Appendix A. More examples

By Junehyuk Jung and Sug Woo Shin

In this appendix we give more examples of families $(\mathscr{F}, \mathcal{N})$ to which the main results of the paper apply. As in [27], the examples are constructed via the Langlands functoriality (Hypothesis A.1).

A.1. **Families to be considered.** Let G be a split reductive group over $\mathbb{Q}$ such that $G(\mathbb{R})$ admits discrete series. We assume that the center $Z(G)$ is anisotropic over $\mathbb{Q}$ for simplicity.[3] For $n \in \mathbb{Z}_{\geq 1}$, define an open compact subgroup of $G(\mathbb{A}^\infty)$

$$U(n) := \ker(G(\widehat{\mathbb{Z}}) \to G(\mathbb{Z}/n\mathbb{Z})).$$

Let $r : \widehat{G} \to GL_m(\mathbb{C})$ be a faithful irreducible representation of the dual group of G such that

$$r \simeq r^\vee.$$

Let ξ be an irreducible algebraic representation of $G \otimes_{\mathbb{Q}} \mathbb{C}$. Assume that the highest weight of ξ is regular. Let $\Pi_\infty(\xi)$ be the set of discrete series of $G(\mathbb{R})$ whose infinitesimal character and central character are the same as $\xi^\vee$. Then $\Pi_\infty(\xi)$ is an L-packet. For an automorphic representation π of $G(\mathbb{A})$, define $m(\pi)$ to be the multiplicity in the discrete L^2-spectrum of $G(\mathbb{Q})\backslash G(\mathbb{A})$, and $N(\pi)$ to be the least $N \in \mathbb{Z}_{\geq 1}$ such that $\pi^{U(N)} \neq 0$. Let $\{n_k\}$ be an increasing sequence of positive integers. Assume

- for each prime p, $p \nmid n_k$ for $k \gg 0$.

Define $\mathcal{F}_k$ to be the multi-set of all discrete automorphic representations π of $G(\mathbb{A})$ such that $\pi_\infty \in \Pi_\infty(\xi)$ in which π appears with multiplicity

$$a_{\mathcal{F}_k}(\pi) := m(\pi) \dim(\pi^\infty)^{U(n_k)}.$$

Hypothesis A.1. For each $k \geq 1$ and $\pi \in \mathcal{F}_k$, there is an isobaric automorphic representation Π of $GL_m(\mathbb{A})$ (the functorial lift of π under r) such that

(1) at all finite places v where G, r and π are unramified, the Satake parameter for π_v transfers to that of Π_v via r,
(2) the L-parameter for π_∞ transfers to that for Π_∞ via r (where the L-parameters are given by the local Langlands correspondence for real reductive groups).

This is the same as the Hypothesis 10.1 of [27], to which we refer the reader for more details on conditions (1) and (2). The functorial lift Π as above is denoted $r_*\pi$. Put $\mathscr{F}_k := r_*\mathcal{F}_k$ (as a multi-set) and $\mathscr{F} := \{\mathscr{F}_k\}_{k\geq 1}$.

Example A.2. Let G be a split symplectic group or a split orthogonal group in n variables where n is either odd or divisible by 4. Then $G(\mathbb{R})$ contains a compact maximal torus so admits discrete series. Moreover Hypothesis A.1 is known in this case by Arthur [1] conditionally on the stabilization of the twisted trace formula, the weighted fundamental lemma (being written up by Chaudouard and Laumon),

[3] In fact [27] (in the level aspect) works with a reductive group over a totally real field which admits discrete series at all infinite places without assuming that $Z(G)$ is anisotropic or that G is split. Though our results should extend to that setting without difficulty (in particular should include the case of quasi-split unitary groups by using [19]), we chose to restrict ourselves to split groups in favor of simplicity and clarity.

and a technical result in harmonic analysis. (See [4, 1.18] for a detailed discussion of these conditions.)

