GALOIS REPRESENTATIONS ARISING FROM SOME COMPACT SHIMURA VARIETIES

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Abstract. Our aim is to establish some new cases of the global Langlands correspondence for GL_m. Along the way we obtain a new result on the description of the cohomology of some compact Shimura varieties. Let F be a CM field with complex conjugation c and Π be a cuspidal automorphic representation of $GL_m(\mathbb{A}_F)$. Suppose that $\Pi^\vee \simeq \Pi \circ c$ and that Π_∞ is cohomological. A very mild condition on Π_∞ is imposed if m is even. We prove that for each prime l there exists a continuous semisimple representation $R_l(\Pi) : \mathrm{Gal}(\overline{F}/F) \to GL_m(\overline{\mathbb{Q}}_l)$ such that Π and $R_l(\Pi)$ correspond via the local Langlands correspondence (established by [HT01] and [Hen00]) at every finite place $w \nmid l$ of F ("local-global compatibility"). We also obtain several additional properties of $R_l(\Pi)$ and prove the Ramanujan-Petersson conjecture for Π. (See Theorem 1.2 and Corollary 1.3 below.) This improves the previous results obtained by [Kot92a], [Clo91], [HT01] and [TY07] where it was assumed in addition that Π is square integrable at a finite place. It is worth noting that the mild condition on Π_∞ in our theorem is removed by a p-adic deformation argument, thanks to Chenevier and Harris ([CH]).

Our approach generalizes that of [HT01], which constructs Galois representations by studying the l-adic cohomology and bad reduction of certain compact Shimura varieties attached to unitary similitude groups. The central part of our work is the computation of the cohomology of the so-called Igusa varieties. Some of the main tools are the stabilized counting point formula for Igusa varieties ([Shi09], [Shi10]) and techniques in the stable and twisted trace formulas.

Recently there have been results by Morel ([Mor10]) and Clozel-Harris-Labesse ([CHLa]) in a similar direction as ours. Our result is stronger in a few aspects. Most notably, we obtain information about $R_l(\Pi)$ at ramified places.

1. Introduction

A version of the global Langlands conjecture states that

Conjecture 1.1. *Let F be a number field and Π be a cuspidal automorphic representation of $GL_m(\mathbb{A}_F)$ which is algebraic in the sense of* [Clo90, Def 1.8]. *For each prime l, with the choice of an isomorphism $\iota_l : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, there exists an irreducible continuous semisimple representation $R_{l,\iota_l}(\Pi) : \mathrm{Gal}(\overline{F}/F) \to GL_m(\overline{\mathbb{Q}}_l)$ such that $R_{l,\iota_l}(\Pi)$ is potentially semistable at every place y of F dividing l and*

$$\mathrm{WD}(R_{l,\iota_l}(\Pi)|_{\mathrm{Gal}(\overline{F}_y/F_y)})^{\mathrm{F-ss}} \simeq \iota_l^{-1}\mathscr{L}_{m,F_y}(\Pi_y) \tag{1.1}$$

for every finite place y of F (including $y|l$).

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Here WD(·) denotes the associated Weil-Deligne representation for local Galois representations and $(\cdot)^{\mathrm{F-ss}}$ means the Frobenius semisimplification. (See [TY07, §1] for instance, to review these notions.) The notation $\mathscr{L}_{m,F_y}(\Pi_y)$ means the local Langlands image of Π_y where the geometric normalization is used (§2.3). Since Π is unramified at all but finitely many places, the conjecture implies that $R_{l,\iota_l}(\Pi)$ has the same property. The representation $R_{l,\iota_l}(\Pi)$ is unique up to isomorphism by Cebotarev density theorem, if it exists. For simplicity of notation, we write $R_l(\Pi)$ for $R_{l,\iota_l}(\Pi)$ later on.

If $m = 1$, Conjecture 1.1 is completely known by class field theory. If $m = 2$ and F is totally real, a lot is known about the conjecture. (See [BR93], [Tay89], [Sai06] and the references therein.) We will be mostly concerned with the case $m \geq 3$. In general the conjecture is still out of reach, but there are favorable circumstances where more tools are available in attacking the conjecture. Let F be a CM field. Use c to denote the complex conjugation. Suppose that $\Pi^\vee \simeq \Pi \circ c$ and that Π is regular algebraic ([Clo90, Def 3.12]). The latter is equivalent to the condition that Π_∞ is cohomological for an irreducible algebraic representation of GL_m. These assumptions on Π essentially ensure that Π "descends" to a representation of a unitary group and that the descended representation can be "seen" in the l-adic cohomology of a relevant PEL Shimura variety of unitary type. In particular many techniques in arithmetic geometry become available. There are some solid results in this setting. If we further assume that

- Π is square integrable at some finite place,

then Conjecture 1.1 is known by a series of works [Kot92a], [Clo91], [HT01] and [TY07] for every $y \nmid l$. More precisely, the assertions of Theorem 1.2 below, without any condition on Π_∞ when m is even, are known under the additional assumption on Π as above. (Although the assertion (vi) is not explicitly recorded, it follows easily from the contents of [TY07].)

It has been conceived for some time that the additional condition on Π might be superfluous. However, it has also been realized by many people that it would require techniques in the trace formula and a better understanding of endoscopy to remove the superfluous assumption on Π. One of the most conspicuous obstacles was the fundamental lemma, which had only been known in some special cases. Thanks to the recent work of Laumon-Ngô ([LN08]), Waldspurger ([Wal97], [Wal06]) and Ngô ([Ngo]) the fundamental lemma (and the transfer conjecture of Langlands and Shelstad) are now fully established. This opened up a possibility for our work.

Our paper is aimed at proving Conjecture 1.1 at $y \nmid l$, without assuming that Π is square integrable at a finite place, but with a very mild assumption on Π_∞ when m is even. (See the third assumption on Π of Theorem 1.2.) This last assumption has been removed by a p-adic deformation argument by Chenevier and Harris ([CH, Thm 3.2.5]), so it should not be regarded as a serious condition. (However, the equality (i) of the theorem is preserved only up to semisimplification in the p-adic deformation argument.) No such assumption on Π_∞ is necessary when m is odd.

The main theorem is the following. (See Remark 7.6 for the case where F is a totally real field.) Note that we also prove the assertions (v) and (vi) below, which are predicted by Conjecture 1.1 at $y|l$, as well as a few additional properties of $R_l(\Pi)$. Unfortunately we do not prove that $R_l(\Pi)$ is irreducible. (If Π is square integrable at a finite place, the irreducibility is known by [TY07, Cor 1.3].)

Theorem 1.2. *(Theorem 7.5, Theorem 7.11, Corollary 7.13) Let $m \in \mathbb{Z}_{\geq 2}$. Let F be any CM field. Let Π be a cuspidal automorphic representation of $GL_m(\mathbb{A}_F)$ such that*

- *$\Pi^\vee \simeq \Pi \circ c$ and*
- *Π_∞ has the same infinitesimal character as some irreducible algebraic representation $\Xi^\vee$ of the restriction of scalars $R_{F/\mathbb{Q}} GL_m$.*
- *Ξ is slightly regular, if m is even.*

Then for each prime l and an isomorphism $\iota_l : \overline{\mathbb{Q}}_l \stackrel{\sim}{\to} \mathbb{C}$, there exists a continuous semisimple representation $R_l(\Pi) = R_{l,\iota_l}(\Pi) : \mathrm{Gal}(\overline{F}/F) \to GL_m(\overline{\mathbb{Q}}_l)$ such that

(i) *for any place y of F not dividing l, there is an isomorphism of Weil-Deligne representations*

$$\mathrm{WD}(R_l(\Pi)|_{\mathrm{Gal}(\overline{F}_y/F_y)})^{\mathrm{F-ss}} \simeq \iota_l^{-1} \mathscr{L}_{m,F_y}(\Pi_y).$$

(ii) *Suppose $y \nmid l$. For any $\sigma \in W_{F_y}$, each eigenvalue α of $R_l(\Pi)(\sigma)$ satisfies $|\alpha|^2 \in |k(y)|^{\mathbb{Z}}$ under any embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.*

(iii) *Let y be a prime of F not dividing l where Π_y is unramified. Then $R_l(\Pi)$ is unramified at y, and for all eigenvalues α of $R_l(\Pi)(\mathrm{Frob}_y)$ and for all embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ we have $|\alpha|^2 = |k(y)|^{m-1}$.*
(iv) *For every $y|l$, $R_l(\Pi)$ is potentially semistable at y with distinct Hodge-Tate weights, which can be described explicitly.*
(v) *If Π_y is unramified at $y|l$, then $R_l(\Pi)$ is crystalline at y.*
(vi) *If Π_y has a nonzero Iwahori fixed vector at $y|l$, then $R_l(\Pi)$ is semistable at y.*

In fact, our method allows to prove a stronger assertion that there exists a compatible system of λ-adic representations associated to Π. That is to say, for each Π as above, there is a number field L such that the representations $R_{l,\iota_l}(\Pi)$ for varying l and ι_l are realized on L_λ-vector spaces for varying finite places λ of L. This can be done by realizing ξ on an L-vector space (where L is large enough to contain the field of definition of Π, cf. [Clo90, 3.1]) and $\mathscr{L}_\xi$ as a smooth L_λ-sheaf rather than a $\overline{\mathbb{Q}}_l$-sheaf (cf. [Kot92a, p.655]), where ξ and $\mathscr{L}_\xi$ are as in §5.2.

It is standard that the theorem implies the Ramanujan-Petersson conjecture for Π as above, but it is worth remarking on the order of proof. First we prove Theorem 1.2 with a weaker version of the first assertion, namely that (i) holds only up to semisimplification. This is enough for deducing the corollary below. Then the temperedness of Π, among others, is used to strengthen the statement of (i).

Corollary 1.3. *(Corollary 7.9) Let m, F, Π be as in the previous theorem. Then Π_w is tempered at every finite place w of F.*

We sketch the strategy of proof of Theorem 1.2. In fact we content ourselves with explaining the proof of only the first assertion as it is more or less standard to prove the other parts. Our strategy relies on the theory of Shimura varieties, whose cohomology is expected to realize the global Langlands correspondence in an appropriate sense. Since there are no Shimura varieties for GL_n if $n > 2$, the next best thing is to use the Shimura variety for a unitary similitude group G. Suppose that the CM field F contains an imaginary quadratic field E. We find a $\mathbb{Q}$-group G such that

- G is quasi-split at all finite places,
- $G(\mathbb{R})$ is isomorphic to $U(1, n-1) \times U(0, n)^{[F:\mathbb{Q}]/2-1}$ up to multiplier factor, and
- $G(\mathbb{A}_E) \simeq GL_1(\mathbb{A}_E) \times GL_n(\mathbb{A}_F)$.

Note that the first assumption is not satisfied by the groups considered in [Kot92a], [Clo91] and [HT01]. When n is odd, such a group G always exists. When n is even, G exists if and only if $n \equiv 2 \pmod 4$ and $[F:\mathbb{Q}]/2$ is odd. In our work it is enough to consider the case when n is odd. Indeed, in order to construct m-dimensional Galois representations, we use $n = m$ if m is odd and $n = m+1$ if m is even. In case m is odd (resp. even), $R_l(\Pi)$ will be realized in the stable (resp. endoscopic) part of the cohomology of Shimura varieties attached to G as above. These correspond to (Case ST) and (Case END) below. Before elaborating on this point, let us give more details about the setup.

Consider a projective system of Shimura varieties, denoted by Sh, whose associated group is G. If $F \neq E$ then Sh is a projective system of smooth projective varieties over F, which arise as the moduli spaces of abelian schemes with additional structure. The projectivity of Sh and the fact that $G/Z(G)$ is anisotropic over $\mathbb{Q}$ are related to each other and essential in our argument. Let ξ be an irreducible algebraic representation of G over $\overline{\mathbb{Q}}_l$, which gives rise to a lisse l-adic sheaf $\mathscr{L}_\xi$ on Sh. The étale cohomology space $H^k(\mathrm{Sh}, \mathscr{L}_\xi) := H^k(\mathrm{Sh} \times_F \overline{F}, \mathscr{L}_\xi)$ is a smooth representation of $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\overline{F}/F)$. We have a decomposition

$$H^k(\mathrm{Sh}, \mathscr{L}_\xi) = \bigoplus_{\pi^\infty} \pi^\infty \otimes R^k_{\xi,l}(\pi^\infty)$$

as π^∞ runs over the set of irreducible admissible representations of $G(\mathbb{A}^\infty)$. Write $H(\mathrm{Sh}, \mathscr{L}_\xi) := \sum_k (-1)^k H^k(\mathrm{Sh}, \mathscr{L}_\xi)$.

Fix a prime p split in E as well as a place w of F above p. The Shimura variety Sh has an integral model over $\mathcal{O}_{F_w}$ and its special fiber $\overline{\mathrm{Sh}}$ has the Newton polygon stratification into $\overline{\mathrm{Sh}}^{(b)}$ where b is a parameter for an isogeny class of p-divisible groups with additional structure. We can define a smooth variety Ig_b over $\overline{\mathbb{F}}_p$ (which is also a projective system of varieties) from $\overline{\mathrm{Sh}}^{(b)}$. Also defined is a $\mathbb{Q}_p$-group J_b which is an inner form of a Levi subgroup of $G_{\mathbb{Q}_p}$. The cohomology space $H(\mathrm{Ig}_b, \mathscr{L}_\xi)$ is naturally a virtual representation of $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$. On the other hand, there is a functor $\mathrm{Mant}_{b,\mu} : \mathrm{Groth}(J_b(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{F_w})$, which is defined in terms of the cohomology of a certain moduli space of p-divisible groups. Mantovan's formula ([Man05, Thm 22], [Man, Thm 1]) is the following identity in $\mathrm{Groth}(G(\mathbb{A}^\infty) \times W_{F_w})$, which generalizes [HT01, Thm IV.2.8].

$$H(\mathrm{Sh}, \mathscr{L}_\xi) = \sum_b \mathrm{Mant}_{b,\mu}(H_c(\mathrm{Ig}_b, \mathscr{L}_\xi)). \tag{1.2}$$

An important point is that $\mathrm{Mant}_{b,\mu}$ is purely local in nature and well-understood thanks to Harris and Taylor. (See §2.4).

Consider a regular algebraic automorphic representation $\Pi = \psi \otimes \Pi^1$ of $G(\mathbb{A}_E) \simeq GL_1(\mathbb{A}_E) \times GL_n(\mathbb{A}_F)$ where Π_∞ is determined by ξ. We deal with two possibilities for Π^1 as follows (§6.1).

- (Case ST) Π^1 is cuspidal, or
- (Case END) $\Pi^1 = \text{n-ind}(\Pi_1 \otimes \Pi_2)$ where Π_i is a cuspidal automorphic representation of $GL_{n_i}(\mathbb{A}_F)$ and $n_1 > n_2 > 0$. $(n_1 + n_2 = n)$

For simplicity of exposition, we assume that the local base change from the representations of $G(\mathbb{A}^\infty)$ to those of $G(\mathbb{A}_E^\infty)$ is well-defined at every finite place. (In practice we work under simplifying assumptions to make sense of base change unconditionally, as in §4.1. To our knowledge, this idea is due to Harris and Labesse (e.g. [Lab]).) We would like to define the "$\Pi^{\infty,p}$-part" of $H_c(\mathrm{Ig}_b, \mathscr{L}_\xi)$. Write

$$H_c(\mathrm{Ig}_b, \mathscr{L}_\xi) = \sum_{\pi^{\infty,p} \otimes \rho_p} n(\pi^{\infty,p} \otimes \rho_p) \cdot [\pi^{\infty,p} \otimes \rho_p]$$

where $n(\pi^{\infty,p} \otimes \rho_p) \in \mathbb{Z}$ and the sum runs over irreducible admissible representations of $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$. Then define

$$H_c(\mathrm{Ig}_b, \mathscr{L}_\xi)\{\Pi^{\infty,p}\} := \sum_{\substack{\pi^{\infty,p} \otimes \rho_p \\ BC(\pi^{\infty,p}) \simeq \Pi^{\infty,p}}} n(\pi^{\infty,p} \otimes \rho_p) \cdot [\rho_p].$$

Also define $\widetilde{R}_l(\Pi) := \sum_{\pi^\infty} R_{\xi,l}(\pi^\infty)$ where π^∞ are representations such that $BC(\pi^{\infty,p}) \simeq \Pi^{\infty,p}$.

