D(W_2 \otimes \mathbb{A}^\infty, \langle \cdot, \cdot \rangle_{2})_S = G_2 \times F_0 S \\
\text{Aut}(VA, \tilde{c}^L, \iota) & \simeq \text{Sp}_{F_0}(W \otimes \mathbb{A}^\infty, \langle \cdot, \cdot \rangle)_S = G \times F_0 S.
\end{align*}
\]

Using the above isomorphisms along with Lemma 3.11, one precisely recovers the classical reductive dual pair of §3.1. Now when $S$ is an $\mathbb{F}_p$-scheme, the level structure in the second case of [Shi, §6.5] can be likewise adapted to produce a “dual pair”, which is classical outside $p$ but looks different at the $p$-components.

3.4. The $p$-adic case. Only in §3.4 (and in §4.3) we change the notation from §3.2. Let $F_0$ be a finite extension of $\mathbb{Q}_p$. Let $D$ be a division $\mathbb{Q}_p$-algebra with involution $\ast$ such that $F_0 \subset D^{\ast =1}$ and $\dim_{\mathbb{Q}_p} D < \infty$. Let $W_1$ be a finite right $D$-module, and let $\langle \cdot, \cdot \rangle_1 : W_1 \times W_1 \to D$ be a $(D, \ast)$-Hermitian pairing. Let $(A_2, L_2, \iota_2)$ be as in §3.2, except that we take $\iota_2 : D \to \text{End}^0(A_2) \otimes \mathbb{Q}_p$. The contents of the previous subsection carries over with obvious changes. For instance, the analogue of Lemma 3.11 produces a dual pair

$$\text{Aut}_D(W_1, \langle \cdot, \cdot \rangle_1) \times \text{Aut}(VA_2, \tilde{c}^L, \iota_2) \to \text{Sp}(V_p A, \tilde{c}^L, \iota) \mapsto \text{Sp}(V_p A, \tilde{c}^L_p)$$

which is injective on $\text{Aut}_D(W_1, \langle \cdot, \cdot \rangle_1) \times \{1\}$ and $\{1\} \times \text{Aut}(VA_2, \tilde{c}^L, \iota_2)$.

When $S = \text{Spec} \overline{\mathbb{F}}_p$ (and $p \neq 2$ for safety), we can formulate everything also in terms of $p$-divisible groups using [Shi, §6.4]. As this is not part of the classical theory, we provide some examples in this unfamiliar territory. Let $D_{1/2}$ be a quaternion division algebra with center $\mathbb{Q}_p$ and $\Sigma_{1/2}$ a simple $p$-divisible group over $\overline{\mathbb{F}}_p$ with slope 1/2. It is well known that $\text{End}^0(\Sigma_{1/2}) \simeq D_{1/2}$.

Example 3.12. Let $S = \text{Spec} \overline{\mathbb{F}}_p$, $(W_1, \langle \cdot, \cdot \rangle_1)$ be an orthogonal $\mathbb{Q}_p$-vector space of dimension $g_1$ and $\Sigma_2 = (\Sigma_{1/2})^{g_2}$. Then the resulting dual pair may be identified with

$$O(W_1, \langle \cdot, \cdot \rangle_1) \times \text{Sp}_{g_2}(D_{1/2}) \to \text{Sp}_{g_1 g_2}(D_{1/2}).$$
**Example 3.13.** Let $S = \text{Spec} \overline{\mathbb{F}}_p$ and $E$ be a quadratic extension of $\mathbb{Q}_p$ with $1 \neq \ast \in \text{Gal}(E/\mathbb{Q}_p)$. Let $(W_1, \langle \cdot, \cdot \rangle_1)$ be a Hermitian $E$-vector space of dimension $g_1$. Let $\Sigma_2 = (\Sigma_{1/2})^{g_2}$ be equipped with $\iota_2 : E \hookrightarrow \text{End}^0(\Sigma_2)$ and a $\mathbb{Q}_p$-linear nondegenerate pairing $\langle \cdot, \cdot \rangle_0 : \Sigma_2 \times \Sigma_2 \rightarrow \mathbb{Q}_p$ such that $\langle ex, y \rangle_2 = \langle x, e^* y \rangle_2$ for all $e \in E$ and $x, y \in \Sigma_2$. Let $\langle \cdot, \cdot \rangle$ be the pairing on $V_p\Sigma_2$ obtained from $\langle \cdot, \cdot \rangle_0$ as in [Shi, §6.3]. This produces a dual pair

$$U(W_1, \langle \cdot, \cdot \rangle_1) \times U(V_p\Sigma_2, \langle \cdot, \cdot \rangle, \iota_2) \rightarrow \text{Sp}_{p,94}(D_{1/2}),$$

where $U(V_p\Sigma_2, \langle \cdot, \cdot \rangle, \iota_2)$ is an inner form of the quasi-split unitary group in $g_2$ variables over $\mathbb{Q}_p$.

4. **Reductive dual pairs of type II**

4.1. **Classical reductive dual pairs of type II.** We only recall the number field case as the $p$-adic case is completely analogous. Let

- $F_0$ be a finite extension field of $\mathbb{Q}$,
- $W$ be a finite dimensional $F_0$-vector space of even dimension,
- $\langle \cdot, \cdot \rangle : W \times W \rightarrow F_0$ be a non-degenerate alternating $F_0$-linear pairing.

Recall ([How79, §5], [MWW87, 1.19-1.20]) that a type II dual reductive pair arises from the following data.

- a division algebra $D$ whose center $E$ contains $F_0$,
- a right $D$-module $W_1$ and a left $D$-module $W_2$, such that $W_1 \otimes_D W_2$ embeds into $W$ as a maximal isotropic subspace for $\langle \cdot, \cdot \rangle$. For $i = 1, 2$, define $G_i := GL_D(W_i)$. (Namely $G_i(R) = GL_{D\otimes_{F_0} R}(W_i \otimes_{F_0} R)$ for any $F_0$-algebra $R$.)

The two groups $G_1$ and $G_2$ form a reductive dual pair of type II in $\text{Sp}_{F_0}(W, \langle \cdot, \cdot \rangle)$.

