Appendix A. More examples

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In this appendix we give more examples of families \((\mathcal{F}, N)\) to which the main results of the paper apply. As in [27], the examples are constructed via the Langlands functoriality (Hypothesis A.1).

A.1. Families to be considered. Let \(G\) be a split reductive group over \(\mathbb{Q}\) such that \(G(\mathbb{R})\) admits discrete series. We assume that the center \(Z(G)\) is anisotropic over \(\mathbb{Q}\) for simplicity.\(^3\) For \(n \in \mathbb{Z}_{\geq 1}\), define an open compact subgroup of \(G(\mathbb{A}_{\infty})\)
\[
U(n) := \ker(G(\hat{\mathbb{Z}}) \to G(\mathbb{Z}/n\mathbb{Z})).
\]
Let \(r : \hat{G} \to GL_m(\mathbb{C})\) be a faithful irreducible representation of the dual group of \(G\) such that \(r \simeq r^\vee\). Let \(\xi\) be an irreducible algebraic representation of \(G \otimes_{\mathbb{Q}} \mathbb{C}\). Assume that the highest weight of \(\xi\) is regular. Let \(\Pi_\infty(\xi)\) be the set of discrete series of \(G(\mathbb{R})\) whose infinitesimal character and central character are the same as \(\xi^\vee\). Then \(\Pi_\infty(\xi)\) is an \(L\)-packet. For an automorphic representation \(\pi\) of \(G(\mathbb{A})\), define \(m(\pi)\) to be the multiplicity in the discrete \(L^2\)-spectrum of \(G(\mathbb{Q}) \backslash G(\mathbb{A})\), and \(N(\pi)\) to be the least \(N \in \mathbb{Z}_{\geq 1}\) such that \(\pi \uparrow_{U(N)} \neq 0\). Let \(\{n_k\}\) be an increasing sequence of positive integers. Assume

- for each prime \(p\), \(p \nmid n_k\) for \(k \gg 0\).

Define \(\mathcal{F}_k\) to be the multi-set of all discrete automorphic representations \(\pi\) of \(G(\mathbb{A})\) such that \(\pi_\infty \in \Pi_\infty(\xi)\) in which \(\pi\) appears with multiplicity
\[
a_{\mathcal{F}_k}(\pi) := m(\pi) \dim(\pi_\infty) \uparrow_{U(n_k)}.
\]

**Hypothesis** A.1. For each \(k \geq 1\) and \(\pi \in \mathcal{F}_k\), there is an isobaric automorphic representation \(\Pi\) of \(GL_m(\mathbb{A})\) (the functorial lift of \(\pi\) under \(r\)) such that

1. at all finite places \(v\) where \(G\), \(r\) and \(\pi\) are unramified, the Satake parameter for \(\pi_v\) transfers to that of \(\Pi_v\) via \(r\),
2. the \(L\)-parameter for \(\pi_\infty\) transfers to that for \(\Pi_\infty\) via \(r\) (where the \(L\)-parameters are given by the local Langlands correspondence for real reductive groups).

This is the same as the Hypothesis 10.1 of [27], to which we refer the reader for more details on conditions (1) and (2). The functorial lift \(\Pi\) as above is denoted \(r_\ast \pi\). Put \(\mathcal{F}_k := r_\ast \mathcal{F}_k\) (as a multi-set) and \(\mathcal{F} := \{\mathcal{F}_k\}_{k \geq 1}\).

**Example A.2.** Let \(G\) be a split symplectic group or a split orthogonal group in \(n\) variables where \(n\) is either odd or divisible by 4. Then \(G(\mathbb{R})\) contains a compact maximal torus so admits discrete series. Moreover Hypothesis A.1 is known in this case by Arthur [1] conditionally on the stabilization of the twisted trace formula, the weighted fundamental lemma (being written up by Chaudouard and Laumon),

\(^3\)In fact [27] (in the level aspect) works with a reductive group over a totally real field which admits discrete series at all infinite places without assuming that \(Z(G)\) is anisotropic or that \(G\) is split. Though our results should extend to that setting without difficulty (in particular should include the case of quasi-split unitary groups by using [19]), we chose to restrict ourselves to split groups in favor of simplicity and clarity.
and a technical result in harmonic analysis. (See [4, 1.18] for a detailed discussion of these conditions.)