We are ready to state our results on the cohomology of Igusa varieties and Shimura varieties. First, $H_c(\mathrm{Ig}_b, \mathscr{L}_\xi)\{\Pi^{\infty,p}\}$ is explicitly described in terms of Π_p. (For a precise statement, see Theorem 6.1.) In fact, in (Case END), the description depends on not only Π_p but also $\Pi_{1,p}$ and $\Pi_{2,p}$. This result, together with (1.2) and our knowledge of $\mathrm{Mant}_{b,\mu}$, leads to a description of $R_l(\Pi)$ in $\mathrm{Groth}(W_{F_w})$. (In fact, as a by-product, we know not only $\widetilde{R}_l(\Pi)$ but also the contribution to $\widetilde{R}_l(\Pi)$ from each Newton polygon stratum.) Up to some explicit nonzero multiplicity and character twist, it turns out that (Theorem 6.4)

- (Case ST) $\widetilde{R}_l(\Pi)|_{W_{F_w}}$ is the local Langlands image of Π^1.
- (Case END) $\widetilde{R}_l(\Pi)|_{W_{F_w}}$ is the local Langlands image of Π_1 or Π_2.

In particular $\dim \widetilde{R}_l(\Pi) = n$ in (Case ST) whereas $\dim \widetilde{R}_l(\Pi) = n_1$ or n_2 in (Case END), up to multiplicity. Moreover, it can be shown that $\widetilde{R}_l(\Pi)$ is a true representation concentrated in $H^{n-1}(\mathrm{Sh}, \mathscr{L}_\xi)$. So far we indicated how the local-global compatibility is established at w on the condition that $p = w|_{\mathbb{Q}}$ splits in E. This can be extended to all places not dividing l. (See the proof of Proposition 7.4.)

With the above result on the cohomology of Shimura varieties, it is not too difficult to deduce Theorem 1.2. Harris proposed a strategy generalizing [BR93] (which may be regarded as the case with $m = 2$ and $n = 3$) and its outline is as follows. As the notation Π is already being used, let Π^0 denote the cuspidal automorphic representation of $GL_m(\mathbb{A}_F)$ in the theorem. If m is odd, use $n = m$ and $\Pi^1 = \Pi^0$. Then $\widetilde{R}_l(\Pi)$ is essentially the desired Galois representation. If m is even, use $n = m + 1$ and $\Pi_1 = \Pi^0$. In this case, it is possible to choose Π_2 so that $\widetilde{R}_l(\Pi)$ is essentially the desired representation, namely it corresponds to Π_1 rather than Π_2. To prove this, we carry out explicit computation of signs in real endoscopy. The slight regularity assumption of Theorem 1.2 ensures that a good choice of Π_2 exists. Actually our construction of Galois representations a priori relies on additional assumptions on F and Π, for technical reasons including the issue of local base change. To remove these assumptions we apply a "patching" argument as in [BR89] and [HT01]. (See the proof of Theorem 7.5.)

We have explained how a result on $H_c(\mathrm{Ig}_b, \mathscr{L}_\xi)\{\Pi^{\infty,p}\}$ implies a result on $\widetilde{R}_l(\Pi)$, thus enabling us to prove Theorem 1.2. The remaining problem is the computation of $H_c(\mathrm{Ig}_b, \mathscr{L}_\xi)\{\Pi^{\infty,p}\}$, which is at the core of our work. The starting point is the following stable trace formula ([Shi09]), which stabilizes the counting point formula for Igusa varieties ([Shi10]).

$$\mathrm{tr}\,(\phi^{\infty,p}\cdot\phi'_p|\iota_l H_c(\mathrm{Ig}_b,\mathscr{L}_\xi)) \quad = \quad |\ker^1(\mathbb{Q},G)|\sum_{G_{\vec{n}}}\iota(G,G_{\vec{n}})ST_e^{G_{\vec{n}}}(\phi^{\vec{n}}_{\mathrm{Ig}}) \tag{1.3}$$

The notations should be explained. The function $\phi^{\infty,p}\cdot\phi'_p \in C_c^\infty(G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p))$ is any acceptable function in the sense of [Shi09, Def 6.2]. The sum is taken over elliptic endoscopic groups $G_{\vec{n}}$ for G (§3.2). The test functions $\phi^{\vec{n}}_{\mathrm{Ig}}$ away from p,∞ are the Langlands-Shelstad transfer of $\phi^{\infty,p}$. See §5.3 for $\phi^{\vec{n}}_{\mathrm{Ig},p}$ and $\phi^{\vec{n}}_{\mathrm{Ig},\infty}$. We remark that an analogous formula for Shimura varieties was obtained earlier by Kottwitz ([Kot92b], [Kot90]) and plays a central role in the computation of Frobenius action on the cohomology of Shimura varieties at the primes of good reduction. Kottwitz' formula is a key input in [Kot92a], [Mor10], [CHLa], to name a few. However, his formula is not needed in [HT01] and our work, where the trace formula for Igusa varieties is importantly used.

We can proceed from (1.3) using similar techniques as in work of Clozel, Harris and Labesse ([Lab], [CHLb]) on the base change and endoscopic transfer for unitary groups. The point is that each summand in (1.3) is (up to constant) equal to the geometric side of the twisted trace formula for $G\times_{\mathbb{Q}} E$ with respect to the Galois action of the nontrivial element $\theta\in\mathrm{Gal}(E/\mathbb{Q})$. This in turn equals the spectral side of the trace formula, expanded in terms of θ-stable automorphic representations of $G_{\vec{n}}(\mathbb{A}_E)$. By a result of Jacquet-Shalika, we can separate a string of Hecke eigenvalues, or the $\Pi^{\infty,p}$-part from the spectral expansion. It turns out that this process singles out a unique term in the spectral expansion in (Case ST) and two terms in (Case END). Using various character identities, an explicit description of $H_c(\mathrm{Ig}_b,\mathscr{L}_\xi)\{\Pi^{\infty,p}\}$ is finally obtained. In doing so, the most interesting and perhaps mysterious character identities are those at p (Lemma 5.10). These arise naturally from the stabilization process for (1.3) at p and reflect the structure of Newton stratification of Sh.

So far we sketched the proof of Theorem 1.2. We end by mentioning latest work of others in a similar direction. Recently Morel announced a result ([Mor10, Cor 8.4.9, 8.4.10]) similar to Theorem 1.2 and its corollary, as an application of her study of *non-compact* unitary Shimura varieties. (In contrast, our work offers no information about the geometry or cohomology of those Shimura varieties.) When m is odd, she constructed $R_l(\Pi)$ up to multiplicity and proved (i) of Theorem 1.2 at the places y where Π_y is unramified, as well as (iii). Now suppose that m is even. If $m\equiv 2 \mod 4$ and $[F^+:\mathbb{Q}]$ is odd, she obtains the same result as in the case of odd m. Otherwise, she can still construct $\wedge^2 R_l(\Pi)$ up to multiplicity and prove an analogue of (i) and (iii) at unramified places. (Actually Morel states the main results only in the case $F^+=\mathbb{Q}$, but it seems that her results extend to the cases mentioned above without much difficulty.) Perhaps the most important input in Morel's work is the counting point formula (and its stabilization) for the special fibers of noncompact Shimura varieties (cf. [Mor05], [Mor08]), which generalizes [Kot92b] and [Kot90] to the noncompact setting. On the other hand, Clozel, Harris and Labesse ([CHLa]) have succeeded to construct even dimensional Galois representations attached to Π as in our work under a similar restriction on Π_∞. Their method shares some common features with ours in that they use the same compact Shimura varieties and the endoscopic transfer from $U(m)\times U(1)$ to $U(m+1)$ as well as the twisted trace formula. The essential difference is that they employ (the stabilization of) Kottwitz' counting point formula ([Kot92b], [Kot90]) and obtain information only at *unramified* (good) places. In contrast, our method makes use of the counting point formula for Igusa varieties and can deal with bad places. Actually we can even describe the compact support cohomology of each Newton stratum (at a possibly bad place) in the endoscopic setting, in a suitable sense.

It is worth noting that there has been a precise conjecture about the cohomology of PEL Shimura varieties of type (A) or (C) for many years. (See the formula on page 201 of [Kot90]. Compare with [LR92, Thm B, p.293] in the case of $U(3)$.) If fully established, the conjecture would imply our result on $\widetilde{R}_l$ as well as our main theorem. So the issue has been not to speculate what should be true in general but to justify what is already expected about the cohomology of Shimura varieties, in as many cases as possible. To our knowledge, our work is the first to unconditionally describe the Galois representations in the endoscopic part of the cohomology at bad places, even in the case of $U(3)$.

We briefly outline the structure of the article. We review background materials in §2-§6. In section 2 we define the functor $\mathrm{Mant}_{b,\mu}$ and recall the results of [HT01] on $\mathrm{Mant}_{b,\mu}$. Sections 3 and 4 are devoted to the discussion of endoscopy, local base change and the twisted trace formula for unitary similitude groups. It is worth remarking that the functions at infinity reviewed in §3.5 and §4.3 play an important role in the study of the cohomology of Shimura varieties and Igusa varieties. In (Case END), the sign calculation of §3.6 is crucial. On the other hand, the functions at infinity allow us to simplify the geometric and the spectral sides of the twisted trace formula (§4.5). In section 5 we recall the definition of Shimura varieties and Igusa varieties, Mantovan's formula and the stable trace formula for Igusa varieties (Propositions 5.2 and 5.3) as well as some other facts. It is important to allow the prime p (where the local structure is to be analyzed) to be ramified in F. As some of our references ([Man05], [Shi09] and [Shi10]) assume that p is unramified in F, we explain how the results there can be extended to our setting. We also need a stable trace formula for G, which will be used to control automorphic multiplicity (Corollary 6.5.(iv).) This is essentially used in obtaining later corollaries. The subsections 5.5 and 5.6 are devoted to an explicit version of "endoscopy for Igusa varieties" at p. Although the local endoscopy of G at p is banal, our discussion clarifies how the global endoscopy for G interacts with the $J_b(\mathbb{Q}_p)$-representations in $H_c(\mathrm{Ig}_b, \mathscr{L}_\xi)$, which encode certain information about bad reduction. The main body of argument is given in §6 and §7. We mainly consider (Case ST) and (Case END), which are introduced in the beginning of §6.1. (See Remark 6.11 for a comment on other cases.) The stable trace formula and the twisted trace formula are combined in the proof of Theorem 6.1, which is a key result of our paper. It is pleasant to see that Theorem 6.4 is derived from Theorem 6.1. Although this may not be very surprising in (Case ST), the computation is more curious in (Case END). In §6.2, we deduce several consequences from Theorem 6.1, Mantovan's formula and the known facts about the functor $\mathrm{Mant}_{b,\mu}$. In the proof of Corollaries 6.5, 6.7, 6.8 and 6.10 we borrow important ideas from Harris and Taylor. The last two corollaries yield the desired Galois representation by removing an unwanted multiplicity (and multiplying an obvious character), under the technical assumptions made in §5 and §6. In §7.1 and §7.2, we prove the main results on $R_l(\Pi)$. In the case of even dimensional Galois representations, it is crucial to make a good choice of an auxiliary Hecke character (Lemma 7.3). This relies on our computation of §3.6. Another important idea is to remove all extra technical assumptions by using patching argument for many quadratic extensions, which is due to [BR89] and [HT01]. In Corollary 7.9 we prove relevant cases of the Ramanujan-Petersson conjecture. Finally in section 7.3, we imitate the argument of [TY07] to prove a stronger result on the local-global compatibility and the last assertion of Theorem 1.2.

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1.1. **Notation and Convention.** Suppose that F is a number field or a local field. By this we mean that F is a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_v$ for some place v of $\mathbb{Q}$. (We allow $v = \infty$.) The Weil group W_F of F is defined in [Tat79]. Let G be a connected reductive group over F. Denote by $\widehat{G}$ the dual group of G, which is a complex Lie group. Define the L-group ${}^L G := \widehat{G} \rtimes W_F$ of G via a semi-product. (See [Bor79] for precise definition.) If F is a finite extension of K, then $R_{F/K}G$ denotes the Weil restriction of scalars (whose set of K-points is the same as $G(F)$). Let $H^1(F,G) := H^1(\mathrm{Gal}(\overline{F}/F), G(\overline{F}))$. When F is a number field, write $\ker^1(F,G)$ for the kernel of $H^1(F,G) \to \prod_v H^1(F_v,G)$ where v runs over all places of F. Similarly define $\ker^1(F,H)$ for any complex Lie group H equipped with the action of $\mathrm{Gal}(\overline{F}/F)$ factoring through a finite quotient.

Let F be a number field and y be a place of F. Write $k(y)$ for the residue field of F_y. Let I_{F_y} denote the inertia group of W_{F_y}. Denote by Frob_y the geometric Frobenius element of W_{F_y}/I_{F_y}, namely the element inducing $x \mapsto x^{-|k(y)|}$ in $\mathrm{Gal}(\overline{k(y)}/k(y))$.

When L is a finite extension of a number field F, we denote by $\mathrm{Ram}_{L/F}$ (resp. $\mathrm{Unr}_{L/F}$, $\mathrm{Spl}_{L/F}$) the set of finite places of F which are ramified (resp. unramified, completely split) in L. When $\Pi \in \mathrm{Irr}(G(\mathbb{A}))$, let $\mathrm{Ram}_{\mathbb{Q}}(\Pi)$ denote the set of primes p of $\mathbb{Q}$ such that there exists a place v dividing p where Π_v is ramified.

Suppose that F is a local non-archimedean field. Denote by $D_{F,\lambda}$ the central division algebra over F with Hasse invariant $\lambda \in \mathbb{Q}/\mathbb{Z}$. Let $\mathrm{Art}_F : F^\times \xrightarrow{\sim} W_F^{\mathrm{ab}}$ be the local Artin map normalized so that a uniformizer of $F^\times$ maps to a lift of a geometric Frobenius element. Let $|\cdot|_F : F^\times \to \mathbb{R}^\times_{>0}$ denote the character which is trivial on $\mathcal{O}_F^\times$ and maps the inverse of any uniformizer to the cardinality of the residue field. Set $|\cdot|_{W_F} := |\cdot|_F \circ \mathrm{Art}_F^{-1}$. There is a unique way to choose $|\cdot|_F^{1/2} : F^\times \to \mathbb{R}^\times_{>0}$. When $\iota_l : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ is fixed, we often write $|\cdot|_F^{1/2}$ for $\iota_l^{-1}|\cdot|_F^{1/2}$ by abuse of notation.