4.2. **Setup for a geometric analogue.** Consider the data at the start of §3.2. We further assume that

- $\ast$ is an involution of the first kind,
- $W_1 = \oplus_{i=1}^m \epsilon_i : D$ is a finite right $D$-module,
- an orthogonal pairing $\langle \cdot, \cdot \rangle_1$ is chosen for $W_1$ such that Hypothesis 3.4 is satisfied,
- there is a complete polarization $V_A \cong V_2 \times V_2''$ with respect to $\tilde{\epsilon}^{L,2}$ (so that $V_2$ and $V_2''$ are isotropic) such that $V_2'$ and $V_2''$ are stable under the action of $\iota_2(D)$.

Construct $(A, L, \iota)$ exactly as in §3.2. Then we have $VA \cong W_1 \otimes_D V_A$, where $\tilde{\epsilon}^{L, \text{Weil}}$ is matched with $\text{tr}_{D/\mathbb{Q}}(\langle \cdot, \cdot \rangle_1 \otimes \tilde{\epsilon}^{L, \text{Weil}})$. As we explained in that subsection, there is a map (3.5). On the other hand, there is a map $\text{Aut}(V_2', \iota_2) \rightarrow \text{Aut}(V_2'', \iota_2)$ given by $\alpha \mapsto (\alpha, \alpha')$, where $\alpha'$ comes from the duality between $V_2'$ and $V_2''$ via $\tilde{\epsilon}^{L}$. Composing with $\text{Aut}(V_2', \iota_2) \rightarrow \text{Aut}(V_2'', \iota_2) \rightarrow \text{Aut}(V_A, \iota_2)$, we obtain $\text{Aut}(V_2', \iota_2) \rightarrow \text{Aut}(V_A, \iota_2)$. Moreover, the image lies in $\text{Aut}(V_A, \tilde{\epsilon}^{L,2}, \iota_2)$. Together with (3.5), we have

$$\text{Aut}_D(W_1) \times \text{Aut}(V_2', \iota_2) \rightarrow \text{Aut}_D(W_1) \times \text{Aut}(V_A, \tilde{\epsilon}^{L,2}, \iota_2) \rightarrow \text{Aut}(V_A, \tilde{\epsilon}^{L})$$

**Lemma 4.1.** The map (4.1) induces a map of group functors on $(\text{Sch}/S)$

$$\text{Aut}_D(W_1) \times \text{Aut}(V_2', \iota_2) \rightarrow \text{Sp}(VA, \tilde{\epsilon}^{L}, \iota) \rightarrow \text{Sp}(VA, \tilde{\epsilon}^{L})$$

injective on $\text{Aut}_D(W_1) \times \{1\}$ and $\{1\} \times \text{Aut}(V_2', \iota_2)$.

**Proof.** The essentially same argument as in the proof of Lemma 3.11 works here. \hfill \square

4.3. **The $p$-adic case.** Let $F_0$, $F$ and $D$ be as in §3.4. Just like we adapted §3.2 to §3.4, we can rework §4.2 in the $p$-adic case. When $S = \text{Spec} \overline{\mathbb{F}}_p$ ($p \neq 2$), things can be reformulated in terms of $p$-divisible groups. We would like to elaborate on this point, as the special case of ordinary $p$-divisible groups will be investigated further in §§5.4-5.5. (Similarly the situation of type I pairs can be recast in terms of $p$-divisible groups.)

- $W_1 = \oplus_{i=1}^m \epsilon_i : D$ is a finite right $D$-module,
- $(\Sigma_2', \iota_2')$ is a $p$-divisible group over $\overline{\mathbb{F}}_p$ with a $\mathbb{Q}_p$-algebra map $\iota_2' : D \hookrightarrow \text{End}^0(\Sigma_2')$.
- $\Sigma^\vee = (\Sigma_2')^\vee$, $\Sigma := (\Sigma^\vee)^\vee \times \Sigma'$.
- $\iota' : D \hookrightarrow \text{End}^0(\Sigma')$ is induced by the diagonal action via $\iota_2'$.
- $\iota : D \hookrightarrow \text{End}^0(\Sigma)$ is the composition of $\iota'$ with $\text{End}^0(\Sigma^\vee) \hookrightarrow \text{End}^0(\Sigma)$ given by $\beta \mapsto (\beta, \beta')$. 

Let \( \alpha \in \text{Aut}_D(W_1) \). For \( 1 \leq i \leq g_1 \), \( \alpha_i = \sum_{j=1}^{g_i} \epsilon_j \cdot d_{ji} \) for some \( d_{ji} \in D \). Make \( \alpha \) act on \( \Sigma' \) in the isogeny category of \( p \)-divisible groups by the same formula as (3.4). This induces \( \text{Aut}_D(W_1) \rightarrow \text{Aut}_D^0(\Sigma', i') \simeq \text{Aut}(V_p\Sigma', i') \). Together with the diagonal action \( \text{Aut}(V_p\Sigma', i^0) \simeq \text{Aut}(V_p\Sigma', i') \), this induces a morphism of group functors
\[
\text{Aut}_D(W_1) \times \text{Aut}(V_p\Sigma', i^0) \rightarrow \text{Aut}(V_p\Sigma', i') \tag{4.2}
\]
This is essentially the desired local type II pair. However \( \Sigma' \) is (in general) neither self-dual nor equipped with a symplectic pairing, without which there is no construction of the Weil representation. Thus we interpret the latter automorphism group in terms of \( (\Sigma, i) \).