A.2. **Satake transforms.** Let p be a prime and G be a Chevalley reductive group over $\mathbb{Z}_p$. Let T be a split maximal torus of G over $\mathbb{Q}_p$ in a good relative position to $G(\mathbb{Z}_p)$ and $B \supset T$ a Borel subgroup. Let $X_*(T)$ and $X_*(T)^+$ be the cocharacter group of T and its subset of B-dominant members, respectively. Denote by Ω the associated Weyl group, which is equipped with sign character $\mathrm{sgn} : \Omega \to \{\pm 1\}$. Write ρ for the half sum of all B-positive roots of T in G. Choose a Put $K_p := G(\mathbb{Z}_p)$. Write $\mathcal{H}_p^{\mathrm{ur}}(G)$ for the unramified Hecke algebra of bi-K_p-invariant functions on $G(\mathbb{Q}_p)$ with values in $\mathbb{C}$. Similarly $\mathcal{H}_p^{\mathrm{ur}}(T)$ is the algebra of functions on $T(\mathbb{Q}_p)$ bi-invariant under $T(\mathbb{Q}_p) \cap K_p$. There is an obvious action of Ω on each of $\mathcal{H}_p^{\mathrm{ur}}(T)$ and $X_*(T)$. For $\mu \in X_*(T)^+$, define $\tau_\mu^G \in \mathcal{H}_p^{\mathrm{ur}}(G)$ to be the characteristic function on $K_p\mu(p)K_p$, and define $\chi_\mu \in \mathbb{C}[X_*(T)]^\Omega$ by the formula

$$\chi_\mu \sum_{\omega \in \Omega} \mathrm{sgn}(\omega)\omega\mu = \sum_{\omega \in \Omega} \mathrm{sgn}(\omega)\omega(\rho + \mu)$$

in the group algebra $\mathbb{C}[X_*(T)]$. It is known (cf. [13, p.465]) that $\{\chi_\mu\}_{X_*(T)^+}$ forms a $\mathbb{C}$-basis of $\mathbb{C}[X_*(T)]^\Omega$. The Satake isomorphism is a canonical $\mathbb{C}$-algebra isomorphism

$$\mathcal{S} : \mathcal{H}_p^{\mathrm{ur}}(G) \xrightarrow{\sim} \mathbb{C}[X_*(T)]^\Omega.$$

We refer the reader to [6] or [8] for details. If we write $\widehat{G}_{\mathrm{ss}}$ for the set of semisimple elements of $\widehat{G}$, there is a canonical isomorphism between $\mathbb{C}[X_*(T)]^\Omega$ and the sub $\mathbb{C}$-algebra in the space of functions on $\widehat{G}_{\mathrm{ss}}$ generated by the finite dimensional irreducible characters of $\widehat{G}$ ([6, §6]). When $r : \widehat{G} \to GL_n(\mathbb{C})$ be an irreducible representation of complex Lie groups, write tr r for its character viewed as an element of $\mathbb{C}[X_*(T)]^\Omega$.

Lemma A.3. *Assume that r be as above. Let $\mu \in X_*(T)^+$.*

1. *Suppose that r has highest weight μ. Then $\mathcal{S}^{-1}(\mathrm{tr}\ r) = \chi_\mu$.*
2. *Suppose that $\mu \neq 0$ (equivalently r is not the trivial representation). Then there exists a constant $C(\mu) > 0$ depending only on μ and the root datum of G such that*
$$|\chi_\mu(1)| \leq C(\mu)p^{-1}.$$
3. *If $\mu = 0$ then $\chi_\mu(1) = 1$.*

Remark A.4. The point of (2) is that $C(\mu)$ is independent of p when one starts with a $\mathbb{Q}$-split group and considers it over $\mathbb{Q}_p$ as p varies.

Proof. Parts (1) and (2) are the lemmas 2.1 and 2.9 in [27]. The last assertion is obvious since χ_0 is the identity in the group algebra $\mathbb{C}[X_*(T)]$, which corresponds to the characteristic function on K_p. □

A.3. **Sparsity of positive definite members.** Going back to the setup of §A.1, we aim to show that almost all members of $\mathscr{F}_k$ are not positive definite as $k \to \infty$. For any function F on the set of automorphic representations of $GL_n(\mathbb{A})$, let us define

$$E_{\mathscr{F}}(F(\pi)) \stackrel{\mathrm{def}}{=} \lim_{k\to\infty} \frac{1}{|\mathscr{F}_k|} \sum_{\pi \in \mathscr{F}_k} F(\pi),$$

which will play the role of $E_{(\mathscr{F},\mathscr{N})}(F(\pi))$ of §2.1.[4] Note that properties Ⓐ and Ⓑ of Definition 2.2 still make sense. Also, for the family given in §A.1, we have $\iota_1 = 1$, hence almost all members in $\mathscr{F}_k$ are cuspidal as k goes to $+\infty$ [25]. Therefore from Remark 4.4 we have an analogue of Lemma 1.1 in our setting using exactly the same argument presented in §2, 3, and 4.