Keep assuming that F is a local non-archimedean field. We denote by $\mathrm{Irr}(G(F))$ (resp. $\mathrm{Irr}_l(G(F))$) the set of all isomorphism classes irreducible admissible representations of $G(F)$ on vector spaces over $\mathbb{C}$ (resp. $\overline{\mathbb{Q}}_l$). When π is an irreducible unitary representation of $G(F)$ (modulo split component in the center), π may also be viewed as an irreducible admissible representation by taking smooth vectors, so we may say $\pi \in \mathrm{Irr}(G(F))$. The subset $\mathrm{Irr}^2(G(F))$ of $\mathrm{Irr}(G(F))$ is the one consisting of (essentially) square-integrable representations. Let $C_c^\infty(G(F))$ denote the space of smooth and compactly supported $\mathbb{C}$-valued functions on $G(F)$. Let P be an F-rational parabolic subgroup of G with a Levi subgroup M. For each $\pi_M \in \mathrm{Irr}(M(F))$ and $\pi \in \mathrm{Irr}(G(F))$, we can define the normalized Jacquet module $J_P^G(\pi)$ and the normalized parabolic induction $\text{n-ind}_P^G(\pi_M)$ so that $J_P^G(\pi)$ (resp. $\text{n-ind}_P^G(\pi_M)$) is an admissible representation of $M(F)$ (resp. $G(F)$). The induced representation $\text{n-ind}_P^G(\pi_M)$ will often be written as $\text{n-ind}_M^G(\pi_M)$ when working inside of $\mathrm{Groth}(G(F))$ or computing traces, since different choices of P give the same result. Define a function $D_{G/M}$ on $M(F)$ by $D_{G/M}(m) = \det(1 - \mathrm{ad}(m))|_{\mathrm{Lie}\,(G)/\mathrm{Lie}\,(M)}$ and a character $\delta_P : M(F) \to \mathbb{R}^\times_{>0}$ by $\delta_P(m) = |\det(\mathrm{ad}(m))|_{\mathrm{Lie}\,(P)/\mathrm{Lie}\,(M)}|_F$. In case $G = GL_n$ and $M = \prod_i GL_{n_i}$ ($\sum_i n_i = n$), consider $\pi_i \in \mathrm{Irr}(GL_{n_i}(F))$. Denote by $\boxplus_i \pi_i$ the Langlands subquotient of $\text{n-ind}_P^G(\otimes_i \pi_i)$ (cf. [BW00, Ch IV], [Sil78]), which is independent of the choice of P. For any $s \in \mathbb{Z}_{>0}$ and a supercuspidal $\pi \in \mathrm{Irr}(GL_n(F))$, let $\mathrm{Sp}_s(\pi) \in \mathrm{Irr}^2(GL_{sn}(F))$ denotes the generalized Steinberg representation ([HT01, p.32]). Let $e(G) \in \{\pm 1\}$ denote the Kottwitz sign defined in [Kot83]. When $F = \mathbb{Q}_v$, we often write $e_v(G)$ for $e(G)$. The definitions in this paragraph make sense for $F = \mathbb{R}$ (except $\mathrm{Sp}_s(\pi)$) using the usual absolute value $|\cdot|$ on $\mathbb{R}$ and the infinitesimal equivalence between representations of $G(\mathbb{R})$.

Assume that G is an unramified group over a non-archimedean field F. Choose a hyperspecial group $K \subset G(F)$. Define a Haar measure on $G(F)$ so that K has volume 1. Define $\mathscr{H}^{\mathrm{ur}}(G(F))$ to be the $\mathbb{C}$-subspace of $C_c^\infty(G(F))$ consisting of bi-K-invariant functions. The convolution equips $\mathscr{H}^{\mathrm{ur}}(G(F))$ with $\mathbb{C}$-algebra structure with char_K being the multiplicative identity. Let $\mathrm{Irr}^{\mathrm{ur}}(G(F))$ denote the subset of $\mathrm{Irr}(G(F))$ consisting of unramified representations of $G(F)$. For each $\pi \in \mathrm{Irr}^{\mathrm{ur}}(G(F))$, define $\chi_\pi : \mathscr{H}^{\mathrm{ur}}(G(F)) \to \mathbb{C}$ by $f \mapsto \mathrm{tr}\,\pi(f)$. The association $\pi \mapsto \chi_\pi$ gives a natural bijection from $\mathrm{Irr}^{\mathrm{ur}}(G(F))$ onto the set of $\mathbb{C}$-algebra morphisms $\mathscr{H}^{\mathrm{ur}}(G(F)) \to \mathbb{C}$. (To see the inverse exists, use [Bor79, 7.1, 9.5].)

For a number field F and a finite set S consisting of places of F, we denote by $\mathbb{A}_F^S$ the restricted product of F_v for $v \notin S$. In case $F = \mathbb{Q}$, write $\mathbb{A}^S$ for $\mathbb{A}_{\mathbb{Q}}^S$ and $\mathbb{A}_S$ for $\prod_{v\in S} \mathbb{Q}_v$. Define $\mathrm{Irr}(G(\mathbb{A}_F^S))$, $C_c^\infty(G(\mathbb{A}_F^S))$ and $\mathscr{H}^{\mathrm{ur}}(G(\mathbb{A}_F^S))$ via restricted product, where the last one makes sense under the assumption that G_{F_v} is unramified for all $v \notin S$. The normalized induction is defined in this adelic context. Let $\mathrm{Art}_F : \mathbb{A}_F^\times/F^\times \stackrel{\sim}{\to} W_F^{\mathrm{ab}}$ denote the global Artin map, which is compatible with the local Artin map defined above.

Let G be a connected reductive group over $\mathbb{Q}$. Write A_G for the maximal $\mathbb{Q}$-split torus in the center of G and define $A_{G,\infty} := A_G(\mathbb{R})^0$. Let K_∞ be a maximal compact subgroup of $G(\mathbb{R})$. Let ξ be an irreducible finite dimensional representation of $G(\mathbb{C})$. Then the restriction of ξ to $A_{G,\infty}$ gives a character $\chi_\xi : A_{G,\infty} \to \mathbb{C}^\times$. Define $C_c^\infty(G(\mathbb{R}), \chi_\xi)$ to be the space of smooth $\mathbb{C}$-valued bi-K_∞-finite functions f on $G(\mathbb{R})$ which are compactly supported modulo $A_{G,\infty}$ and such that $f(ag) = \chi_\xi(a)f(g)$ for all $a \in A_{G,\infty}$ and $g \in G(\mathbb{R})$.

We frequently confuse an isomorphism class or an equivalence class with its member. For instance, when we write $\pi \in \mathrm{Irr}_l(G(\mathbb{Q}_p))$, it means that π is an irreducible admissible representation of $G(\mathbb{Q}_p)$ on a $\overline{\mathbb{Q}}_l$-vector space.

Finally let us agree that $(z/\overline{z})^{N/2}$ $(N \in \mathbb{Z})$ denotes $e^{iN\theta}$ for $z = re^{i\theta} \in \mathbb{C}^\times$ with $r \in \mathbb{R}_{>0}$ and $\theta \in \mathbb{R}/2\pi i\mathbb{Z}$.

2. Rapoport-Zink spaces of EL-type

Let p and l be prime numbers such that $p \neq l$. The aim of §2 is to recollect the description of the cohomology of certain Rapoport-Zink spaces, which will be incorporated into various versions of "Mant" functors defined below. We describe these functors in terms of the local Langlands correspondence and study their properties in the cases which are relevant to the Shimura varieties of §5. We will freely adopt notations from §3.1 such as I_n, $P_{n-h,h}$, $GL_{n-h,h}$, and so on.

2.1. $B(G)$ and isocrystals.

Let G be a connected reductive group over $\mathbb{Q}_p$. Let $L := \mathrm{Frac}W(\overline{\mathbb{F}}_p)$. Denote by σ the Frobenius on L which induces the p-th power map on the residue field. Define $B(G)$ to be the set of equivalence classes in $G(L)$ where $x, y \in G(L)$ are equivalent if there exists $g \in G(L)$ such that $x = g^{-1}yg^\sigma$. The set $B(G)$ classifies the isomorphism classes of isocrystals (over $\overline{\mathbb{F}}_p$) with G-structure in the sense of Rapoport and Richartz ([RR96, 3.3, 3.4.(i)]). For a $\overline{\mathbb{Q}}_p$-morphism $\mu : \mathbb{G}_m \to G$, Kottwitz defined a finite subset $B(G, \mu)$ of $B(G)$. The set $B(G, \mu)$ often provides parameters for the Newton polygon stratification in the context of Shimura varieties ([Har01, §4], [Man05], cf. [Shi09, §5]).

Let T be a maximal torus of G defined over $\mathbb{Q}_p$. Let $\Omega = \Omega(G, T)$ be the Weyl group over $\overline{\mathbb{Q}}_p$. Put $N(G) := ((X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q})/\Omega)^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$. There is a Newton map ([Kot85, §4], [RR96, 1.7-1.9])

$$\bar{\nu}_G : B(G) \to N(G)$$

which is useful in describing the set $B(G)$.

Suppose that G is a finite product of connected reductive $\mathbb{Q}_p$-groups G_i. Write $\mu = \prod_i \mu_i$ for $\mu_i : \mathbb{G}_m \to G_i$. Then we have a natural identification

$$B(G, \mu) = \prod_i B(G_i, \mu_i).$$

2.2. $\mathrm{Mant}_{b,\mu}$ functor.

Let $n \in \mathbb{Z}_{>0}$ and $\Phi_p(F) := \mathrm{Hom}_{\mathbb{Q}_p}(F, \overline{\mathbb{Q}}_p)$. Consider a quadruple (F, V, μ, b) where

(i) F is a finite extension of $\mathbb{Q}_p$. (We do not assume that F is unramified over $\mathbb{Q}_p$.)
(ii) $V = F^n$ is an F-vector space. Let $G := \mathrm{Res}_{F/\mathbb{Q}_p} GL_F(V)$.
(iii) $\mu : \mathbb{G}_m \to G$ is a homomorphism over $\overline{\mathbb{Q}}_p$ (up to $G(\overline{\mathbb{Q}}_p)$-conjugacy) which induces a weight decomposition $V \otimes_F \overline{F} = V_0 \oplus V_1$, defined over a finite extension of $\mathbb{Q}_p$, where $\mu(z)$ acts on V_i by z^i for $i = 0, 1$.
(iv) $b \in B(G, -\mu)$.

Giving μ is equivalent to giving a pair of nonnegative integers (p_σ, q_σ) for each $\sigma \in \Phi_p(F)$ such that $p_\sigma + q_\sigma = n$. Given such data, the corresponding μ is represented by the homomorphism $\overline{\mathbb{Q}}_p^\times \to \prod_{\sigma\in\Phi_p(F)} GL_n(\overline{\mathbb{Q}}_p)$ given by

$$z \mapsto \prod_\sigma \mathrm{diag}(\underbrace{z,\dots,z}_{p_\sigma},\underbrace{1,\dots,1}_{q_\sigma}).$$

Roughly speaking, $B(G,-\mu)$ classifies isocrystals with F-action up to isomorphism (or Barsotti-Tate groups with $\mathcal{O}_F$-action up to isogeny, via covariant Dieudonné theory) whose Hodge polygons are determined by μ. Note that $N(G)$ may be identified with the set of unordered n-tuple of rational numbers. In fact $\bar\nu_G$ is injective, thus each $b \in B(G)$ is uniquely characterized by its image under the Newton map

$$\bar\nu_G(b) = (\underbrace{\lambda_1,\dots,\lambda_1}_{m_1},\underbrace{\lambda_2,\dots,\lambda_2}_{m_2},\dots,\underbrace{\lambda_r,\dots,\lambda_r}_{m_r})$$

where $r \in \mathbb{Z}_{>0}$ and $\lambda_i \in \mathbb{Q}$ and $m_i \in \mathbb{Z}_{>0}$ for $1 \le i \le r$. We may and will assume $\lambda_1 < \cdots < \lambda_r$. See [Shi09, Ex 4.3] for the explicit condition on r, $\{\lambda_i\}$ and $\{m_i\}$ in order that $b \in B(G,-\mu)$.

The reflex field E is by definition the fixed field in $\overline{\mathbb{Q}}_p$ of the stabilizer in $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of the pairs $\{(p_\sigma,q_\sigma)\}_{\sigma\in\Phi_p(F)}$. We will be only concerned with the case $\sum_\sigma p_\sigma \le 1$. If $p_\sigma = 0$ for all σ then $E = \mathbb{Q}_p$. If $p_\sigma = 1$ for a unique σ then E is identified with $\sigma(F) \subset \overline{\mathbb{Q}}_p$.

The datum (F,V,μ,b) gives rise to a formal scheme $\mathcal{M}_{b,\mu}$ over $\mathrm{Spf}\,\mathcal{O}_{\widehat{F}^{\mathrm{ur}}}$ representing a moduli problem for Barsotti-Tate groups with $\mathcal{O}_F$-action. In fact $\mathcal{M}_{b,\mu}$ is non-canonically isomorphic to $\mathbb{Z}$-copies of the Lubin-Tate deformation space for formal $\mathcal{O}_F$-modules of dimension 1 and height n, where the latter is studied in [Car90], [HG94] and [HT01, Ch 2], for instance. (See [RZ96, 3.78-3.79] for details. In the description of $\mathcal{M}_{b,\mu}$ in Proposition 3.79, replace $\mathrm{Spf}\,W(\overline{\mathbb{F}}_p)$ with $\mathrm{Spf}\,\mathcal{O}_{\widehat{F}^{\mathrm{ur}}}$. Although Rapoport and Zink discuss the same moduli space as in our case, there is a difference in the choice of b. See (1.47) there. The source of the difference is that Barsotti-Tate groups of $\mathcal{O}_F$-slope λ correspond to isocrystals of slope $-\lambda$ in our convention but to isocrystals of slope $1-\lambda$ in that book.)

There is a standard construction to obtain a tower of rigid analytic spaces $\mathcal{M}^{\mathrm{rig}}_{b,\mu,U}$ over $\widehat{F}^{\mathrm{ur}}$ for open compact subgroups U of $GL_n(\mathcal{O}_F)$ ([RZ96, Ch 5]), where $\mathcal{M}^{\mathrm{rig}}_{b,\mu,GL_n(\mathcal{O}_F)}$ coincides with the rigid analytic space attached to $\mathcal{M}_{b,\mu}$. (Here $GL_n(\mathcal{O}_F)$ is regarded as the stabilizer of the standard lattice $\mathcal{O}_F^n$ inside V.) We consider the étale cohomology of Rapoport-Zink spaces in the sense of Berkovich ([Ber93]), for which we use the following abbreviated notation:

$$H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu,U}) := H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu,U} \times_{\widehat{E^{\mathrm{ur}}}} \widehat{\overline{E^{\mathrm{ur}}}}, \overline{\mathbb{Q}}_l).$$

This $\overline{\mathbb{Q}}_l$-vector space has the structure of a smooth representation of $J_b(\mathbb{Q}_p) \times W_E$. The last action commutes with the action of $G(\mathbb{Q}_p)$ on the tower of $\mathcal{M}^{\mathrm{rig}}_{b,\mu,U}$ via Hecke correspondences. Details about these actions can be found in [RR96, Ch 5], [Far04, Ch 4] and [Man04].

Define[1] the functor $\mathrm{Mant}_{b,\mu} : \mathrm{Groth}(J_b(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p)\times W_E)$ by

$$\mathrm{Mant}_{b,\mu}(\rho) := \sum_{i,j\ge 0} (-1)^{i+j} \varinjlim_U \mathrm{Ext}^i_{J_b(\mathbb{Q}_p)\text{-smooth}}(H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu,U}),\rho))(-D). \tag{2.1}$$

in the notation of [Man05] and [Man]. (Our $J_b(\mathbb{Q}_p)$ is denoted by T_b in [Man05]. Our $\mathrm{Mant}_{b,\mu}$ is $\mathcal{E}_b$ in [Man].) Here D is the dimension of $\mathcal{M}^{\mathrm{rig}}_{b,\mu,U}$ and $(-D)$ is the Tate twist. The Ext-groups are taken in the category of smooth representations of $J_b(\mathbb{Q}_p)$ and the limit is over open compact subgroups U as above. Since the Ext-groups in (2.1) vanish beyond a certain degree and yield finite length representations for each U ([Far04, §4.4]), $\mathrm{Mant}_{b,\mu}$ is well-defined.

[1] Although we named this functor after Mantovan's work clarifying its relationship with the cohomology of Shimura varieties, it should be noted that (variants of) $\mathrm{Mant}_{b,\mu}$ were considered previously by several authors, as in [Rap95], [Har01], [Far04].