Consider any \( F_0 \)-linear \((i)\) symplectic pairing \( \langle \cdot, \cdot \rangle : V_p\Sigma \times V_p\Sigma \rightarrow F_0 \otimes_{\mathbb{Q}_p} V_p\mu_{p^\infty} \) for which \( V_p(\Sigma')^\vee \) and \( V_p\Sigma' \) are totally isotropic. Then \( \langle \cdot, \cdot \rangle \) induces \( \delta : \text{Aut}(V_p\Sigma') \simeq \text{Aut}(V_p(\Sigma')^\vee) \) so that the two elements corresponding via \( \delta \) are an adjoint pair with respect to \( \langle \cdot, \cdot \rangle \). Then \( (\delta, \text{id}) \) defines an embedding \( \text{Aut}(V_p\Sigma', i') \hookrightarrow \text{Aut}(V_p\Sigma, \langle \cdot, \cdot \rangle, i) \). Composing with (4.2), obtain
\[
\text{Aut}_D(W_1) \times \text{Aut}(V_p\Sigma', i^0) \rightarrow \text{Aut}(V_p\Sigma, \langle \cdot, \cdot \rangle, i) \rightarrow \text{Aut}(V_p\Sigma, \langle \cdot, \cdot \rangle) \tag{4.3}
\]
injective on \( \text{Aut}_D(W_1) \times \{1\} \) and \( \{1\} \times \text{Aut}(V_p\Sigma', i^0) \).

**Example 4.2.** Let \( S = \text{Spec} \mathbb{F}_p \). Let \( |D : F_0| = d, \dim_D W_1 = n_1 \geq 1 \). The height of \( \Sigma_2' \) is \( d n_2 \) for some \( n_2 \geq 1 \). Suppose that \( \Sigma_2' \) (also \( \Sigma' \)) is etale, in which case we may view \( V_1' \Sigma_2' \) simply as a left \( D \)-module \( W_2 \), and \( V_p\Sigma' \) as a left \( D \)-module \( W_1 \otimes_D W_2 \). Then (4.2) may be identified with
\[
GL_D(W_1) \times GL_D(W_2) \rightarrow GL_{F_0}(W_1 \otimes_D W_2).
\]
Let us explicitly describe the restriction of the Weil representation via the above map.

For simplicity, assume \( D = F_0 = \mathbb{Q}_p \) for now. We can find \( \langle \cdot, \cdot \rangle \) starting from a perfect pairing \( \langle \cdot, \cdot \rangle_0 : \Sigma \times \Sigma \rightarrow \mu_{p^\infty} \). The Schrodinger model is
\[
\omega = C^\infty_c(W_1 \otimes_{\mathbb{Q}_p} W_2, \mathbb{F}_p)
\]
equipped with the action of an ind \( \mathbb{F}_p \)-group scheme \( P \) (which is analogous to the Siegel parabolic subgroup) as in [Shi, Cor 7.7]. Put \( G_1 := GL_{\mathbb{Q}_p}(W_1), G_2 := GL_{\mathbb{Q}_p}(W_2) \). We need not recall the definition of \( P \) here. It is enough to record that by restricting via \( G_1 \times G_2 \rightarrow GL_{\mathbb{Q}_p}(W_1 \otimes_{\mathbb{Q}_p} W_2) \rightarrow P \), we obtain a \( G_1 \times G_2 \)-representation \( \omega|_{G_1 \times G_2} \) satisfying
\[
((g_1, g_2) \cdot \phi)(w_1 \otimes w_2) = \phi(g_1^{-1}w_1 \otimes g_2^{-1}w_2), \quad \phi \in C^\infty_c(W_1 \otimes_{\mathbb{Q}_p} W_2, \mathbb{F}_p).
\]
By identifying \( G_1 = GL_{\mathbb{Q}_p}(W_1^\vee) \) via \( g_1 \mapsto (g_1^{-1})^\vee \) and using the canonical isomorphism \( W_1 \otimes_{\mathbb{Q}_p} W_2 \simeq \text{Hom}_{\mathbb{Q}_p}(W_1^\vee, W_2) \), we can also view \( \omega|_{G_1 \times G_2} \) as a representation given by
\[
((g_1, g_2) \cdot \phi)(f) = \phi(g_2^{-1} f g_1), \quad \phi \in C^\infty_c(\text{Hom}_{\mathbb{Q}_p}(W_1^\vee, W_2), \mathbb{F}_p), \quad f \in \text{Hom}_{\mathbb{Q}_p}(W_1^\vee, W_2). \tag{4.4}
\]

More generally, when the assumption that \( D = F_0 = \mathbb{Q}_p \) is dropped, we have \( G_1 = GL_D(W_1^\vee), G_2 = GL_D(W_2) \) and \( \omega|_{G_1 \times G_2} \) is a representation on \( C^\infty_c(\text{Hom}_D(W_1^\vee, W_2), \mathbb{F}_p) \) given by the same formula as (4.4).

5. A Remark on Mod p Theta Correspondence

The last section begins with some recollection of classical conjectures and results (§§5.1-5.2), which could be omitted but were included for the reader’s convenience. Then we take the liberty to speculate on the possibility of a theta correspondence for \( p \)-adic groups when the coefficient field is \( \mathbb{F}_p \). There are one negative and one positive results. A somewhat surprising fact, though it is easy to prove, is that the naive analogue of Howe’s conjecture on the bijective correspondence is almost vacuous already for the \( (GL_1, GL_1) \)-pair (Proposition 5.9). On the other hand, we verify that a weak analogue of Howe’s unramified theta correspondence still works for type II pairs (Theorem 5.14).

Often \( G_i \) will denote the group of \( F \)-points rather than the underlying algebraic group by abuse of notation. Now \( F_0 \) and \( F \) are finite extensions of \( \mathbb{Q}_p \) (rather than \( Q \)).
5.1. Review of Howe’s conjecture. Let $(G_1, G_2)$ be a (classical) reductive dual pair of type 1 or 2 as in §3.1 or §4.1 which comes with an inclusion

$$G_1 \times G_2 \hookrightarrow \text{Sp}_{F_0}(W, \langle \cdot, \cdot \rangle).$$

Write $\tilde{G}_1$ (resp. $\tilde{G}_2$) for the preimage of $G_1 \times \{1\}$ (resp. $\{1\} \times G_2$) via the classical double covering $\text{Sp}_{F_0}(W, \langle \cdot, \cdot \rangle) \to \text{Sp}_{F_0}(W, \langle \cdot, \cdot \rangle)$. The subgroup of $\text{Sp}_{F_0}(W, \langle \cdot, \cdot \rangle)$ generated by $\tilde{G}_1$ and $\tilde{G}_2$ is denoted $\tilde{G}_1 \tilde{G}_2$. It is known ([MVW87, Ch2,III.1]) that

Lemma 5.1. Let $\tilde{g}_1, \tilde{g}_2 \in \text{Sp}_{F_0}(W, \langle \cdot, \cdot \rangle)$ with images $g_1, g_2$ in $\text{Sp}_{F_0}(W, \langle \cdot, \cdot \rangle)$, respectively. If $g_1 g_2 = g_2 g_1$ then $\tilde{g}_1 \tilde{g}_2 = \tilde{g}_2 \tilde{g}_1$.