A.2. Satake transforms. Let $p$ be a prime and $G$ be a Chevalley reductive group over $\mathbb{Z}_p$. Let $T$ be a split maximal torus of $G$ over $\mathbb{Q}_p$ in a good relative position to $G(\mathbb{Z}_p)$ and $B \supset T$ a Borel subgroup. Let $X_*(T)$ and $X_+(T)$ be the cocharacter group of $T$ and its subset of $B$-dominant members, respectively. Denote by $\Omega$ the associated Weyl group, which is equipped with sign character $\text{sgn} : \Omega \to \{\pm 1\}$. Write $\rho$ for the half sum of all $B$-positive roots of $T$ in $G$. Choose a Put $K_p := G(\mathbb{Z}_p)$. Write $\mathcal{H}_p^\text{ur}(G)$ for the unramified Hecke algebra of bi-$K_p$-invariant functions on $G(\mathbb{Q}_p)$ with values in $\mathbb{C}$. Similarly $\mathcal{H}_p^\text{ur}(T)$ is the algebra of functions on $T(\mathbb{Q}_p)$ bi-invariant under $T(\mathbb{Q}_p) \cap K_p$. There is an obvious action of $\Omega$ on each of $\mathcal{H}_p^\text{ur}(T)$ and $X_*(T)$. For $\mu \in X_+(T)$, define $\tau_\mu^G \in \mathcal{H}_p^\text{ur}(G)$ to be the characteristic function on $K_p \mu(p)K_p$, and define $\chi_\mu \in \mathbb{C}[X_*(T)]^\Omega$ by the formula

$$\chi_\mu(\sum_{\omega \in \Omega} \text{sgn}(\omega)\omega\mu) = \sum_{\omega \in \Omega} \text{sgn}(\omega)(\rho + \mu)$$

in the group algebra $\mathbb{C}[X_*(T)]$. It is known (cf. [13, p.465]) that $\{\chi_\mu\}_{\mu \in \Omega}$ forms a $\mathbb{C}$-basis of $\mathbb{C}[X_*(T)]^\Omega$. The Satake isomorphism is a canonical $\mathbb{C}$-algebra isomorphism

$$S : \mathcal{H}_p^\text{ur}(G) \xrightarrow{\sim} \mathbb{C}[X_*(T)]^\Omega.$$ We refer the reader to [6] or [8] for details. If we write $G_{\text{ss}}$ for the set of semisimple elements of $G$, there is a canonical isomorphism between $\mathbb{C}[X_*(T)]^\Omega$ and the sub $\mathbb{C}$-algebra in the space of functions on $G_{\text{ss}}$ generated by the finite dimensional irreducible characters of $G$ ([6, §6]). When $r : G \to GL_n(\mathbb{C})$ be an irreducible representation of complex Lie groups, write $\text{tr } r$ for its character viewed as an element of $\mathbb{C}[X_*(T)]^\Omega$.

Lemma A.3. Assume that $r$ be as above. Let $\mu \in X_+(T)$.

1. Suppose that $r$ has highest weight $\mu$. Then $S^{-1}(\text{tr } r) = \chi_\mu$.
2. Suppose that $\mu \neq 0$ (equivalently $r$ is not the trivial representation). Then there exists a constant $C(\mu) > 0$ depending only on $\mu$ and the root datum of $G$ such that

$$|\chi_\mu(1)| \leq C(\mu)p^{-1}.$$  

3. If $\mu = 0$ then $\chi_\mu(1) = 1$.

Remark A.4. The point of (2) is that $C(\mu)$ is independent of $p$ when one starts with a $\mathbb{Q}$-split group and considers it over $\mathbb{Q}_p$ as $p$ varies.

Proof. Parts (1) and (2) are the lemmas 2.1 and 2.9 in [27]. The last assertion is obvious since $\chi_0$ is the identity in the group algebra $\mathbb{C}[X_*(T)]$, which corresponds to the characteristic function on $K_p$.  