2.3. **Local Langlands correspondence.** Let F be a finite extension of $\mathbb{Q}_p$. Harris-Taylor ([HT01]) and Henniart ([Hen00]) proved that there is a natural bijection

$$\mathrm{rec}_{n,F} : \mathrm{Irr}(GL_n(F)) \to \text{WD-Rep}_n(W_F)$$

where $\text{WD-Rep}_n(W_F)$ is the set of isomorphism classes of Frobenius semisimple n-dimensional Weil-Deligne representations of W_F on $\mathbb{C}$-vector spaces. See [HT01, p.2] for the characterizing properties of $\mathrm{rec}_{n,F}$. We will often use the following normalization.

$$\mathscr{L}_{n,F}(\pi) := \mathrm{rec}_{n,F}(\pi^\vee) \otimes |\cdot|_{W_F}^{-(n-1)/2}$$

2.4. **The case of dimension 0 and 1.** Let $n \in \mathbb{Z}_{>0}$. Fix a finite extension F over $\mathbb{Q}_p$, an embedding $\tau^0 : F \hookrightarrow \overline{\mathbb{Q}}_p$ and an isomorphism $\iota_l : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. By abuse of notation, $\iota_l^{-1}|\cdot|_F^{1/2} : F^\times \to \overline{\mathbb{Q}}_l^\times$ will be denoted by $|\cdot|_F^{1/2}$ or $|\cdot|^{1/2}$.

Write $\mathcal{G}_n := R_{F/\mathbb{Q}_p} GL_n$. Let $\mu_{n,\tau^0} : \mathbb{G}_m \to \mathcal{G}_n \times_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \simeq \prod_{\sigma\in\Phi_p(F)} (GL_n)_{\overline{\mathbb{Q}}_p}$ be the $\overline{\mathbb{Q}}_p$-morphism given by

$$z \mapsto \left(\begin{pmatrix} z & 0 \\ 0 & I_{n-1} \end{pmatrix}_{\sigma=\tau^0}, \ (I_n)_{\sigma\neq\tau^0} \right).$$

Let $\mu_{n,\text{ét}} : \mathbb{G}_m \to \mathcal{G}_n \times_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ denote the trivial map. Define $b_{n,0}, b_{0,n} \in B(\mathcal{G}_n)$ so that $\nu_{\mathcal{G}_n}(b_{n,0}) = (-1/n, \dots, -1/n)$ and $b_{0,n} = 1$. Observe that $b_{n,0} \in B(\mathcal{G}_n, -\mu_{n,\tau^0})$ and $b_{0,n} \in B(\mathcal{G}_n, -\mu_{n,\text{ét}})$. For $1 \le h \le n-1$, define $b_{n-h,h} \in B(\mathcal{G}_n)$ to be the image of $(b_{n-h,0}, b_{0,h})$ under the block diagonal embedding $\mathcal{G}_{n-h} \times \mathcal{G}_h \hookrightarrow \mathcal{G}_n$. Then $b_{n-h,h} \in B(\mathcal{G}_n, -\mu_{n,\tau^0})$. For any $0 \le h \le n-1$, define an F-group $J_{n-h,h} := D^\times_{F,1/(n-h)} \times GL_h$, where $D^\times_{F,1/(n-h)}$ is an inner form of GL_{n-h} coming from a division algebra with invariant $1/(n-h)$. Set $J_{0,n} := GL_n$. We see that $R_{F/\mathbb{Q}_p} J_{n-h,h}$ is isomorphic to J_b for $b = b_{n-h,h}$ $(0 \le h \le n)$. Let $P_{n-h,h}$ be the parabolic subgroup of GL_n whose (i,j)-component is zero exactly when $i > n-h$ and $j \le n-h$. Define a character

$$\bar{\delta}^{1/2}_{P_{n-h,h}} : J_{n-h,h}(F) \to \overline{\mathbb{Q}}_l^\times$$

by the relation $\bar{\delta}^{1/2}_{P_{n-h,h}}(g) = \delta^{1/2}_{P_{n-h,h}}(g^*)$ where $g^* \in GL_{n-h,h}(F)$ is any element whose conjugacy class is transferred from that of g.

Let us write $\mathrm{Mant}_{n-h,h}$ $(0 \le h \le n-1)$ and $\mathrm{Mant}_{0,n}$ for $\mathrm{Mant}_{b,\mu}$ when $(b,\mu) = (b_{n-h,h}, \mu_{n,\tau^0})$ and $(b,\mu) = (b_{0,n}, \mu_{n,\text{ét}})$, respectively. For each $0 \le h \le n$, define $\text{n-Mant}_{n-h,h}$ by the relation

$$\text{n-Mant}_{n-h,h}(\rho) := e(J_{n-h,h}) \cdot \mathrm{Mant}_{n-h,h}(\rho \otimes \bar{\delta}^{1/2}_{P_{n-h,h}}) \tag{2.2}$$

for every $\rho \in \mathrm{Groth}(J_{n-h,h}(F))$. Note that the Kottwitz sign $e(J_{n-h,h})$ equals $(-1)^{n-h-1}$ if $0 \le h \le n-1$ and 1 if $h = n$.

Lemma 2.1. *For each* $\pi \in \mathrm{Irr}_l(GL_n(F))$, $\mathrm{Mant}_{0,n}(\pi) = [\pi][\mathbf{1}]$ *where* $\mathbf{1}$ *is the trivial character of* $W_{\mathbb{Q}}$.

Proof. Follows from [Far04, Ex 4.4.8]. □

Recall that there exists a natural bijection $JL_n : \mathrm{Irr}^2(D^\times_{F,1/n}) \xrightarrow{\sim} \mathrm{Irr}^2(GL_n(F))$ uniquely characterized by a character identity ([DKV84]).

Proposition 2.2. (i) *If* $\pi \in \mathrm{Irr}_l(GL_n(F))$ *is supercuspidal,* $\text{n-Mant}_{n,0}(JL_n^{-1}(\pi)) = [\pi][\mathscr{L}_{n,F}(\pi)]$.

(ii) *For* $s \in \mathbb{Z}_{>0}$, $g = n/s \in \mathbb{Z}_{>0}$ *and a supercuspidal* $\pi \in \mathrm{Irr}_l(GL_g(F))$, $\text{n-Mant}_{n,0}(JL_n^{-1}(Sp_s(\pi)))$ *equals*

$$\sum_{j=1}^{s} \left([Sp_j(\pi) \boxplus \pi|\det|^j \boxplus \cdots \boxplus \pi|\det|^{s-1}] \otimes [\mathscr{L}_{g,F}(\pi|\det|^{j-1}) \otimes |\cdot|^{g(1-s)/2}] \right)$$

(iii) *For each* $\rho_1 \in \mathrm{Irr}_l(J_{n-h,0}(F))$ *and* $\rho_2 \in \mathrm{Irr}_l(J_{0,h}(F))$,

$$\text{n-Mant}_{n-h,h}(\rho_1 \otimes \rho_2) = \text{n-ind}^{GL_n(F)}_{P_{n-h,h}(F)}(\text{n-Mant}_{n-h,0}(\rho_1) \otimes \text{n-Mant}_{0,h}(\rho_2)) \otimes |\cdot|^{-h/2}_{W_F}$$

in $\mathrm{Groth}(GL_n(F) \times W_F)$.

Proof. Both (i) and (ii) follow from a reinterpretation of [HT01, Thm VII.1.3, VII.1.5] in our language. We elaborate on this point.

Let $J := D^{\times}_{F,1/n}$. According to [Har05, Thm 4.3.11], in his notation,

$$\Psi_n(\rho) := \sum_i (-1)^i \mathrm{Hom}_J(\Psi^i_{c,n}, \rho)$$

coincides with $[\pi][\mathscr{L}_{n,F}(\pi)]$ if $\rho = JL_n^{-1}(\pi)$ for a supercuspidal representation π. On the other hand, $\Psi_n(\rho)$ is identified with $\mathrm{Mant}_{n,0}(\rho)$ for any $\rho \in \mathrm{Irr}_l(J)$ by adapting [Man04, Thm 8.7] to our case. Indeed, in the identity of that theorem, the right hand side is nothing but $\mathrm{Mant}_{n,0}(\rho)$ whereas the left hand side is easily seen to be the same as $\Psi_n(\rho)$ since the special fibers of the relevant Rapoport-Zink spaces are zero dimensional. Let us compare

$$\Psi_{F,l,n}(\rho) = \sum_i (-1)^{n-1-i} \Psi^i_{F,l,n}(\rho)$$

of [HT01, p.87-88] with $\Psi_n(\rho)$ above. Note that $\Psi_{F,l,n}(\rho)$ (resp. $\Psi_n(\rho)$) is defined via the Lubin-Tate deformation spaces (resp. the Rapoport-Zink spaces). Note that the Rapoport-Zink space of each fixed level is non-canonically isomorphic to $\mathbb{Z}$-copies of the Lubin-Tate space of the same level and that one of the copies is canonically isomorphic to the Lubin-Tate space. ([Str05, 2.3], cf. [Har05, p.49].) From this fact, it is not difficult to prove that $\Psi^i_{F,l,n}(\rho) \stackrel{\sim}{\to} \mathrm{Hom}_J(\Psi^i_{c,n}, \rho)$. In other words,

$$\Psi_{F,l,n}(\rho) = (-1)^{n-1}\Psi_n(\rho) = (-1)^{n-1}\mathrm{Mant}_{n,0}(\rho) = \text{n-Mant}_{n,0}(\rho)$$

in $\mathrm{Groth}(GL_n(F) \times W_F)$. Therefore [HT01, Thm VII.1.3, VII.1.5] imply our first two assertions. Note that $r_l(\pi)$ in their notation is isomorphic to $\mathscr{L}_{n,F}(\pi)$ in view of the relation of r_l with $\mathrm{rec}_{n,F}$ on page 237 of [HT01].

It remains to prove the third assertion. This result can be derived from [Har05, Prop 4.3.14, 4.3.17] (where p is allowed to ramify in F), which is already implicit in [HT01]. For simplicity of notation, we derive it from [Man08, Cor 5] (in case p is unramified in F), which implies in our case that

$$\mathrm{Mant}_{n-h,h}(\rho_1 \otimes \rho_2) = \mathrm{Ind}^{GL_n(F)}_{P_{n-h,h}(F)}(\mathrm{Mant}_{n-h,0}(\rho_1) \otimes \mathrm{Mant}_{0,h}(\rho_2)).$$

Here Ind is the non-normalized parabolic induction. From the above formula it is straightforward to deduce the assertion (iii) in view of Lemma 2.1 and the fact that (cf. [HT01, Lem II.2.9])

$$\text{n-Mant}_{n-h,0}(\rho_1 \otimes |\mathrm{Nm}|^{1/2}) = \text{n-Mant}_{n-h,0}(\rho_1) \otimes |\det|^{1/2} \otimes |\cdot|^{-1/2}_{W_F}$$

where $\mathrm{Nm} : D^{\times}_{F,1/(n-h)} \to F^{\times}$ denotes the reduced norm map. $\square$

We define a morphism $\mathrm{Red}^{n-h,h}$ as the composition

$$\mathrm{Groth}(GL_n(F)) \stackrel{J^{GL_n}_{P^{\mathrm{op}}_{n-h,h}} \otimes \delta^{1/2}_{P_{n-h,h}}}{\longrightarrow} \mathrm{Groth}(GL_{n-h,h}(F)) \stackrel{LJ_{n-h}\otimes \mathrm{id}}{\longrightarrow} \mathrm{Groth}(J_{n-h,h}(F))$$

where $J^{GL_n}_{P^{\mathrm{op}}_{n-h,h}}$ is the normalized Jacquet module and $LJ_{n-h} : \mathrm{Groth}(GL_{n-h}(F) \to \mathrm{Groth}(J_{n-h,0}(F))$ is the map defined by Badulescu ([Bad07]), which extends the inverse of the usual Jacquet-Langlands correspondence JL_{n-h}. Define

$$\text{n-Red}^{n-h,h} := \mathrm{Red}^{n-h,h} \otimes \bar{\delta}^{-1/2}_{P_{n-h,h}}.$$

Equivalently, $\text{n-Red}^{n-h,h} = (LJ_{n-h} \otimes \mathrm{id}) \circ J^{GL_n}_{P^{\mathrm{op}}_{n-h,h}}$. It is easy to see that $\mathrm{Mant}_{n-h,h} \circ \mathrm{Red}^{n-h,h} = e(J_{n-h,h}) \cdot \text{n-Mant}_{n-h,h} \circ \text{n-Red}^{n-h,h}$ for $0 \le h \le n$.

Proposition 2.3. *For any* $\pi \in \mathrm{Irr}_l(GL_n(F))$*, the following holds in* $\mathrm{Groth}(GL_n(F) \times W_F)$.

$$\sum_{h=0}^{n-1} \text{n-Mant}_{n-h,h}(\text{n-Red}^{n-h,h}(\pi)) = [\pi][\mathscr{L}_{n,F}(\pi)] \tag{2.3}$$

Proof. It is enough to check the proposition when π is the full parabolic induction from (essentially) square integrable representations of Levi subgroups (including $GL_n(F)$ itself), since such representations π generate $\mathrm{Groth}(GL_n(F))$ as a $\mathbb{Z}$-module ([Zel80, Cor 7.5]). By using Lemma 2.1 and Proposition 2.2, the left hand side of (2.3) can be computed in terms of Jacquet module and parabolic induction with help of the Bernstein-Zelevinsky classification. The computational detail is essentially the same as in the proof of [HT01, Thm VII.1.7]. □

3. Endoscopy of unitary similitude groups

3.1. **Setting.** We use the following notation.

- $\vec{n} = (n_i)_{i\in[1,r]}$ where $n_i, r \in \mathbb{Z}_{>0}$ and $[1,r] := \{1,2,\dots,r\}$.
- $GL_{\vec{n}} := \prod_{i\in[1,r]} GL_{n_i}$ and $i_{\vec{n}} : GL_{\vec{n}} \hookrightarrow GL_N$ ($N = \sum_i n_i$) is the embedding

$$(A_1,\dots,A_r) \mapsto \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \dots & \dots & \vdots \\ 0 & \cdots & 0 & A_r \end{pmatrix}.$$

  Define $\det : GL_{\vec{n}} \to GL_1$ by $\det(g) := \det(i_{\vec{n}}(g))$.
- Φ_n and I_n are the matrices in GL_n with entries $(\Phi_n)_{ij} = (-1)^{i+1}\delta_{i,n+1-j}$ and $(I_n)_{ij} = \delta_{i,j}$. Put $\Phi_{\vec{n}} := i_{\vec{n}}(\Phi_{n_1},\dots,\Phi_{n_r})$.
- ${}^tg$ denotes the transpose when g is a matrix.
- $P_{\vec{n}}$ is the upper triangular parabolic subgroup of GL_n containing $i_{\vec{n}}(GL_{\vec{n}})$ as a Levi subgroup.
- $\epsilon : \mathbb{Z} \to \{0,1\}$ is the unique map such that $\epsilon(n) \equiv n \mod 2$.
- $F = EF^+$ where F^+ (resp. E) is a totally real (resp. imaginary quadratic) extension of $\mathbb{Q}$.
- $\mathrm{Spl}_{F/F^+,\mathbb{Q}}$ is the set of all rational primes p such that every place of F^+ above p splits in F.
- $\mathrm{Unr}_{F/\mathbb{Q}}$ (resp. $\mathrm{Ram}_{F/\mathbb{Q}}$) is the set of all primes p which are unramified (resp. ramified) in F.
- $\tau : F \hookrightarrow \mathbb{C}$ is a $\mathbb{Q}$-algebra embedding and $\tau_E := \tau|_E$.
- c denotes the complex conjugation on $\mathbb{C}$ or any CM field.
- $\Phi_{\mathbb{C}} := \mathrm{Hom}_{\mathbb{Q}}(F,\mathbb{C})$, $\Phi_{\mathbb{C}}^+ := \mathrm{Hom}_{E,\tau_E}(F,\mathbb{C})$ and $\Phi_{\mathbb{C}}^- := c\Phi_{\mathbb{C}}^+$.
- $w^\star$ is a fixed element in $W_{\mathbb{Q}}\backslash W_E$.
- $\varpi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ is any Hecke character such that $\varpi|_{\mathbb{A}^\times/\mathbb{Q}^\times}$ equals the composite of $\mathrm{Art}_{\mathbb{Q}}$ and the natural surjective character $W_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(E/\mathbb{Q}) \stackrel{\sim}{\to} \{\pm 1\}$. Using the Artin map Art_E, we view ϖ also as a character $W_E \to \mathbb{C}^\times$.
- $\mathrm{Ram}_{\mathbb{Q}}(\varpi)$ is the set of all primes p such that ϖ is ramified at some place above p.