In particular, $\tilde{G}_1$ and $\tilde{G}_2$ are centralizing each other in $\text{Sp}_{F_0}(W, \langle \cdot, \cdot \rangle)$. Thus the pullback of the Weil representation via

$$\tilde{G}_1 \times \tilde{G}_2 \overset{\text{product}}{\longrightarrow} \tilde{G}_1 \tilde{G}_2 \hookrightarrow \text{Sp}_{F_0}(W, \langle \cdot, \cdot \rangle)$$

makes a good sense and is going to be written as $\omega|_{\tilde{G}_1 \times \tilde{G}_2}$. Define

$$R(\tilde{G}_1 \tilde{G}_2) = \{ \pi_1 \otimes \pi_2 \in \text{Irr}(\tilde{G}_1 \times \tilde{G}_2) \mid \text{Hom}_{\tilde{G}_1 \times \tilde{G}_2}(\omega|_{\tilde{G}_1 \times \tilde{G}_2}, \pi_1 \otimes \pi_2) \neq 0 \}.$$ 

For $i = 1, 2$, let $R(\tilde{G}_i)$ denote the image of $R(\tilde{G}_1 \tilde{G}_2)$ in $\text{Irr}(\tilde{G}_i)$ under the projection $\pi_1 \otimes \pi_2 \mapsto \pi_i$. The following theorem was conjectured by Howe in [How79, §7] (also see [MVW87, III.2]). Parts (i) and (ii) were established by Waldspurger ([Wal90]) and Mínguez ([Min08a]), respectively.

Theorem 5.2. Suppose that either

(i) $F$ has residue field characteristic different from 2 or
(ii) $(G_1, G_2)$ is a type 2 pair.

Then $R(\tilde{G}_1 \tilde{G}_2)$ is a graph of bijection in $R(\tilde{G}_1) \times R(\tilde{G}_2)$.

Remark 5.3. Note that we only consider $F$ as a finite extension of $\mathbb{Q}_p$. The function field case of the conjecture is known to be true except type I pairs with $\text{char} F = 2$.

5.2. Howe’s conjecture in the unramified case. This subsection follows [How79, §7], where the reader can find more detail. Assume that $F$ is an unramified extension of $F_0$, that $G_1, G_2$ are unramified groups, that $D$ splits over $F$, and that the residue field characteristic of $F$ is odd. Then there exists an $\mathcal{O}_{F_0}$-lattice $\Lambda \subset W$ which is self-dual with respect to $\langle \cdot, \cdot \rangle$. Set $J := \text{Sp}_{F_0}(\Lambda, \langle \cdot, \cdot \rangle)$ and $\tilde{J}$ to be the preimage of $J$ in $\text{Sp}_{F_0}(W, \langle \cdot, \cdot \rangle)$. Then $\tilde{J} \simeq J \times \{1, -1\}$. Let $K_i \subset G_i$ be a hyperspecial maximal compact subgroup for $i = 1, 2$. By conjugation one can assume that $K_1, K_2 \subset J$. Thereby the inclusions $K_i \hookrightarrow \tilde{G}_i$ are obtained. Following Howe, define for $i = 1, 2$,

$$R(\tilde{G}_i, K_i) := \{ \pi_i \in \text{Irr}(\tilde{G}_i) : \pi_i|_{K_i} \neq 0 \}$$

$$R(\tilde{G}_1 \tilde{G}_2, K_1 K_2) := \{ (\pi_1, \pi_2) \in R(\tilde{G}_1 \tilde{G}_2) : \pi_i|_{K_i} \neq 0, i = 1, 2 \}.$$

Let $\mathcal{H}(\tilde{G}_i, /K_i)$ denote the Hecke algebra of $K_i$-bi-invariant $\mathbb{C}$-valued functions on $\tilde{G}_i$. Each element of $\mathcal{H}(\tilde{G}_i, /K_i)$ defines an endomorphism of $(\omega|_{\tilde{G}_1 \times \tilde{G}_2})|_{K_1 K_2}$.

Theorem 5.4. ([How79, Thm 7.1], cf. [MVW87, §5])

(i) $R(\tilde{G}_1 \tilde{G}_2, K_1 K_2)$ is a graph of bijection in $R(\tilde{G}_1, K_1) \times R(\tilde{G}_2, K_2)$.

(ii) Suppose $(\pi_1, \pi_2) \in R(\tilde{G}_1 \tilde{G}_2)$. Then $\pi_1 \in R(\tilde{G}_1, K_1)$ if and only if $\pi_2 \in R(\tilde{G}_2, K_2)$.

(iii) The images of $\mathcal{H}(\tilde{G}_i, K_i)$ in $\text{End}_\mathbb{C}((\omega|_{\tilde{G}_1 \times \tilde{G}_2})|_{K_1 K_2})$ are the same for $i = 1, 2$. 
5.3. Commuting elements in the metaplectic group. In order to formulate a theta correspondence in our geometric context, it is fundamental to have the analogue of Lemma 5.1 for $\text{Mp}(V_p A, \tilde{e}_p^L)$ and $\text{Sp}(V_p A, \tilde{e}_p^L)$. We will state this as a hypothesis. Recall from [Shi, §5.1, §5.5] that for $(A, L)$ (as in this article) there is a natural sequence of group functors on $(\text{Sch}/S)$

$$1 \to \mathbb{G}_m \to \text{Mp}(V_p A, \tilde{e}_p^L) \to \text{Sp}(V_p A, \tilde{e}_p^L) \to 1,$$

whose $T$-valued points yield an exact sequence of groups for each $S$-scheme $T$. For a $p$-divisible group $\Sigma$ with a perfect alternating pairing $\langle \cdot, \cdot \rangle : \Sigma \times \Sigma \to \mu_{p\infty}$ over $S$ there is a similar sequence

$$1 \to \mathbb{G}_m \to \text{Mp}(V_p \Sigma, \langle \cdot, \cdot \rangle) \to \text{Sp}(V_p \Sigma, \langle \cdot, \cdot \rangle) \to 1.$$