Lemma A.5. *Let $\mathscr{F}$ be a family as in §A.1 satisfying Ⓐ and Ⓑ. Then almost all members in $\mathscr{F}$ are not positive definite (as $k \to \infty$) in the following sense: Let $B_k \subset \mathscr{F}_k$ be the sub multi-set of positive-definite members. Then $\lim_{k\to\infty} |B_k|/|\mathscr{F}_k| = 0$.*

Our final task is to verify properties Ⓐ and Ⓑ for the family $\{\mathscr{F}_k\}$. Actually we prove stronger assertions as can be easily seen from the proofs.

Lemma A.6. *The family $\mathscr{F}$ satisfies Ⓐ.*

Proof. Take μ_p in Ⓐ to be the Plancherel measure on the unramified unitary dual of $G(\mathbb{Q}_p)$. For each prime p, note that p doesn't divide level for $k \gg 0$ by assumption. So the lemma is exactly the level aspect in the corollary 9.22 of [27]. □

Lemma A.7. *The family $\mathscr{F}$ satisfies Ⓑ.*

Proof. The corollary 9.22 of [27] (+ Lemma A.3.(i)) gives us

$$E_{\mathscr{F}}(\lambda(p)) = \widehat{\mu}_p^{\mathrm{pl}}(\chi_r), \tag{A.1}$$

$$E_{\mathscr{F}}(\lambda(p)^2) = \widehat{\mu}_p^{\mathrm{pl}}(\chi_{r\otimes r}). \tag{A.2}$$

Here $\chi_{r\otimes r} := \sum_{r'} a_{r'}\chi_{r'}$ where $r\otimes r = \oplus_{r'} a_{r'} r'$ is the decomposition into irreducible representations with multiplicity $a_{r'} \in \mathbb{Z}_{\geq 0}$. The Plancherel formula satisfied by the Plancherel measure tells us that $\widehat{\mu}_p^{\mathrm{pl}}(\chi_r) = \chi_r(1)$ and $\widehat{\mu}_p^{\mathrm{pl}}(\chi_{r\otimes r}) = \sum_{r'} a_{r'}\chi_{r'}(1)$. From (A.1) and Lemma A.3.(2)

$$E_{\mathscr{F}}(\lambda(p)) = O(p^{-1}).$$

Since r is self-dual, $a_{r'} = 1$ when r' is the trivial representation. (To see this, observe that $\mathrm{Hom}_{\widehat{G}}(\mathrm{r}, \mathrm{r}^\vee)$ is one-dimensional if nonzero, provided that r is irreducible.) From (A.2) and Lemma A.3.(2)(3),

$$E_{\mathscr{F}}(\lambda(p)^2) = 1 + O(p^{-1})$$

where the implicit constant is dependent only on the decomposition $r\otimes r = \oplus_{r'} a_{r'} r'$ and the constants $C(\mu)$ as μ ranges over the highest weights corresponding to r' with $a_{r'} > 0$. The latter two are clearly independent of p. □

Theorem A.8. *Let $\{\mathscr{F}_k\}$ be a family of §A.1. Under Hypothesis A.1, almost all members of $\mathscr{F}$ are not positive definite.*

Proof. Apply Lemma A.5 along with Lemmas A.6 and A.7. □

In particular the conclusion of the theorem is true for Example A.2, conditional on the expected results as described in that example. This provides a large number of examples in addition to Theorem 1.2, 1.4, and 1.5.

[4] The only difference in the setting is that we have $S(X) \subset S(Y)$ whenever $X < Y$, where there is no such relation among $\mathscr{F}_k$. Note that we do not make use of this fact in the proof of Lemma 1.1.

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