Define a $\mathbb{Q}$-group $G_{\vec{n}}$ by

$$G_{\vec{n}}(R) := \{(\lambda, g) \in GL_1(R) \times GL_{\vec{n}}(F\otimes_{\mathbb{Q}} R) : g\Phi_{\vec{n}}{}^tg^c = \lambda\Phi_{\vec{n}}\} \tag{3.1}$$

for any $\mathbb{Q}$-algebra R, where $g_i \in GL_{\vec{n}}(F\otimes_{\mathbb{Q}} R)$. Note that $G_{\vec{n}}$ is quasi-split over $\mathbb{Q}$. Also define

$$\mathbb{G}_{\vec{n}} := R_{E/\mathbb{Q}}(G_{\vec{n}} \times_{\mathbb{Q}} E)$$

and let θ denote the action on $\mathbb{G}_{\vec{n}}$ induced by (id, c) on $G_{\vec{n}} \times_{\mathbb{Q}} E$. We can identify the dual groups as follows.

$$\widehat{G}_{\vec{n}} \simeq \mathbb{C}^\times \times \prod_{\sigma\in\Phi_{\mathbb{C}}^+} GL_{\vec{n}}(\mathbb{C}) \qquad \text{and} \qquad \widehat{\mathbb{G}}_{\vec{n}} \simeq \mathbb{C}^\times \times \mathbb{C}^\times \times \prod_{\sigma\in\Phi_{\mathbb{C}}} GL_{\vec{n}}(\mathbb{C}). \tag{3.2}$$

The L-group ${}^LG_{\vec{n}} := \widehat{G}_{\vec{n}} \rtimes W_{\mathbb{Q}}$ is defined by the relation that $w(\lambda, g_\sigma)w^{-1} = (\lambda', g'_\sigma)$ where

$$(\lambda', g'_\sigma) \quad = \quad (\lambda, g_{w^{-1}\sigma}) \quad \text{or} \quad \left(\lambda \prod_{\sigma\in\Phi_{\mathbb{C}}^+} \det g_\sigma,\ \Phi_{\vec{n}}\, {}^tg^{-1}_{cw^{-1}\sigma}\Phi_{\vec{n}}^{-1}\right)$$

according as $w \in W_E$ or $w \notin W_E$, respectively. Similarly, ${}^L\mathbb{G}_{\vec{n}} := \widehat{\mathbb{G}}_{\vec{n}} \rtimes W_{\mathbb{Q}}$ is requiring that $w(\lambda_+, \lambda_-, g_\sigma)w^{-1}$ equal

$$(\lambda_+, \lambda_-, g_{w^{-1}\sigma}) \quad \text{or} \quad \left(\lambda_- \prod_{\sigma\in\Phi_{\mathbb{C}}^-} \det g_\sigma,\ \lambda_+ \prod_{\sigma\in\Phi_{\mathbb{C}}^+} \det g_\sigma,\ \Phi_{\vec{n}}\, {}^t g_{cw^{-1}\sigma}^{-1} \Phi_{\vec{n}}^{-1}\right)$$

according as $w \in W_E$ or $w \notin W_E$, respectively. Consider the map $BC_{\vec{n}} : {}^LG_{\vec{n}} \to {}^L\mathbb{G}_{\vec{n}}$ given by

$$(\lambda, (g_{\sigma,i})_{\sigma\in\Phi_{\mathbb{C}}^+}) \rtimes w \mapsto (\lambda, \lambda,\ (g_{\sigma,i})_{\sigma\in\Phi_{\mathbb{C}}^+},\ (g_{c\sigma,i})_{\sigma\in\Phi_{\mathbb{C}}^-}) \rtimes w.$$

Note that $(G_{\vec{n}},\ {}^LG_{\vec{n}}, 1, BC_{\vec{n}})$ is an endoscopic datum for $(\mathbb{G}_{\vec{n}}, \theta, 1)$ in the context of twisted endoscopy ([KS99, §2.1]).

Note that (3.1) may be used to equip $G_{\vec{n}}$ with a $\mathbb{Z}$-scheme structure by allowing R to be a $\mathbb{Z}$-algebra, and the same is true for $\mathbb{G}_{\vec{n}}$. For each prime p, put $K_p^{\vec{n}} := G_{\vec{n}}(\mathbb{Z}_p)$ and $\mathbb{K}_p^{\vec{n}} := G_{\vec{n}}(\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p) = \mathbb{G}_{\vec{n}}(\mathbb{Z}_p)$. If $p \in \mathrm{Spl}_{F/F^+,\mathbb{Q}}$ then $K_p^{\vec{n}}$ (resp. $\mathbb{K}_p^{\vec{n}}$) is a special subgroup of $G_{\vec{n}}(\mathbb{Q}_p)$ (resp. $\mathbb{G}_{\vec{n}}(\mathbb{Q}_p)$). In case $p \in \mathrm{Unr}_{F/\mathbb{Q}}$, $K_p^{\vec{n}}$ (resp. $\mathbb{K}_p^{\vec{n}}$) is a hyperspecial subgroup of $G_{\vec{n}}(\mathbb{Q}_p)$ (resp. $\mathbb{G}_{\vec{n}}(\mathbb{Q}_p)$). In that case we define unramified Hecke algebras $\mathscr{H}^{\mathrm{ur}}(G_{\vec{n}}(\mathbb{Q}_p))$ and $\mathscr{H}^{\mathrm{ur}}(\mathbb{G}_{\vec{n}}(\mathbb{Q}_p))$ using $K_p^{\vec{n}}$ and $\mathbb{K}_p^{\vec{n}}$ (§1.1).

Let us fix Haar measures. For every prime p, choose measures $\mu_{G_{\vec{n}},p}$ on $G_{\vec{n}}(\mathbb{Q}_p)$ and $\mu_{\mathbb{G},p}$ on $\mathbb{G}_{\vec{n}}(\mathbb{Q}_p)$ such that $\mu_{G_{\vec{n}},p}(K_p^{\vec{n}}) = 1$ and $\mu_{\mathbb{G}_{\vec{n}},p}(\mathbb{K}_p^{\vec{n}}) = 1$. Choose Haar measures $\mu_{A_{G_{\vec{n}},\infty}}$ on $A_{G_{\vec{n}},\infty} \simeq \mathbb{R}_{>0}^\times$ and $\mu_{A_{\mathbb{G}_{\vec{n}},\infty}}$ on $A_{\mathbb{G}_{\vec{n}},\infty} \simeq (\mathbb{R}_{>0}^\times)^{r+1}$ using the standard measure dx/x on $\mathbb{R}_{>0}^\times$. Finally choose Haar measures $\mu_{G_{\vec{n}},\infty}$ and $\mu_{\mathbb{G}_{\vec{n}},\infty}$ such that the quotient measures $(\prod_v \mu_{G_{\vec{n}},v})/\mu_{A_{G_{\vec{n}},\infty}}$ and $(\prod_v \mu_{\mathbb{G}_{\vec{n}},v})/\mu_{A_{\mathbb{G}_{\vec{n}},\infty}}$ are the Tamagawa measures ([Ono66, §2]) on $G_{\vec{n}}(\mathbb{A})/A_{G_{\vec{n}},\infty}$ and $\mathbb{G}_{\vec{n}}(\mathbb{A})/A_{\mathbb{G}_{\vec{n}},\infty}$, respectively.

Lemma 3.1. *Let r be the number of components in $\vec{n}$. Then*

$$\tau(G_{\vec{n}}) = 2^r \text{ or } 2^{r-1} \qquad \text{and} \qquad \tau(\mathbb{G}_{\vec{n}}) = 1.$$

Proof. For any reductive group G_0 over $\mathbb{Q}$,

$$\tau(G_0) = |\pi_0(Z(\widehat{G}_0)^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})})| / |\ker^1(\mathbb{Q}, G_0)| \tag{3.3}$$

([Kot88, p.629]). It is easy to see that $\tau(\mathbb{G}_{\vec{n}}) = 1$. Indeed, $Z(\widehat{\mathbb{G}}_{\vec{n}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$ is a product of copies of $\mathbb{C}^\times$ and $\ker^1(\mathbb{Q}, \mathbb{G}_{\vec{n}})$ is trivial by Shapiro's lemma and Hilbert 90.

Recall from [Kot84, (4.2.2)] that $|\ker^1(\mathbb{Q}, G_{\vec{n}})| = |\ker^1(\mathbb{Q}, Z(\widehat{G}_{\vec{n}}))|$. Using the description of $\widehat{G}_{\vec{n}}$ in (3.2) we identify $Z(\widehat{G}_{\vec{n}})$ with $\mathbb{C}^\times \times \prod_{\sigma,i} \mathbb{C}^\times$ where σ runs over $\Phi_{\mathbb{C}}^+$ and i over $\{1, \ldots, r\}$. It is easy to see that $Z(\widehat{G}_{\vec{n}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$ is identified with the set of $(\lambda, (g_i))$ where $\lambda \in \mathbb{C}^\times$, $g_i \in \{\pm 1\}$ and $\lambda(\prod_i g_i^{n_i})^{[F^+:\mathbb{Q}]} = 1$. Therefore

$$|\pi_0(Z(\widehat{G}_{\vec{n}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})})| = \begin{cases} 2^r, & 2|[F^+ : \mathbb{Q}] \text{ or } \forall i,\ 2|n_i \\ 2^{r-1}, & \text{otherwise} \end{cases}$$

On the other hand, $\ker^1(E, G_{\vec{n}})$ is trivial by Shapiro's lemma and Hilbert 90, which implies that $\ker^1(E, Z(\widehat{G}_{\vec{n}}))$ is also trivial. So we have an injection

$$\ker^1(\mathbb{Q}, Z(\widehat{G}_{\vec{n}})) \hookrightarrow H^1(E/\mathbb{Q}, Z(\widehat{G}_{\vec{n}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/E)})$$

via the inverse of the inflation map for H^1. Note that $Z(\widehat{G}_{\vec{n}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/E)}$ is isomorphic to $\mathbb{C}^\times \times (\mathbb{C}^\times)^r$. The group Z^1 of 1-cocycles consists of those $(\lambda, (g_i))$ which satisfy $\lambda^2(\prod_i g_i^{n_i})^{[F^+:\mathbb{Q}]} = 1$. The group B^1 of 1-coundaries precisely contains $(\lambda, (g_i))$ which has the form $\lambda = (\prod_i a_i^{n_i})^{[F^+:\mathbb{Q}]}$ and $g_i = a_i^{-2}$ for some $a_i \in \mathbb{C}^\times$ $(1 \le i \le r)$. Both Z^1 and B^1 surject onto $(\mathbb{C}^\times)^r$ via projection maps. Comparing the numbers of fibers for these projection maps, we obtain

$$|H^1(E/\mathbb{Q}, Z(\widehat{G}_{\vec{n}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/E)})| = \begin{cases} 2, & 2|[F^+ : \mathbb{Q}] \text{ or } \forall i,\ 2|n_i \\ 1, & \text{otherwise} \end{cases}$$

Therefore $\tau(G_{\vec{n}})$ equals 2^r or 2^{r-1}.

□

Remark 3.2. Although we have not pursued the precise value of $\tau(G_{\vec{n}})$, it can be easily determined in some cases. If $2|n_i$ for all i, we can prove that $\ker^1(\mathbb{Q}, G_{\vec{n}}) = 1$ using the argument in the second paragraph of [Kot92b, §7]. So $\tau(G_{\vec{n}}) = 2^r$ if every n_i is even. In case $[F^+ : \mathbb{Q}]$ is odd and some n_i is odd, the above proof shows that $\ker^1(\mathbb{Q}, G_{\vec{n}}) = 1$ and $\tau(G_{\vec{n}}) = 2^{r-1}$.

3.2. **Endoscopic triples and L-morphisms.** Let $\mathcal{E}^{\mathrm{ell}}(G_n)$ be a set of representatives for isomorphism classes of endoscopic triples for G_n over $\mathbb{Q}$ ([Kot84, §7]). We can identify $\mathcal{E}^{\mathrm{ell}}(G_n)$ with the set of triples

$$\{(G_n, s_n, \eta_n)\} \cup \{(G_{n_1,n_2}, s_{n_1,n_2}, \eta_{n_1,n_2} : n_1 + n_2 = n, n_1 \geq n_2 > 0\},$$

where (n_1, n_2) may be excluded in some cases if both n_1 and n_2 are odd numbers. (This is to satisfy the condition (7.4.3) of [Kot84]. As we will mainly work with odd n, we will not be concerned with the possible exclusion of such (n_1, n_2).) Here $s_n = 1 \in \widehat{G}_n$, $s_{n_1,n_2} = (1, (I_{n_1}, -I_{n_2})) \in \widehat{G}_{n_1,n_2}$ and $\eta_n : \widehat{G}_n \to \widehat{G}_n$ is the identity map whereas η_{n_1,n_2} is the embedding

$$(\lambda, (g_{\sigma,1}, g_{\sigma,2})) \mapsto \left(\lambda, \begin{pmatrix} g_{\sigma,1} & 0 \\ 0 & g_{\sigma,2} \end{pmatrix}\right)$$

The above description of $\mathcal{E}^{\mathrm{ell}}(G_n)$ can be verified as proposition 4.6.1 of [Rog90], which deals with the case of unitary groups.

We can extend η_{n_1,n_2} to an L-morphism $\widetilde{\eta}_{n_1,n_2}$ by[2]

$$\begin{array}{rcl} w \in W_E & \mapsto & \left(\varpi(w)^{-N(n_1,n_2)}, \begin{pmatrix} \varpi(w)^{\epsilon(n-n_1)} \cdot I_{n_1} & 0 \\ 0 & \varpi(w)^{\epsilon(n-n_2)} \cdot I_{n_2} \end{pmatrix}\right) \rtimes w \\ w^\star & \mapsto & \left(a_{n_1,n_2}, \Phi_{n_1,n_2}\Phi_n^{-1}\right) \rtimes w^\star \end{array}$$

where $N(n_1, n_2) := [F^+ : \mathbb{Q}](n_1\epsilon(n - n_1) + n_2\epsilon(n - n_2))/2 \in \mathbb{Z}$. The constant a_{n_1,n_2} is chosen to be a square root of the number $(-1)^{-N(n_1,n_2)} \det(\Phi_{n_1,n_2}\Phi_n)$. It is readily checked that $\widetilde{\eta}_{n_1,n_2}$ is indeed an L-morphism.

Let $\widetilde{\zeta}_{n_1,n_2} : {}^L\mathbb{G}_{n_1,n_2} \to {}^L\mathbb{G}_n$, be the map defined on $\widehat{\mathbb{G}}_{n_1,n_2}$ by

$$(\lambda_+, \lambda_-, (g_{\sigma,1}, g_{\sigma,2})) \mapsto \left(\lambda_+, \lambda_-, \begin{pmatrix} g_{\sigma,1} & 0 \\ 0 & g_{\sigma,2} \end{pmatrix}\right)$$

and sending $w \in W_E$ and $w^\star$ respectively to

$$\left(\varpi(w)^{-N(n_1,n_2)}, \varpi(w)^{-N(n_1,n_2)}, \begin{pmatrix} \varpi(w)^{\epsilon(n-n_1)} \cdot I_{n_1} & 0 \\ 0 & \varpi(w)^{\epsilon(n-n_2)} \cdot I_{n_2} \end{pmatrix}\right) \rtimes w$$

$$\left(a_{n_1,n_2}, a_{n_1,n_2}, \Phi_{n_1,n_2}\Phi_n^{-1}\right) \rtimes w^\star$$

We have the following commutative diagram of L-morphisms.

$$\begin{array}{ccc} {}^LG_{n_1,n_2} & \xrightarrow{\widetilde{\eta}_{n_1,n_2}} & {}^LG_n \\ \Big\downarrow{\scriptstyle BC_{n_1,n_2}} & & \Big\downarrow{\scriptstyle BC_n} \\ {}^L\mathbb{G}_{n_1,n_2} & \xrightarrow[\widetilde{\zeta}_{n_1,n_2}]{} & {}^L\mathbb{G}_n \end{array} \tag{3.4}$$

3.3. **Constant terms for $GL_{\vec{n}}$.** We record a well-known lemma, which will be applied later to explicit endoscopic transfer. For simplicity we state the lemma only for general linear groups. In §3.3 only, we use the following notation. Let L be a nonarchimedean field of characteristic 0. For $r > 1$, fix $\vec{n} = (n_1, \ldots, n_r)$ such that $\sum_i n_i = n$. Let $G := GL_n$ and $M := GL_{\vec{n}}$. (Later we will also consider a group G which is a finite product of general linear groups. The lemma below obviously extends to this case.) Let P be any conjugate of $P_{\vec{n}}$ containing M. Denote by N the unipotent radical of P. For each $f \in C_c^\infty(G(L))$, define the constant term along P by

$$f^P(m) := \delta_P^{1/2}(m) \int_{N(L)} \int_{G(\mathcal{O}_L)} f(kmnk^{-1})\, dk\, dn, \qquad m \in M(L). \tag{3.5}$$

[2]We chose to write $n - n_1$ and $n - n_2$ rather than n_2 and n_1 so that the formula readily generalizes if one is to define $\widetilde{\eta}_{\vec{n}}$ for arbitrary $\vec{n} = (n_1, \ldots, n_r)$ such that $\sum_{i=1}^r n_i = n$. cf. [Rog92, §1].