**Hypothesis 5.5.** Let $\tilde{g}_1, \tilde{g}_2 \in \text{Mp}(V_p A, \tilde{e}_p^L)(T)$ for an $S$-scheme $T$. If the images of $\tilde{g}_1$ and $\tilde{g}_2$ in $\text{Sp}(V_p A, \tilde{e}_p^L)(T)$ commute then $\tilde{g}_1 \tilde{g}_2 = \tilde{g}_2 \tilde{g}_1$. The same holds in place of $(\Sigma, \langle \cdot, \cdot \rangle)$ in the case of $\text{Sp}(V_p \Sigma, \langle \cdot, \cdot \rangle)$.

The same analogue for $\text{Mp}(V A, \tilde{e}^L)$ should follow from the hypothesis at every prime $p$. In this subsection we focus on one prime $p$ at a time. Although it is tempting to conjecture that the hypothesis is always true, we do not have good evidence other than analogy with the case of classical dual pairs. Nevertheless in some simple cases we can show:

**Lemma 5.6.** Hypothesis 5.5 holds true

(i) for $\text{Mp}(V_p A, \tilde{e}_p^L)(T)$ if $p$ is invertible in $S$ and there is a level structure $\eta : V_p \simeq V_p A$ for a symplectic $\mathbb{Q}$-vector space $V_p$ with $\langle \cdot, \cdot \rangle : V_p \times V_p \to \mathbb{G}_m$ (cf. §3.3, [Shi, §6.2]), or

(ii) for $\text{Mp}(V_p \Sigma, \langle \cdot, \cdot \rangle)$ if $S$ is an $\mathbb{F}_p$-scheme and $\Sigma$ is an ordinary $p$-divisible group over $\mathbb{F}_p$, or

(iii) for $\text{Mp}(V_p \Sigma, \langle \cdot, \cdot \rangle)$ if $S$ and $(\Sigma, \langle \cdot, \cdot \rangle)$ are as in (ii) and there is a level structure $\zeta : V_p \Sigma \simeq V_p A$ (cf. [Shi, §6.3]).

Proof. We may assume that $T$ is connected. In case (i), $\text{Mp}(V_p A, \tilde{e}_p^L)$ may be identified with the classical metaplectic group $\text{Mp}(V_p, \langle \cdot, \cdot \rangle \circ \eta)$ (as a constant group scheme), for which the assertion is well known ([MVW87, Ch 2, Lem II.5]). In case (ii), the proof is obvious as $\text{Mp}(V_p \Sigma, \langle \cdot, \cdot \rangle) \simeq \mathbb{G}_m \times \text{Sp}(V_p \Sigma, \langle \cdot, \cdot \rangle)$ ([Shi, Cor 7.8]). It is easy to deduce (iii) from (ii). \qed

In the notation of Lemmas 3.11 and 4.1, let $\tilde{G}_1$ (resp. $\tilde{G}_2$) denote the pullback of $\text{Mp}(V_p A, \tilde{e}_p^L)$ along $\text{Aut}_D(W_1, \langle \cdot, \cdot \rangle_1) \times \{1\} \hookrightarrow \text{Sp}(V_p A, \tilde{e}_p^L)$ (resp. $\{1\} \times \text{Aut}(V_p A, \tilde{e}_p^L, \omega_1) \hookrightarrow \text{Sp}(V_p A, \tilde{e}_p^L)$) for type I pairs and along $\text{Aut}_D(W_1) \times \{1\} \hookrightarrow \text{Sp}(V_p A, \tilde{e}_p^L)$ (resp. $\{1\} \times \text{Aut}(V_p, \tilde{e}_2) \hookrightarrow \text{Sp}(V_p A, \tilde{e}_p^L)$) for type II pairs. Thus $\tilde{G}_1$ and $\tilde{G}_2$ are subgroup functors of $\text{Mp}(V_p A, \tilde{e}_p^L)$. We have a similar definition of $\tilde{G}_1$ and $\tilde{G}_2$ in the setting of $p$-divisible groups, cf. (4.3).

**Corollary 5.7.** Suppose that Hypothesis 5.5 is true for $(A, L)$ or $(\Sigma, \langle \cdot, \cdot \rangle)$. (For instance suppose that one of the conditions of Lemma 5.6 holds.) Then $\tilde{G}_1$ and $\tilde{G}_2$ defined above commute.

Proof. This is immediate since the images of $\tilde{G}_1$ and $\tilde{G}_2$ in $\text{Sp}(V_p A, \tilde{e}_p^L)$ commute by construction. \qed

**Remark 5.8.** Deligne informed us that Hypothesis 5.5 is false for an arbitrary metaplectic group. More precisely, let $G$ be a connected reductive group, say over $\mathbb{Q}_p$. Let $\tilde{G} \to G(\mathbb{Q}_p)$ be an arbitrary metaplectic extension of $G(\mathbb{Q}_p)$ by $\mathbb{G}_m$ or $\mu_n$ for some $n \geq 2$. Then there are counterexamples where $\tilde{g}_1, \tilde{g}_2$ commute in $\tilde{G}$ but their images in $G(\mathbb{Q}_p)$ do not commute. In this regard, the double covering of a symplectic group may be somewhat special, and we are hoping that its generalization $\text{Mp}(V_p A, \tilde{e}_p^L)$ also has this special property.

5.4. Quotients of the mod $p$ Weil representation of a $p$-adic metaplectic group. Here we explore a mod $p$ analogue of Howe’s conjecture for $p$-adic groups. A mod $l$ analogue of Howe’s conjecture for a prime $l \neq p$ was considered in [Min08b]. Interestingly Mínguez observed that there is a counterexample to the naive analogue of Theorem 5.2 in the mod $l$ setting when $l$ is not a so-called banal prime.