A.3. Sparsity of positive definite members. Going back to the setup of §A.1, we aim to show that almost all members of $\mathcal{F}_k$ are not positive definite as $k \to \infty$. For any function $F$ on the set of automorphic representations of $GL_n(\mathbb{A})$, let us define

$$E_\mathcal{F}(F(\pi)) \overset{\text{def}}{=} \lim_{k \to \infty} \frac{1}{|\mathcal{F}_k|} \sum_{\pi \in \mathcal{F}_k} F(\pi),$$
which will play the role of $E_{\mathcal{F}_{\mathcal{X}}}(\pi)$ of §2.1.\footnote{The only difference in the setting is that we have $S(X) \subset S(Y)$ whenever $X < Y$, where there is no such relation among $\mathcal{F}_k$. Note that we do not make use of this fact in the proof of Lemma 1.1.} Note that properties (A) and (B) of Definition 2.2 still make sense. Also, for the family given in §A.1, we have $\iota_1 = 1$, hence almost all members in $\mathcal{F}_k$ are cuspidal as $k$ goes to $+\infty$ [25]. Therefore from Remark 4.4 we have an analogue of Lemma 1.1 in our setting using exactly the same argument presented in §§2, 3, and 4.

**Lemma A.5.** Let $\mathcal{F}$ be a family as in §A.1 satisfying (A) and (B). Then almost all members in $\mathcal{F}$ are not positive definite (as $k \to \infty$) in the following sense: Let $B_k \subset \mathcal{F}_k$ be the sub multi-set of positive-definite members. Then $\lim_{k \to \infty} |B_k|/|\mathcal{F}_k| = 0$.

Our final task is to verify properties (A) and (B) for the family $\{\mathcal{F}_k\}$. Actually we prove stronger assertions as can be easily seen from the proofs.

**Lemma A.6.** The family $\mathcal{F}$ satisfies (A).

**Proof.** Take $\mu_p$ in (A) to be the Plancherel measure on the unramified unitary dual of $G(\mathbb{Q}_p)$. For each prime $p$, note that $p$ doesn’t divide level for $k \gg 0$ by assumption. So the lemma is exactly the level aspect in the corollary 9.22 of [27]. □

**Lemma A.7.** The family $\mathcal{F}$ satisfies (B).

**Proof.** The corollary 9.22 of [27] (+ Lemma A.3.(i)) gives us
(A.1) $E_{\mathcal{F}}(\lambda(p)) = \hat{\mu}_p^{pl}(\chi_r)$,
(A.2) $E_{\mathcal{F}}(\lambda(p)^2) = \hat{\mu}_p^{pl}(\chi_{r \otimes r})$.

Here $\chi_{r \otimes r} := \sum r' a_{r'} \chi_{r'}$ where $r \otimes r = \oplus r' a_{r'} r'$ is the decomposition into irreducible representations with multiplicity $a_{r'} \in \mathbb{Z}_{\geq 0}$. The Plancherel formula satisfied by the Plancherel measure tells us that $\hat{\mu}_p^{pl}(\chi_r) = \chi_r(1)$ and $\hat{\mu}_p^{pl}(\chi_{r \otimes r}) = \sum r' a_{r'} \chi_{r'}(1)$. From (A.1) and Lemma A.3.(2)
$$E_{\mathcal{F}}(\lambda(p)) = O(p^{-1}).$$

Since $r$ is self-dual, $a_{r'} = 1$ when $r'$ is the trivial representation. (To see this, observe that $\text{Hom}_G(r, r')$ is one-dimensional if nonzero, provided that $r$ is irreducible.) From (A.2) and Lemma A.3.(2)(3),
$$E_{\mathcal{F}}(\lambda(p)^2) = 1 + O(p^{-1})$$

where the implicit constant is dependent only on the decomposition $r \otimes r = \oplus r' a_{r'} r'$ and the constants $C(\mu)$ as $\mu$ ranges over the highest weights corresponding to $r'$ with $a_{r'} > 0$. The latter two are clearly independent of $p$. □

**Theorem A.8.** Let $\{\mathcal{F}_k\}$ be a family of §A.1. Under Hypothesis A.1, almost all members of $\mathcal{F}$ are not positive definite.

**Proof.** Apply Lemma A.5 along with Lemmas A.6 and A.7. □

In particular the conclusion of the theorem is true for Example A.2, conditional on the expected results as described in that example. This provides a large number of examples in addition to Theorem 1.2, 1.4, and 1.5.
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