Let $\widetilde{i}_{\vec{n}} : {}^L M \hookrightarrow {}^L G$ be the L-morphism which trivially extends $i_{\vec{n}} : \widehat{M} \hookrightarrow \widehat{G}$.

Lemma 3.3. *The following are true.*

(i) *For any semisimple $m \in M(L)$ which is regular in $G(L)$,*

$$O_m^{M(L)}(f^P) = D_{G/M}(m)^{1/2} O_m^{G(L)}(f). \tag{3.6}$$

(ii) *For any $\pi \in \mathrm{Irr}(M(L))$, $\mathrm{tr}\,\pi(f^P) = \mathrm{tr}\,\text{n-ind}_M^G(\pi)(f)$.*

(iii) *If $f \in \mathscr{H}^{\mathrm{ur}}(G(L))$ then f^P is the image of f under the map $\mathscr{H}^{\mathrm{ur}}(G(L)) \to \mathscr{H}^{\mathrm{ur}}(M(L))$ which is dual to $\widetilde{i}_{\vec{n}}$.*

Proof. The first assertion is Lemma 9 of [vD72] and the second assertion is the first formula on page 237 thereof. The last assertion is an easy consequence of the Satake transform for general linear groups (cf. [AC89, p.32-33]). □

This lemma is a special case of the Langlands-Shelstad transfer, with respect to the L-morphism $\widetilde{i}_{\vec{n}}$. Indeed, it is easy to verify that $D_{G/M}(m)^{1/2}$ coincides with the transfer factor of [LS87] up to constant.

3.4. **Explicit transfer at finite places.** We begin with a brief reminder of the Langlands-Shelstad transfer in general. Let (H, s, η) be an endoscopic triple for a connected reductive $\mathbb{Q}$-group G. Suppose that there is an L-morphism $\widetilde{\eta} : {}^L H \to {}^L G$. Langlands and Shelstad ([LS87], [LS90]) defined a complex-valued function $\Delta_v(\cdot,\cdot)_H^G$, called the (local) transfer factor, on a pair (γ_H, γ) where $\gamma_H \in G_{\vec{n}}(\mathbb{Q}_v)$ is a semisimple $(G_n, G_{\vec{n}})$-regular element and $\gamma \in G(\mathbb{Q}_v)$ is such that the stable conjugacy classes of γ_H and γ are matching. Such a pair (γ_H, γ) will be called a *matching pair* for convenience. The local transfer factor is well-defined up to constant. Moreover, it depends not only on (H, s, η) but also on $\widetilde{\eta}$. Langlands and Shelstad conjectured that for each function $\phi_v \in C_c^\infty(G(\mathbb{Q}_v))$, there exists $\phi_v^H \in C_c^\infty(H(\mathbb{Q}_v))$ satisfying an identity about the transfer of orbital integrals ([LS90, 2.1], [Kot86, Conj 5.5]). We will refer to ϕ_v^H as a Δ_v-matching function for ϕ_v or simply a Δ_v-transfer of ϕ_v. In the unramified situation, Langlands ([Lan83, III.3]) proposed a more precise conjecture about the transfer, called the fundamental lemma. (See also [Hal95, §2], which states the fundamental lemma for unramified Hecke algebras and reduces its proof to the case of unit elements.)

Before going further, we point out that the Langlands-Shelstad conjecture on the existence of Δ_v-transfer is proved as well as the fundamental lemma (for unit elements) in all cases, due to Waldspurger, Laumon-Ngô and Ngô ([LN08], [Wal97], [Wal06], [Ngo]).

Remark 3.4. Actually Walspurger and Ngô prove the fundamental lemma (for any $\mathbb{Q}_p$-group G_0) over $\mathbb{Q}_p$ only if p is large enough (with respect to the rank of G_0). But the results of Hales (in particular, [Hal95, Thm 6.1]) can be used to prove the fundamental lemma for all primes p, by induction on the rank of G_0. Although the paper of Hales is somewhat sketchy, its main results are reproved by section 9 of [Mor10] which is more detailed.

However, one can avoid the use of the fundamental lemma for small primes p, if one wishes, without weakening our main results. Let P_N be the set of all primes $p < N$ for a sufficiently large N. Impose an additional assumption that $P_N \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$ throughout §5 and §6. The point is that if $p \in \mathrm{Spl}_{F/F^+,\mathbb{Q}}$, the fundamental lemma for $G(\mathbb{Q}_p)$ is known without appealing to Hales, as $G(\mathbb{Q}_p)$ is a product of general linear groups. In §7 we can remove the additional assumption, by adding a condition on $E \in \mathcal{E}(F)$ in the proof of Theorem 7.5 that every $p \in P_N$ splits in E.

Let us return to the situation of §3.1 and §3.2. Let $G_{\vec{n}} \in \mathcal{E}^{\mathrm{ell}}(G)$. Let v be a finite place of $\mathbb{Q}$. Below we will give a particular normalization of the transfer factor $\Delta_v(\cdot,\cdot)_{G_{\vec{n}}}^{G_n}$, which is a complex-valued function on a pair (γ_H, γ) where $\gamma_H \in G_{\vec{n}}(\mathbb{Q}_v)$ is a semisimple $(G_n, G_{\vec{n}})$-regular element and $\gamma \in G(\mathbb{Q}_v)$ is such that the stable conjugacy classes of γ_H and γ are matching. We will also define a map $\widetilde{\eta}^*_{n_1,n_2}$, which gives the $\Delta_v(\cdot,\cdot)_{G_{\vec{n}}}^{G_n}$-transfer (or simply Δ_v-transfer). Moreover, we present an explicit representation-theoretic transfer $\widetilde{\eta}_{n_1,n_2,*}$, which is tied to $\widetilde{\eta}^*_{n_1,n_2}$ via character identity.

For later use in Case 2 and Case 3, we record a natural isomorphism for $v \in \mathrm{Spl}_{F/F^+,\mathbb{Q}}$. Fix an isomorphism $\iota_v : \mathbb{Q}_v \simeq \mathbb{C}$. Let $\mathcal{V}_v^+$ be the set of places x of F such that the composite map $F \overset{x}{\hookrightarrow} \overline{\mathbb{Q}}_v \overset{\iota_v}{\simeq} \mathbb{C}$ belongs to $\Phi_{\mathbb{C}}^+$. (This is the same definition as in the paragraph below (4.1).) Suppose either $\vec{n} = (n)$ or $\vec{n} = (n_1, n_2)$. The group $G_{\vec{n}}(\mathbb{Q}_v)$ is a subgroup of $\mathbb{Q}_v^\times \times GL_{\vec{n}}(F \otimes_{\mathbb{Q}} \mathbb{Q}_v)$ and the projection map onto $\mathbb{Q}_v^\times \times \prod_{x \in \mathcal{V}_v^+} GL_{\vec{n}}(F_x)$ induces an isomorphism

$$G_{\vec{n}}(\mathbb{Q}_v) \quad \simeq \quad \mathbb{Q}_v^\times \times \prod_{x \in \mathcal{V}_v^+} GL_{\vec{n}}(F_x). \tag{3.7}$$

Using the above isomorphism, fix an embedding $G_{n_1,n_2} \hookrightarrow G_n$ via i_{n_1,n_2}. Set $Q_{n_1,n_2} := \mathbb{Q}_v^\times \times \prod_{x \in \mathcal{V}_v^+} P_{n_1,n_2}$ the parabolic subgroup of G_n, containing G_{n_1,n_2} as a Levi subgroup.

Case 1. $v \in \mathrm{Unr}_{F/\mathbb{Q}}$ and $v \notin \mathrm{Ram}(\varpi)$.

In this case, $\widetilde{\eta}_{n_1,n_2}$ induces a $\mathbb{C}$-algebra map of unramified Hecke algebras

$$\widetilde{\eta}^* : \mathscr{H}^{\mathrm{ur}}(G_n(\mathbb{Q}_v)) \to \mathscr{H}^{\mathrm{ur}}(G_{n_1,n_2}(\mathbb{Q}_v))$$

and a transfer of unramified representations

$$\widetilde{\eta}_* : \mathrm{Irr}^{\mathrm{ur}}(G_{n_1,n_2}(\mathbb{Q}_v)) \to \mathrm{Irr}^{\mathrm{ur}}(G_n(\mathbb{Q}_v)).$$

By the proof of the fundamental lemma ([Ngo]) and an earlier work of Hales ([Hal95]), $\Delta_v(\cdot,\cdot)$ can be normalized so that

$$\phi_v^{n_1,n_2} := \widetilde{\eta}^*(\phi_v)$$

is a Δ_v-transfer of ϕ_v for any $\phi_v \in \mathscr{H}^{\mathrm{ur}}(G(\mathbb{Q}_v))$. Denote this normalization by $\Delta_v^0(\cdot,\cdot)$. Then for every $\pi \in \mathrm{Irr}^{\mathrm{ur}}(G_{n_1,n_2}(\mathbb{Q}_v))$, we have

$$\mathrm{tr}\, \pi(\widetilde{\eta}^*(\phi_v)) = \mathrm{tr}\, \widetilde{\eta}_*(\pi)(\phi_v). \tag{3.8}$$

Case 2. $v \in \mathrm{Spl}_{E/\mathbb{Q}}$.

Let $\phi_v \in C_c^\infty(G_n(\mathbb{Q}_v))$. Let $u := x|_E$ for any $x \in \mathcal{V}_v^+$. Define a character $\chi^+_{\varpi,u} : G_{n_1,n_2}(\mathbb{Q}_v) \to \mathbb{C}^\times$ by

$$\chi^+_{\varpi,u}(\lambda, (g_{x,1}, g_{x,2})) := \varpi_u \left(\lambda^{-N(n_1,n_2)} \prod_{x \in \mathcal{V}_v^+} \prod_{1 \le i \le 2} N_{F_x/E_u}(\det(g_{x,i}))^{\epsilon(n-n_i)} \right).$$

(We view λ as an element of $E_u^\times$ via $\mathbb{Q}_v^\times \simeq E_u^\times$.) Denote by $\phi_v^{Q_{n_1,n_2}}$ the constant term along Q_{n_1,n_2} (§3.3). Define

$$\phi_v^{n_1,n_2} := \phi_v^{Q_{n_1,n_2}} \cdot \chi^+_{\varpi,u}.$$

For any (G_n, G_{n_1,n_2})-regular semisimple $g \in G_{n_1,n_2}(\mathbb{Q}_v)$, define

$$\Delta_v^0(g,g) := |D_{G_n/G_{n_1,n_2}}(g)|^{1/2} \cdot \chi^+_{\varpi,u}(g).$$

(Recall that we fix an embedding of G_{n_1,n_2} into G_n as a Levi subgroup.) Note that the above formula pins down the value of $\Delta_v^0(\cdot,\cdot)$ on every matching pair. It is not hard to show that $\Delta_v^0(\cdot,\cdot)$ is equal, up to constant, to the Langlands-Shelstad transfer factor with respect to $\widetilde{\eta}$. We sketch the argument. Let $\widetilde{\eta}' : {}^L G_{n_1,n_2} \hookrightarrow {}^L G_n$ be an L-morphism (canonical up to $\widehat{G}_n$-conjugacy) corresponding to the fixed Levi embedding $G_{n_1,n_2} \hookrightarrow G_n$ via [Bor79, §3]. We may arrange that $\widetilde{\eta}'$ and $\widetilde{\eta}$ are identical on $\widehat{G}_{n_1,n_2}$ by conjugating $\widetilde{\eta}'$ by an element of $\widehat{G}_n$, so that $\widetilde{\eta} = a\widetilde{\eta}'$ for $a \in H^1(W_{\mathbb{Q}_v}, Z(\widehat{G}_{n_1,n_2}))$. Let $\chi_a : G_{n_1,n_2}(\mathbb{Q}_v) \to \mathbb{C}^\times$ denote the character corresponding to a. (As $Z(\widehat{G}_{n_1,n_2})$ is the dual torus of the maximal abelian quotient of G_{n_1,n_2}, the cohomology class a determines χ_a via [Bor79, §9].) Let $\Delta'_v(g,g)$ denote the transfer factor with respect to $\widetilde{\eta}'$. The following facts (which are true after normalization up to constant) are standard and deduced directly[3] from the definition of transfer factors ([LS87, §3]).

- $\Delta'_v(g,g) = |D_{G_n/G_{n_1,n_2}}(g)|^{1/2}$.
- $\Delta_v^0(g,g) = \Delta'_v(g,g) \cdot \chi_a(g)$.

[3]The value $|D_{G_n/G_{n_1,n_2}}(g)|^{1/2}$ (resp. $\chi_a(g)$) comes from the factor Δ_{IV} (resp. Δ_{III_2}) of [LS87]. In the unramified situation, we remark that the first identity in the bullet list is a special case of [Hal93, Lem 9.2] and that the second identity appears in the proof of [Hal95, Lem 3.3].

Finally, one checks that $\chi_a = \chi^+_{\varpi,u}$ by explicitly working out the duality for the torus $Z(\widehat{G}_{n_1,n_2})$.

For any $\pi_v \in \mathrm{Irr}(G_{n_1,n_2})$, define

$$\widetilde{\eta}_*(\pi_v) := \text{n-ind}^{G_n}_{Q_{n_1,n_2}} \left(\pi_v \otimes \chi^+_{\varpi,u}\right)$$

It is easily deduced from Lemma 3.3 that the following identities hold for any g and π_v as above. In particular $\phi_v^{n_1,n_2}$ is a Δ_v^0-transfer of ϕ_v.

$$O_g(\phi_v^{n_1,n_2}) = \Delta_v(g,g)\cdot O_g(\phi_v) \tag{3.9}$$

$$\mathrm{tr}\,\pi_v(\phi_v^{n_1,n_2}) = \mathrm{tr}\,\widetilde{\eta}_*(\pi_v)(\phi_v). \tag{3.10}$$

Case 3. $v \in \mathrm{Spl}_{F/F^+,\mathbb{Q}}$ and $v \notin \mathrm{Spl}_{E/\mathbb{Q}}$

We retain the same notation as in Case 2, but write v for the unique place of E above v by abuse of notation. Things are very similar to Case 2 except that the character $\chi^+_{\varpi,v}$, defined below, is slightly different from $\chi^+_{\varpi,u}$ of Case 2.

$$\begin{aligned}
\chi^+_{\varpi,v}(\lambda,(g_{x,1},g_{x,2})) &:= \varpi_v\left(\prod_{x\in\mathcal{V}_v^+}\prod_{1\le i\le 2} N_{F_x/E_v}(\det(g_{x,i}))^{\epsilon(n-n_i)}\right)\\
\phi_v^{n_1,n_2}(g) &:= \phi_v^{Q_{n_1,n_2}}(g)\cdot\chi^+_{\varpi,v}(g)\\
\Delta_v^0(g,g) &:= |D_{G_n/G_{n_1,n_2}}(g)|^{1/2}\cdot\chi^+_{\varpi,v}(g)\\
\widetilde{\eta}_*(\pi_v) &:= \text{n-ind}^{G_n}_{Q_{n_1,n_2}}\left(\pi_v\otimes\chi^+_{\varpi,v}\right)
\end{aligned}$$

The same argument as in Case 2 shows that $\Delta_v^0(\cdot,\cdot)$ is the Langlands-Shelstad transfer factor with respect to $\widetilde{\eta}_{n_1,n_2}$ (up to constant). As in Case 2, it is easy to check that the same identities as in (3.9) and (3.10) hold. So $\phi_v^{n_1,n_2}$ is a Δ_v^0-transfer of ϕ_v.