In the mod $p$ case the problem is even more serious. The naive analogue of Theorem 5.2 hopelessly fails already in the case of the most elementary type II pair $(\text{GL}_1(\mathbb{Q}_p), \text{GL}_1(\mathbb{Q}_p))$. To study this case, put $k := \mathbb{F}_p$ and recall that recall that $\omega = C^\infty_c(\mathbb{Q}_p, k)$ is a representation of $\text{GL}_1(\mathbb{Q}_p) \times \text{GL}_1(\mathbb{Q}_p)$ such that $(g_1, g_2) \cdot \phi(g) = \phi(g_2^{-1} g g_1)$ (Example (4.2)). For our purpose it suffices to consider the first copy of $\text{GL}_1(\mathbb{Q}_p)$, and we will view $\omega$ as a $\text{GL}_1(\mathbb{Q}_p)$-representation as such.
Proposition 5.9. Let \( V \) be a nonzero finite dimensional smooth representation of \( GL_1(\mathbb{Q}_p) \).

(i) Any \( k[GL_1(\mathbb{Q}_p)] \)-linear map \( C^\infty_c(\mathbb{Q}_p^\infty, k) \to V \) is zero for each \( V \).

(ii) Any nonzero \( k[GL_1(\mathbb{Q}_p)] \)-linear map \( \omega = C^\infty_c(\mathbb{Q}_p, k) \to V \) has 1-dimensional image which is the trivial representation of \( GL_1(\mathbb{Q}_p) \).

Proof. (i) Let \( \xi : C^\infty_c(\mathbb{Q}_p^\infty, k) \to V \) be a \( k[GL_1(\mathbb{Q}_p)] \)-linear map. It suffices to prove that there exists \( N \geq 1 \) such that \( \xi(\mathrm{char}_{a(1+p^nZ_p)}) = 0 \) for all \( a \in \mathbb{Q}_p^\infty \) and all \( n \geq N \). As \( V \) is smooth, there exists \( m \geq 1 \) such that \( 1+p^nZ_p \) acts trivially on \( V \). Take any \( n \geq m \). Let \( b_0, \ldots, b_{p-1} \) be any coset representatives for \( (1+p^nZ_p)/(1+p^n+1Z_p) \). Then

\[
\xi(\mathrm{char}_{a(1+p^nZ_p)}) = \sum_{i=0}^{p-1} \xi(\mathrm{char}_{ab_i(1+p^n+1Z_p)}) = \sum_{i=0}^{p-1} b_i^{-1} \cdot \xi(\mathrm{char}_{a(1+p^n+1Z_p)}) = \sum_{i=0}^{p-1} \xi(\mathrm{char}_{a(1+p^n+1Z_p)}) = 0.
\]

(ii) Let \( 1 \) denotes the trivial representation of \( GL_1(\mathbb{Q}_p) \). The map \( \omega \to 1 \) by \( \phi \mapsto \phi(0) \) induces an exact sequence of \( k[GL_1(\mathbb{Q}_p)] \)-modules

\[
0 \to C^\infty_c(\mathbb{Q}_p^\infty, k) \to \omega \to 1 \to 0.
\]

This and the assertion (i) imply (ii). \( \square \)

Remark 5.10. If \( k = \mathbb{C} \) then any continuous character \( \chi : \mathbb{Q}_p^\infty \to \mathbb{C}^\times \) is realized as a quotient of \( C^\infty_c(\mathbb{Q}_p^\infty, \mathbb{C}) \) (with multiplicity one). If \( \mu \) is a \( \mathbb{C} \)-valued Haar measure on \( \mathbb{Q}_p^\infty \) then \( \phi \mapsto \int_{\mathbb{Q}_p^\infty} \phi(x) \chi^{-1}(x)d\mu \) exhibits a nonzero map in \( \mathrm{Hom}_{\mathbb{C}}(C^\infty_c(\mathbb{Q}_p^\infty, \mathbb{C}), \chi) \). See [MVW87, Ch 3, Lem 2.3] for the study of quotients of \( C^\infty_c(GL_n(\mathbb{Q}_p), \mathbb{C}) \), where a \( \mathbb{C} \)-valued Haar measure on \( GL_n(\mathbb{Q}_p) \) is indispensable. The case \( k = \overline{\mathbb{F}}_p \) is special due to the lack of an \( \overline{\mathbb{F}}_p \)-valued Haar measure. Indeed, the argument of Proposition 5.9.(i) essentially proves the nonexistence of an \( \overline{\mathbb{F}}_p \)-valued Haar measure on any pro-\( p \) subgroup of \( \mathbb{Q}_p^\infty \).

The above proposition tells us that in the mod \( p \) setting, there are no interesting naïve analogues of \( R(G_1 G_2) \) and \( R(G_1 G_2, K_1 K_2) \). We do not know how to overcome this difficulty and make a plausible conjecture in a similar spirit as Howe’s conjecture: Even if we allow \( \pi_1 \) and \( \pi_2 \) to be reducible representations of finite length, the mod \( p \) Weil representation still has few quotients of the form \( \pi_1 \otimes \pi_2 \). It is not immediately clear whether replacing \( \mathrm{Hom} \) with \( \mathrm{Ext} \) would help.

5.5. A weak analogue of Howe’s conjecture in the unramified case. Despite the negative result of §5.4, we would like to ask (cf. Remark 5.15).

Question 5.11. Is a suitable analogue of Theorem 5.4 true in the mod \( p \) case?

The aim of this subsection is to verify a weak analogue of Theorem 5.4.(iii) for ordinary type II pairs. More precisely we will show that the Weil representation admits a finite filtration whose quotients have the property described in Theorem 5.4.(iii). (In fact we prove slightly more; see Theorem 5.14 below.) Our argument was inspired by [MVW87] and [Min08a].

Assume that \( n_1 \geq n_2 \geq 1 \). Let us set up some notation.

- \( F \) is a finite extension of \( \mathbb{Q}_p \) with valuation \( v_F : F^\times \to \mathbb{Z} \); its ring of integers is \( \mathcal{O}_F \).
- \( G_i := GL_{n_i}(F) \), \( K_i := GL_{n_i}(\mathcal{O}_F) \) for \( i = 1, 2 \).
- \( H_r := GL_r(F) \), \( U_r := GL_r(\mathcal{O}_F) \) (where \( 0 \leq r \leq n_2 \)), agreeing \( H_0 = U_0 = \{1\} \).
- \( M_{n_1-r} := H_{n_1-r} \times H_r \), \( K_{n_1-r} := U_{n_1-r} \times U_r \).
- \( T_i \) is the diagonal maximal torus of \( G_i \) for \( i = 1, 2 \).
- \( S_i \) is the diagonal maximal torus of \( H_r \).
- \( B_1 \) (resp. \( B_2 \)) is the Borel subgroup of upper (resp. lower) triangular matrices in \( G_1 \) (resp. \( G_2 \)).
- \( B^+ \) (resp. \( B^- \)) is the Borel subgroup of upper (resp. lower) triangular matrices in \( H_r \).
- \( \Phi_r \) is the set of \( B^+_r \)-positive roots of \( T_r \) in \( G_i \) for \( i = 1, 2 \).
- \( \Phi^+_r \) (resp. \( \Phi^-_r \)) is the set of \( B^+_r \)-positive (resp. \( B^-_r \)-positive) roots of \( T_r \) in \( H_r \).
- All Hecke algebras and spaces of functions (e.g. \( C^\infty(G_1) \)) will have coefficients in \( k = \overline{\mathbb{F}}_p \). The reference to \( k \) will be omitted most of the time.
Let \( \omega|_{G_1 \times G_2} \) denote the restricted Weil representation at the end of Example 4.2 (with \( D = F \)). By choosing a basis of \( W_1^V \) and \( W_2 \) in that example, we can identify \( \omega|_{G_1 \times G_2} = C_\infty^c(M_{n_1,n_2}(F), k) \) with \( G_1 \times G_2 \)-action described by

\[
((g_1, g_2) \phi)(x) = \phi(g_2^{-1} x g_1), \quad \forall x \in M_{n_1,n_2}(F).
\]

For \( 0 \leq r \leq n_2 + 1 \), let \( \Omega_r \) denote the \( k \)-subspace of \( \omega|_{G_1 \times G_2} \) consisting of functions supported on \( n_1 \times n_2 \)-matrices of rank \( \geq r \). Then there is a \( G_1 \times G_2 \)-stable filtration

\[
\{0\} = \Omega_{n_2+1} \subseteq \Omega_{n_2} \subseteq \Omega_{n_2-1} \subseteq \cdots \subseteq \Omega_1 \subseteq \Omega_0 = \omega|_{G_1 \times G_2}.
\]

Then the space of \( K_1 \times K_2 \)-invariant vectors on quotients are equipped with Hecke actions

\[
\mathcal{H}(G_1, K_1) \times \mathcal{H}(G_2, K_2) \to \text{End}_k((\Omega_r/\Omega_{r+1})_{K_1 \times K_2}). \tag{5.1}
\]

For \( 0 \leq r \leq n \), let \( \mu_r \) denote the \( H_r \times H_r \)-representation on \( C_\infty^c(H_r, k) \), where the action is given by

\[
((h_1, h_2) \cdot \phi)(h) = \phi(h_2^{-1} h h_1).
\]

Observe that \( \mu_{U_r \times U_r}^c = C_\infty^c(U_r \backslash H_r/U_r, k) \) comes equipped with a natural action of \( \mathcal{H}(H_r, U_r) \times \mathcal{H}(H_r, U_r) \). Let \( \# \) denote the order 2 automorphism of \( \mathcal{H}(H_r, U_r) \) induced by \( g \mapsto g^{-1} \) on \( H_r \).

**Lemma 5.12.** For every \( f \in \mathcal{H}(H_r, U_r) \), the actions of \((f, 1)\) and \((1, f^\#)\) on \( \mu_{U_r \times U_r}^c \) are the same.

**Proof.** Take \( V_1 = V_2 = 1 \) in [Her, Prop 6.2]. Note that our action of \( \mathcal{H}_k(H_r, U_r) \times \{1\} \) is a left action, which differs from Herzig’s right action by \( f \mapsto f^\# \). \( \square \)

**Lemma 5.13.** For every \( 0 \leq r \leq n_2 \), there is an isomorphism of \( G_1 \times G_2 \)-representations

\[
\Omega_r/\Omega_{r+1} \cong \text{Ind}_{F_2 \times r \cdot r \cdot r}^{G_1 \times G_2} (1 \otimes \mu_r).
\]

**Proof.** It is enough to observe that Lemma 1.3 and the paragraph above Définition 2.1 of [Min08a] carry over to the case of \( \mathbb{F}_p \)-coefficients. Note that our \( G_1 \) (resp. \( G_2 \)) is his \( G_1' \) (resp. \( G_2' \)). \( \square \)

There are partial Satake transforms (as \( \mathcal{S}_{\mathcal{G}}^M \) of [Her, §2.3])

\[
\mathcal{S}_i : \mathcal{H}(G_i, K_i) \to \mathcal{H}(M_{n_i-r,r}, K_{n_i-r,r}), \quad i = 1, 2
\]

defined with respect to \( B_i \). (In other words, require the \( P \) of [Her, §2.3] to contain \( B_i \).) Define

\[
S_r^+ := \{ t \in S_r : \nu_F(\alpha(t)) \leq 0, \forall \alpha \in \Phi_r^+ \}
\]

and also \( S_r^- \), using \( \Phi_r^- \) in place of \( \Phi_r^+ \). Let \( \mathcal{H}_{S_r}^+ \) denote the subalgebra of \( \mathcal{H}_{S_r}(1) = C_\infty((\text{GL}_1(F)/\text{GL}_1(\mathbb{Q}_F))^\nu)^\nu \) consisting of functions whose supports are contained in \( S_r^- \). Similarly define \( \mathcal{H}_{S_r}^+, \mathcal{H}_{T_1}^+, \mathcal{H}_{T_2}^+ \) replacing \( \Phi_r^+ \) in (5.2) respectively with \( \Phi_r^- \), the set of \( B_1 \cap M_{n_1-r,r}\)-positive roots, and the set of \( B_2 \cap M_{n_2-r,r}\)-positive roots. Consider the diagram

\[
\begin{array}{ccc}
\mathcal{H}(M_{n_1-r,r}, K_{n_1-r,r}) & \cong & \mathcal{H}(H_r, U_r) \\
\downarrow & & \downarrow \\
\mathcal{H}_{S_r}^- & \cong & \mathcal{H}_{S_r}^+
\end{array}
\]

where \(* = - \) if \( i = 1 \) and \(* = + \) if \( i = 2 \). The vertical maps are the Satake isomorphisms of [Her11] with respect to \( B_1 \cap M_{n_1-r,r} \) and \( B_2^- \). The bottom horizontal arrow is induced by the inclusion

\[
S_r = \text{GL}_1(F)^+ \to T_i = \text{GL}_1(F)^+, \quad (x_1, \ldots, x_r) \mapsto (x_1, \ldots, x_r, 1, \ldots, 1).
\]