Remark 3.5. There are overlaps between Case 1 and Case 2 and between Case 1 and Case 3, namely when $v \in \mathrm{Unr}_{F/\mathbb{Q}} \cap \mathrm{Spl}_{F/F^+,\mathbb{Q}}$, $v \notin \mathrm{Ram}_{\mathbb{Q}}(\varpi)$, $\phi_v \in \mathscr{H}^{\mathrm{ur}}(G_n(\mathbb{Q}_v))$ and $\pi_v \in \mathrm{Irr}^{\mathrm{ur}}(G_n(\mathbb{Q}_v))$. However it is not hard to see that the definitions are consistent: Consider such v, ϕ_v and π_v. Then $\phi_v^{n_1,n_2}$ in Case 2 or Case 3 is the same as in Case 1. This follows from the fact that constant terms are compatible with Satake transform (cf. [AC89, p.33]). By the same fact we check the consistency of the definition of $\Delta_v^0(g,g)$ and $\widetilde{\eta}_*(\pi_v)$.

3.5. **Transfer of pseudo-coefficients at infinity.** Here we review Shelstad's results on real endoscopy ([She82]) for discrete series representations, based on the summary of Kottwitz ([Kot90, §7]). We will freely use the Langlands correspondence for real reductive groups ([Lan88]). Let G be an $\mathbb{R}$-inner form of G_n. Set $(H,s,\eta) := (G_{\vec{n}}, s_{\vec{n}}, \eta_{\vec{n}}) \in \mathcal{E}^{\mathrm{ell}}(G_n)$, which is also an endoscopic triple for G.

Let ξ be an irreducible algebraic representation of $G_{\mathbb{C}}$. Define $\chi_\xi : A_{G,\infty} \to \mathbb{C}^\times$ to be the character obtained by restricting ξ to $A_{G,\infty}$. Define $\mathrm{Irr}(G(\mathbb{R}),\chi_\xi^{-1})$ to be the set of $\pi \in \mathrm{Irr}(G(\mathbb{R}))$ whose restriction to $A_{G,\infty}$ is χ_ξ^{-1}. Let $\Pi_{\mathrm{unit}}(G(\mathbb{R}),\xi^\vee)$ denote the set of $\pi \in \mathrm{Irr}(G(\mathbb{R}))$ which are unitary (modulo $A_{G,\infty}$) and have the same infinitesimal character and central character as $\xi^\vee$. Denote by $\mathrm{Irr}_{\mathrm{temp}}(G(\mathbb{R}),\chi_\xi^{-1})$ (resp. $\Pi_{\mathrm{disc}}(G(\mathbb{R}),\xi^\vee)$) the subset of $\mathrm{Irr}(G(\mathbb{R}),\chi_\xi^{-1})$ (resp. $\Pi_{\mathrm{unit}}(G(\mathbb{R}),\xi^\vee)$) consisting of those representations which are tempered (resp. square-integrable) modulo $A_{G,\infty}$. Choose any maximal compact subgroups $K_\infty \subset G(\mathbb{R})$ and $\mathbb{K}_\infty \subset \mathbb{G}(\mathbb{R})$ which are admissible in the sense of [Art88b, §1]. Define an integer

$$q(G) := \frac{1}{2}\dim(G(\mathbb{R})/K_\infty A_{G,\infty}). \tag{3.11}$$

Fix real elliptic maximal tori $T \subset G$ and $T_H \subset H$ along with an $\mathbb{R}$-isomorphism $j : T_H \xrightarrow{\sim} T$. Also fix a Borel subgroup B of G over $\mathbb{C}$ such that $B \supset T_{\mathbb{C}}$. Let $\varphi_\xi : W_{\mathbb{R}} \to {}^LG$ be the discrete L-parameter for ξ which corresponds to the L-packet $\Pi_{\mathrm{disc}}(G(\mathbb{R}),\xi^\vee)$. Let Ω (resp. Ω_H) denote the complex Weyl group for T in G (resp. T_H in H) and $\Omega_{\mathbb{R}}$ the real Weyl group for T.

For each $\pi \in \Pi_{\rm disc}(G(\mathbb{R}),\xi^\vee)$, there exists $\phi_\pi \in C_c^\infty(G(\mathbb{R}),\chi_\xi)$ such that for any $\pi' \in \mathrm{Irr}_{\rm temp}(G(\mathbb{R}),\chi_\xi^{-1})$,

$$\mathrm{tr}\,\pi'(\phi_\pi) = \begin{cases} 1, & \text{if } \pi' \simeq \pi \\ 0, & \text{otherwise} \end{cases}$$

Such a function ϕ_π is called a *pseudo-coefficient* for π. Whenever we write the expression ϕ_π in the future, let us agree that a choice of a pseudo-coefficient for π is implicit.

The members π of $\Pi_{\rm disc}(G(\mathbb{R}),\xi^\vee)$ are parametrized by $\omega_\pi \in \Omega/\Omega_{\mathbb{R}}$ so that each $\pi = \pi(\varphi_\xi, \omega_\pi^{-1}B)$ is characterized by the character formula of [Kot90, p.183], which is due to Harish-Chandra. We want to describe the transfer of ϕ_π to $H(\mathbb{R})$ as a linear combination of pseudo-coefficients for discrete series of $H(\mathbb{R})$. Shelstad defined the transfer factor $\Delta_{j,B}$ (depending on $\widetilde{\eta}$) on elliptic regular elements, which is enough for our purpose. (Note that pseudo-coefficients have trivial orbital integrals on non-elliptic semisimple elements and that the case of elliptic singular elements is covered by [LS90, 2.4].)

Remark 3.6. (A similar remark appears in [Shi10, Rem 5.5].) In principle, we have to be careful about the different conventions for transfer factors when we refer to [Kot90] and work of Langlands and Shelstad at the same time. The convention in [Kot90] differs from that of Langlands and Shelstad by $s \mapsto s^{-1}$, as explained on page 178 of that article. Fortunately there is no danger for us to confuse the two conventions, as $s = s^{-1}$ holds for every endoscopic triple in $\mathcal{E}^{\rm ell}(G_n)$.

For any discrete L-parameter φ_H for $H(\mathbb{R})$ and its associated L-packet $\Pi(\varphi_H)$, let

$$\phi_{\varphi_H} := \frac{1}{|\Pi(\varphi_H)|} \sum_{\pi_H \in \Pi(\varphi_H)} \phi_{\pi_H}. \tag{3.12}$$

(In [Kot90, §7], ϕ_{φ_H} was denoted by $h(\varphi_H)$.) Define

$$\phi_\pi^H := (-1)^{q(G)} \sum_{\widetilde{\eta}\varphi_H \sim \varphi_\xi} \langle a_{\omega_*(\varphi_H)\omega_\pi}, s\rangle \det(\omega_*(\varphi_H)) \cdot \phi_{\varphi_H} \tag{3.13}$$

where the sum runs over equivalence classes of φ_H such that $\widetilde{\eta}\varphi_H$ is equivalent to φ_ξ. We remind the reader that we adopted notations of [Kot90]. (In that article, see page 185 for $\omega_*(\varphi_H)$ and page 175 for $a_{\omega_*(\varphi_H)\omega_\pi}$.)

Lemma 3.7. *Let* $\pi = \pi(\varphi_\xi, \omega_\pi^{-1}B)$.

(i) *For any discrete L-parameter φ_H for $H(\mathbb{R})$,*

$$\sum_{\pi_H \in \Pi(\varphi_H)} \mathrm{tr}\,\pi_H(\phi_\pi^H) = \begin{cases} (-1)^{q(G)}\langle a_{\omega_*(\varphi_H)\omega_\pi}, s\rangle \det(\omega_*(\varphi_H)), & \textit{if } \widetilde{\eta}\varphi_H \sim \varphi_\xi \\ 0, & \textit{otherwise} \end{cases}$$

(ii) ϕ_π^H *is a $\Delta_{j,B}$-transfer of* ϕ_π.

Remark 3.8. Compare with [Clo, Thm 3.4], which proves a similar result with a somewhat different approach. It seems that our proof is general enough to work for other groups with little change.

Proof. Note that (i) follows immediately from the definition of ϕ_π^H.

Let us prove (ii). It suffices to prove that for any elliptic regular $\gamma_H \in H(\mathbb{R})$ and $\gamma := j(\gamma_H)$,

$$SO_{\gamma_H}(\phi_\pi^H) = \Delta_{j,B}(\gamma_H,\gamma_0) \sum_{\gamma \sim_{\rm st} \gamma_0} \langle \mathrm{inv}(\gamma_0,\gamma), s\rangle \cdot O_\gamma(\phi_\pi) \tag{3.14}$$

where $\mathrm{inv}(\gamma_0,\gamma)$ is defined in [Kot86, 6.7] and the sum runs over the set of $\gamma \in G(\mathbb{R})$ (up to $G(\mathbb{R})$-conjugacy) which are stably conjugate to γ_0. We import notations and facts from pages 183-186 of [Kot90]. By the third formula of page 186 and the formula for $\Delta_{j,B}$ of page 184,

$$\begin{aligned} SO_{\gamma_H}(\phi_{\varphi_H}) &= (-1)^{q(H)}\mathrm{vol}^{-1} \sum_{\omega_H \in \Omega_H} \chi_{\omega_H(B_H)}(\gamma_H^{-1}) \cdot \Delta_{\omega_H(B_H)}(\gamma_H^{-1})^{-1} \\ &= (-1)^{q(G)}\mathrm{vol}^{-1} \sum_{\omega_H \in \Omega_H} \Delta_{j,\omega_H\omega_*(B)}(\gamma_H,\gamma_0) \cdot \chi_{\omega_H\omega_*(B)}(\gamma_0^{-1}) \cdot \Delta_{\omega_H\omega_*(B)}(\gamma_0^{-1})^{-1} \end{aligned}$$

where we wrote ω_* for $\omega_*(\varphi_H)$. Since $\Delta_{j,\omega_H\omega_*(B)} = \det(\omega_*)\Delta_{j,B}$, we see that $SO_{\gamma_H}(\phi^H_\pi)$ equals (recalling from [Kot90, p.185] that there is a bijection between Ω_* and the set of φ_H)

$$(-1)^{q(G)}\mathrm{vol}^{-1}\sum_{\omega_*\in\Omega_*}\sum_{\omega_H\in\Omega_H}\langle a_{\omega_*\omega_\pi},s\rangle\cdot\Delta_{j,B}(\gamma_H,\gamma_0)\cdot\chi_{\omega_H\omega_*(B)}(\gamma_0^{-1})\cdot\Delta_{\omega_H\omega_*(B)}(\gamma_0^{-1})^{-1}.$$

Using the equality $\langle a_{\omega_*\omega_\pi},s\rangle = \langle a_{\omega_H\omega_*\omega_\pi},s\rangle$ and the bijection $\Omega_H\times\Omega_*\to\Omega$ mapping (ω_H,ω_*) to $\omega_H\omega_*$, we can simplify the above expression as

$$(-1)^{q(G)}\mathrm{vol}^{-1}\Delta_{j,B}(\gamma_H,\gamma_0)\sum_{\omega\in\Omega}\langle a_{\omega\omega_\pi},s\rangle\cdot\chi_{\omega(B)}(\gamma_0^{-1})\cdot\Delta_{\omega(B)}(\gamma_0^{-1})^{-1}.$$

On the other hand, using the computation of orbital integrals in [Kot92a, p.659],

$$\sum_{\gamma\sim_{\mathrm{st}}\gamma_0}\langle\mathrm{inv}(\gamma_0,\gamma),s\rangle O_\gamma(\phi_\pi) = \sum_{\omega\in\Omega/\Omega_{\mathbb{R}}}\langle a_\omega,s\rangle O_{\omega\gamma_0}(\phi_\pi) = \sum_{\omega\in\Omega/\Omega_{\mathbb{R}}}\langle a_\omega,s\rangle\mathrm{vol}^{-1}\cdot\mathrm{tr}\,\pi((\omega\gamma_0)^{-1})$$

Using the formula of [Kot90, p.183] for $\mathrm{tr}\,\pi((\omega\gamma_0)^{-1})$, the last expression can be written as

$$(-1)^{q(G)}\mathrm{vol}^{-1}\sum_{\omega\in\Omega/\Omega_{\mathbb{R}}}\langle a_\omega,s\rangle\sum_{w_0\in\Omega_{\mathbb{R}}}\chi_{\omega_0\omega_\pi^{-1}(B)}(\omega^{-1}\gamma_0^{-1})\cdot\Delta_{\omega_0\omega_\pi^{-1}(B)}(\omega^{-1}\gamma_0^{-1})^{-1}. \tag{3.15}$$

Since $\langle a_\omega,s\rangle = \langle a_{\omega\omega_0},s\rangle$ and the last summand is equal to $\chi_{\omega\omega_0\omega_\pi^{-1}(B)}(\gamma_0^{-1})\cdot\Delta_{\omega\omega_0\omega_\pi^{-1}(B)}(\gamma_0^{-1})^{-1}$, (3.15) is the same as

$$(-1)^{q(G)}\mathrm{vol}^{-1}\sum_{\omega\in\Omega}\langle a_{\omega\omega_\pi},s\rangle\cdot\chi_{\omega(B)}(\gamma_0^{-1})\cdot\Delta_{\omega(B)}(\gamma_0^{-1})^{-1}.$$

Hence (3.14) is proved.

□

Remark 3.9. Note that each φ_H such that $\widetilde{\eta}\varphi_H\sim\varphi_\xi$ corresponds to an L-packet of the form $\Pi_{\mathrm{disc}}(H(\mathbb{R}),\xi(\varphi_H)^\vee)$ where $\xi(\varphi_H)$ is a suitable irreducible algebraic representation of H. The function ϕ_{φ_H} is often called an *Euler-Poincaré function* in the following sense: for each $\pi_H\in\Pi(H(\mathbb{R}),\chi^{-1}_{\xi(\varphi_H)})$, the trace $\mathrm{tr}\,\pi_H(\phi_{\varphi_H})$ computes the Euler-Poincare characteristic of the relative Lie algebra cohomology of $\pi_H\otimes\xi(\varphi_H)$. The existence of an Euler-Poincaré function was proved by Clozel-Delorme ([CD90]). Its explicit realization as (3.12) was used by several authors ([Kot92a, Lem 3.2], cf. [Art89, (3.1)]). The twisted analogue is obtained by Labesse ([Lab91]). (cf. §4.3.)

In view of Remark 3.9, we will sometimes write $\phi_{H,\xi(\varphi_H)}$ for ϕ_{φ_H}.

3.6. **Explicit computation of real endoscopic signs.** We wish to make the discussion of the last subsection explicit in case $G = G(U(1,n-1)\times U(0,n)\times\cdots U(0,n))$, which is an inner form of $G_n\times_{\mathbb{Q}}\mathbb{R}$. A precise definition of G is given below. As in §3.5, we use the notation of [Kot90, §7] without recalling here. Note that a similar computation as ours was obtained earlier by Clozel ([Clo]).