Then there exists a unique map \( \mathcal{J}_r \) which makes the diagram commute. Now consider

\[
\begin{array}{cccc}
\mathcal{H}(G_1, K_1) & \xrightarrow{\mathcal{J}_1} & \mathcal{H}(M_{n_1-r,r}, K_{n_1-r,r}) & \xrightarrow{\mathcal{J}} & \mathcal{H}(H_r, U_r) & \xrightarrow{\#} & \mathcal{H}_{S_r}^- \\
\mathcal{H}(G_2, K_2) & \xrightarrow{\mathcal{J}_2} & \mathcal{H}(M_{n_2-r,r}, K_{n_2-r,r}) & \xrightarrow{\mathcal{J}} & \mathcal{H}(H_r, U_r) & \xrightarrow{\#} & \mathcal{H}_{S_r}^+
\end{array}
\]

\[
(5.4)
\]
The third arrow in each row is the same as in the right vertical arrow of (5.3). The rightmost vertical map of (5.4) is induced by $t \mapsto t^{-1}$ on $S_r$ and denoted # by abuse of notation. Obviously the right rectangle commutes.

**Theorem 5.14.**

(i) Let $f_i \in \mathcal{H}(G_i, K_i)$, $i = 1, 2$. If $f_1$ and $f_2$ have the same image in $\mathcal{H}^+_S$ via (5.4) then $(f_1, 1)$ and $(1, f_2)$ have the same image via (5.1).

(ii) The composite maps $\mathcal{H}(G_1, K_1) \to \mathcal{H}_S$ and $\mathcal{H}(G_2, K_2) \to \mathcal{H}^+_S$ are surjective.

(iii) $\mathcal{H}(G_1, K_1) \times \{1\}$ and $\{1\} \times \mathcal{H}(G_2, K_2)$ have the same image in $\text{End}_k((\Omega_r/\Omega_{r+1})^{K_1 \times K_2})$.

**Proof.** (i) Recall that $\Omega_r/\Omega_{r+1} \cong \text{Ind}_{P_{n_2-r,r} \times P_{n-r,r}}^{G_1 \times G_2} (1 \otimes \mu_r)$ from Lemma 5.13. By sending $\phi$ of the induced representation to $\phi(1)$, we obtain an isomorphism of $k$-vector spaces (cf. (2.12) of [Her])

$$ \left( \Omega_r/\Omega_{r+1} \right)^{K_1 \times K_2} \cong (1 \otimes \mu_r)^{K_{n-r,r} \times K_{n_2-r,r}}. $$

By [Her, Lem 2.13], the above map is equivariant for the action of $\prod_{i=1}^2 \mathcal{H}(G_i, K_i)$, if the latter acts on the right hand side through $\prod_{i=1}^2 \mathcal{H}(M_{n_i-r,r}, K_{n_i-r,r})$ via $(\mathcal{A}_1, \mathcal{A}_2)$. Since the action of $\prod_{i=1}^2 M_{n_i-r,r}$ on $1 \otimes \mu_r$ factors through its projections onto $H_r \times H_r$, we see that $\prod_{i=1}^2 \mathcal{H}(M_{n_i-r,r}, K_{n_i-r,r})$ acts through $\mathcal{H}(H_r, U_r) \times \mathcal{H}(H_r, U_r)$ via $(\mathcal{A}_1, \mathcal{A}_2)$. At this point, observe that the assumption of (i) implies that $\mathcal{A}_1(f_1)$ maps to $\mathcal{A}_2(f_2)$ via #. Lemma 5.12 shows that $\mathcal{A}_1(f_1)$ and $\mathcal{A}_2(f_2)$ have the same action on $(1 \otimes \mu_r)^{K_{n-r,r} \times K_{n_2-r,r}}$.

(ii) The image of $\mathcal{A}_1$, mapped to $\mathcal{H}^-_{T_1}$ via (5.3) consists exactly of

$$ \phi \in C^c_c(T_1), \quad \text{supp } \phi \subset \{ t \in T_1 : v_F(\alpha(t)) \leq 0, \forall \alpha \in \Phi_1 \}. $$

Clearly the set of such $\phi$ maps onto $\mathcal{H}^-_{S_r}$ via (5.3), noting that $\Phi_1$ restricts to $\Phi^+_F$ via $S_r \to T_1$. Therefore $\mathcal{H}(G_1, K_1) \to \mathcal{H}^-_{S_r}$ is onto. The same argument shows that $\mathcal{H}(G_2, K_2) \to \mathcal{H}^+_S$ is onto as well.

(iii) This is immediate from (i) and (ii).

**Remark 5.15.** A variant of Question 5.11 can be asked, replacing the hyperspecial subgroups by either the maximal pro-$p$ subgroups of Iwahori subgroups or the congruence subgroups consisting of elements which are 1 modulo $p$. For instance, one can ask whether one gets a correspondence of “Serre weights”.

**Remark 5.16.** Let us conclude with a speculative remark. Our belief is that the conjectural mod $p$ theta correspondence for pro-$p$ groups should be compatible with the conjectural (global) theta correspondence for mod $p$ automorphic forms. However it is not easy to make sense of this, as no satisfactory representation-theoretic approach seems available to study mod $p$ automorphic forms in general. Some anomalies are discussed in [Ser96], for instance.

**References**