For each $\sigma\in\Phi^+_{\mathbb{C}}$ let

$$J_\sigma := \begin{pmatrix}1 & 0\\ 0 & -I_{n-1}\end{pmatrix}\text{ if }\sigma=\tau\quad\text{and}\quad J_\sigma = I_n\text{ if }\sigma\neq\tau.$$

and define an $\mathbb{R}$-group G and its maximal $\mathbb{R}$-elliptic torus T (via the obvious diagonal embedding) by

$$\begin{array}{rcll} G(A) & := & \{(\lambda,(g_\sigma))\in A^\times\times M_n(\mathbb{C}\otimes_{\mathbb{R}}A)^{\Phi^+_{\mathbb{C}}} & \mid\ \forall\sigma,\quad g_\sigma J_\sigma{}^tg^c_\sigma = \lambda J_\sigma\}\\ T(A) & := & \{(\lambda,(t_{\sigma,i}))\in A^\times\times((\mathbb{C}\otimes_{\mathbb{R}}A)^n)^{\Phi^+_{\mathbb{C}}} & \mid\ \forall\sigma,i,\quad t_{\sigma,i}t^c_{\sigma,i} = \lambda\}\end{array} \tag{3.16}$$

for any $\mathbb{R}$-algebra A, where $\sigma\in\Phi^+_{\mathbb{C}}$ and $1\le i\le n$.

Let $n_1,n_2\in\mathbb{Z}_{>0}$ be such that $n_1>n_2$ and $n_1+n_2=n$. The group $(H,s,\eta) := (G_{n_1,n_2},s_{n_1,n_2},\eta_{n_1,n_2})$ (defined in §3.2) is an endoscopic triple for G, equipped with $\widetilde{\eta}_{n_1,n_2} : {}^LH\to{}^LG$. For our purpose, we may identify H with the $\mathbb{R}$-group given by

$$H(A)\ :=\ \{(\lambda,(h_\sigma))\in A^\times\times M_{n_1,n_2}(\mathbb{C}\otimes_{\mathbb{R}}A)^{\Phi^+_{\mathbb{C}}}\ \mid\ \forall\sigma,\ \ h_\sigma J'_\sigma{}^th^c_\sigma = \lambda J'_\sigma\}$$

for $\mathbb{R}$-algebras A, where J'_σ is a suitable diagonal matrix with entries $+1$ and -1 such that H is quasi-split. Let $T_H := T$, which obviously embeds into H diagonally. Take $j : T_H\xrightarrow{\sim}T$ to be the identity. There is an obvious isomorphism (induced by the map $z_1 \otimes z_2 \mapsto z_1 z_2$ from $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ to $\mathbb{C}$, in view of (3.16))

$$G(\mathbb{C}) \simeq GL_1(\mathbb{C}) \times GL_n(\mathbb{C})^{\Phi^+_{\mathbb{C}}} \tag{3.17}$$

and similarly for H, with GL_n replaced by GL_{n_1,n_2}. Let $B \subset G_{\mathbb{C}}$ and $B_H \subset H_{\mathbb{C}}$ be the Borel subgroups consisting of upper triangular matrices. Note that $B \supset T_{\mathbb{C}}$ and $B_H \supset (T_H)_{\mathbb{C}}$.

Let $\mathcal{S}_N$ denote the symmetric group in N variables. There are natural identifications

$$\Omega = \mathcal{S}_n^{\Phi^+_{\mathbb{C}}}, \qquad \Omega_H = (\mathcal{S}_{n_1} \times \mathcal{S}_{n_2})^{\Phi^+_{\mathbb{C}}}, \qquad \Omega_{\mathbb{R}} = \mathcal{S}_{n-1} \times \mathcal{S}_n^{\Phi^+_{\mathbb{C}} \backslash \{\tau\}}$$

so that any $\omega = (\omega_\sigma) \in \Omega$ acts on T as $(\lambda, (t_{\sigma,i})) \in T \mapsto (\lambda, (t_{\sigma,\omega_\sigma(i)}))$, and similarly for Ω_H acting on T_H. Of course Ω_H is identified with a subgroup of Ω via j. The component $\mathcal{S}_{n-1}$ of $\Omega_{\mathbb{R}}$ is viewed as the group which permutes the sub-indices for $(t_{\tau,2}, t_{\tau,3}, \ldots, t_{\tau,n})$. The set Ω_* is a subset of $(\omega_\sigma) \in \Omega$ such that $\omega_\sigma(1) < \cdots < \omega_\sigma(n_1)$ and $\omega_\sigma(n_1+1) < \cdots < \omega_\sigma(n)$ for every $\sigma \in \Phi^+_{\mathbb{C}}$. The multiplication induces a bijection $\Omega_H \times \Omega_* \to \Omega$.

Let ξ be an irreducible algebraic representation of $G_{\mathbb{C}}$. To ξ there is a way to attach $a_0(\xi) \in \mathbb{Z}$ and $\vec{a}(\xi)_\sigma \in (\mathbb{Z})^n$ for each $\sigma \in \Phi^+_{\mathbb{C}}$ by the following condition: $\xi|_{GL_1}$ is $x \mapsto x^{a_0(\xi)}$ and $\vec{a}(\xi)_\sigma = (a(\xi)_{\sigma,1}, \ldots, a(\xi)_{\sigma,n})$ is the highest weight for the restriction of ξ to the σ-component GL_n, with respect to (3.17), where $a(\xi)_{\sigma,1} \geq \cdots \geq a(\xi)_{\sigma,n}$. (This is different from the convention of [HT01, p.97-98] in that the inequalities are reversed.) Define $w(\xi) \in \mathbb{Z}$ and $\alpha(\xi)_{\sigma,i} \in \frac{1}{2}\mathbb{Z}$ by

$$w(\xi) := -2a_0(\xi) - \sum_{\sigma,i} a(\xi)_{\sigma,i}, \qquad \alpha(\xi)_{\sigma,i} = -a(\xi)_{\sigma,n+1-i} + \frac{n+1-2i}{2}. \tag{3.18}$$

View ϖ_∞ as a character $\mathbb{C}^\times \to \mathbb{C}^\times$ by identifying $E \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{C}$ via τ_E. Note that $\varpi_\infty(z) = (z/\overline{z})^{\delta/2}$ for an odd number $\delta \in \mathbb{Z}$, as ϖ_∞ extends the sign character on $\mathbb{R}^\times$. Let $(\gamma(\xi)_{\sigma,i})$ be any permutation of $(\alpha(\xi)_{\sigma,i})$ by an element of Ω such that

$$\gamma(\xi)_{\sigma,1} > \cdots > \gamma(\xi)_{\sigma,n_1}, \qquad \gamma(\xi)_{\sigma,n_1+1} > \cdots > \gamma(\xi)_{\sigma,n}$$

and put

$$\beta(\xi)_{\sigma,i} = \begin{cases} \gamma(\xi)_{\sigma,i} - \epsilon(n-n_1) \cdot \frac{\delta}{2}, & \text{if } 1 \leq i \leq n_1, \\ \gamma(\xi)_{\sigma,i} - \epsilon(n-n_2) \cdot \frac{\delta}{2}, & \text{if } n_1 < i \leq n. \end{cases}$$

Consider a discrete L-parameter $\varphi_H : W_{\mathbb{R}} \to {}^L H$ sending $z \in W_{\mathbb{C}}$ to

$$\left((z\overline{z})^{-w(\xi)/2} (z/\overline{z})^{(N(n_1,n_2)\delta - \sum_{\sigma,i} a(\xi)_{\sigma,i})/2}, \left((z/\overline{z})^{\beta(\xi)_{\sigma,i}} \right)_{\sigma \in \Phi^+_{\mathbb{C}}, 1 \leq i \leq n} \right) \rtimes z$$

where $((z/\overline{z})^{\beta(\xi)_{\sigma,i}})_{1 \leq i \leq n}$ for each σ embeds into the diagonal of $GL_{n_1,n_2}(\mathbb{C})$ in an obvious way. So $\varphi_\xi := \widetilde{\eta} \circ \varphi_H$ sends $z \in W_{\mathbb{C}}$ to

$$\left((z\overline{z})^{-w(\xi)/2} (z/\overline{z})^{-\sum_{\sigma,i} a(\xi)_{\sigma,i}/2}, \begin{pmatrix} (z/\overline{z})^{\gamma(\xi)_{\sigma,1}} & 0 & \cdots & 0 \\ 0 & (z/\overline{z})^{\gamma(\xi)_{\sigma,2}} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & (z/\overline{z})^{\gamma(\xi)_{\sigma,n}} \end{pmatrix}_{\sigma \in \Phi^+_{\mathbb{C}}} \right) \rtimes z.$$

It is not hard to check that φ_ξ is (up to equivalence) the discrete L-parameter for $\Pi_{\mathrm{disc}}(G(\mathbb{R}), \xi^\vee)$, which justifies our notation for φ_ξ. (Use the characterizing properties of φ_ξ in §4.3.)

The element $\omega_* = \omega_*(\varphi_H) \in \Omega_*$ is easy to describe in terms of $\gamma(\xi)_{\sigma,i}$'s. It is the unique element of Ω_* such that

$$\gamma(\xi)_{\sigma,\omega_*(1)} > \cdots > \gamma(\xi)_{\sigma,\omega_*(n)}, \qquad \forall \sigma \in \Phi^+_{\mathbb{C}}. \tag{3.19}$$

This description of ω_* easily follows from the discussion of [Kot90, p.184-185].

Recall that $\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{R}), \xi^\vee)$ are parametrized by $\omega_\pi \in \Omega/\Omega_{\mathbb{R}}$. (We will confuse ω_π with any of its representatives in Ω.) Note that $|\Omega/\Omega_{\mathbb{R}}| = n$. We may write

$$\Pi_{\mathrm{disc}}(G(\mathbb{R}), \xi^\vee) = \{\pi^1, \ldots, \pi^n\}$$

where π^i is characterized as follows: if we write $\omega_{\pi^i} = (\omega_{\pi^i,\sigma})_{\sigma \in \Phi^+_{\mathbb{C}}}$ then $\omega_{\pi^i,\tau}$ is an element of the permutation group $\mathcal{S}_n$ that takes 1 to i. (The last condition determines ω_{π^i} as an element of $\Omega/\Omega_{\mathbb{R}}$.)

We will consider $h : R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G$ factoring through T. Suppose that on $\mathbb{R}$-points $h : \mathbb{C}^\times \to T(\mathbb{R})$ is given by (compare with (5.1))

$$z \mapsto (z, (z, \underbrace{\overline{z}, \dots, \overline{z}}_{n-1})_{\sigma=\tau}, (\underbrace{\overline{z}, \dots, \overline{z}}_{n})_{\sigma\neq\tau}).$$

We have a natural identification $\widehat{T}^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} = \mathbb{C}^\times \times (\{\pm 1\}^n)^{\Phi^+_{\mathbb{C}}}$ so that $\mu_h \in X^*(\widehat{T}^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})})$ (defined on page 167 of [Kot90]) sends each element $(\lambda, (t_{\sigma,i}))$ of $\widehat{T}$ to $\lambda t_{\tau,1}$. In particular,

$$\langle \mu_h, s_{n_1,n_2}\rangle = 1. \tag{3.20}$$

Recall from [Kot90, p.175] that for each $\omega \in \Omega$, the character $a_\omega \in X^*(\widehat{T}^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})})$ is defined by $a_\omega := \omega\mu_h - \mu_h$. Hence

$$a_\omega(\lambda, (t_{\sigma,i})) = t^{-1}_{\tau,1} t_{\tau,\omega(1)}.$$

The following computation is immediate.

$$\langle a_{\omega_* \omega_\pi^i}, s_{n_1,n_2}\rangle = \begin{cases} 1, & \text{if } 1 \le i \le n_1,\\ -1, & \text{if } n_1 < i \le n. \end{cases} \tag{3.21}$$

4. Twisted trace formula and base change

In section 4 we review the twisted trace formula and the base change for the groups $G_{\vec{n}}$ and $\mathbb{G}_{\vec{n}}$. The twisted trace formula is due to Arthur and various results on base change are due to Clozel and Labesse, who also studied the case of unitary groups in more detail. Our strategy basically follows theirs with minor differences for unitary similitude groups. Throughout §4 we assume that

$$\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\varpi) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}. \tag{4.1}$$

For each prime p, fix a field isomorphism $\iota_p : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Also fix an embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, which gives an embedding $\iota_p^{-1}\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ for each p. Choose $\tau : F \hookrightarrow \mathbb{C}$ and define $\Phi^+_{\mathbb{C}}$ and $\Phi^-_{\mathbb{C}}$ as in the last section. For each prime $p \in \mathrm{Spl}_{F/F^+,\mathbb{Q}}$, define $\Phi_p := \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}_p)$, $\Phi^+_p := \iota_p^{-1}\Phi^+_{\mathbb{C}}$, $\Phi^-_p := \iota_p^{-1}\Phi^-_{\mathbb{C}}$. Let $\mathcal{V}_p$ be the set of places of F above p. Let $\mathcal{V}^+_p$ be the image of Φ^+_p under the natural map $\Phi_p \to \mathcal{V}_p$. Then $\mathcal{V}_p = \mathcal{V}^+_p \coprod c\mathcal{V}^+_p$.

Let $\# : R_{F/\mathbb{Q}}GL_{\vec{n}} \mapsto R_{F/\mathbb{Q}}GL_{\vec{n}}$ denote the map $g \mapsto \Phi_{\vec{n}}\,{}^t g^{-c} \Phi_{\vec{n}}^{-1}$. Define

$$\mathbb{G}^+_{\vec{n}} := (R_{E/\mathbb{Q}}\mathrm{GL}_1 \times R_{F/\mathbb{Q}}GL_{\vec{n}}) \rtimes \{1, \theta\} \tag{4.2}$$

where $\theta(\lambda, g)\theta^{-1} = (\lambda^c, \lambda^c g^\#)$. Denote by $\mathbb{G}^0_{\vec{n}}$ and $\mathbb{G}^0_{\vec{n}}\theta$ the cosets of $\{1\}$ and $\{\theta\}$ in $\mathbb{G}^+_{\vec{n}}$ so that $\mathbb{G}^+_{\vec{n}} = \mathbb{G}^0_{\vec{n}} \coprod \mathbb{G}^0_{\vec{n}}\theta$. Recall that $\mathbb{G}_{\vec{n}}$ was defined in the last section. There is a natural $\mathbb{Q}$-isomorphism $\mathbb{G}_{\vec{n}} \xrightarrow{\sim} \mathbb{G}^0_{\vec{n}}$ which may be described on the R-points (for any $\mathbb{Q}$-algebra R) of the underlying groups as

$$(E \otimes_{\mathbb{Q}} R)^\times \times GL_{\vec{n}}(F \otimes_{\mathbb{Q}} E \otimes_{\mathbb{Q}} R) \to (E \otimes_{\mathbb{Q}} R)^\times \times GL_{\vec{n}}(F \otimes_{\mathbb{Q}} R) \tag{4.3}$$

induced by the linear map $f \otimes e \mapsto fe$ from $F \otimes_{\mathbb{Q}} E$ to F. The isomorphism $\mathbb{G}_{\vec{n}} \xrightarrow{\sim} \mathbb{G}^0_{\vec{n}}$ extends to $\mathbb{G}_{\vec{n}} \rtimes \mathrm{Gal}(E/\mathbb{Q}) \xrightarrow{\sim} \mathbb{G}^+_{\vec{n}}$ so that $c \in \mathrm{Gal}(E/\mathbb{Q})$ maps to θ, where $\mathrm{Gal}(E/\mathbb{Q})$ acts on $\mathbb{G}_{\vec{n}}$ in the obvious way. So we will use $\mathbb{G}_{\vec{n}}$, $\mathbb{G}_{\vec{n}}\theta$ interchangeably with $\mathbb{G}^0_{\vec{n}}$, $\mathbb{G}^0_{\vec{n}}\theta$ by abuse of notation.

From now on, we often write $\mathbb{G}$ for $\mathbb{G}_{\vec{n}}$, G for $G_{\vec{n}}$, Φ for $\Phi_{\vec{n}}$ and BC for $BC_{\vec{n}}$ until the end of this section, unless we specify otherwise. We caution the reader that from §5, the symbol G denotes an inner form of G_n.

4.1. θ-stable representations.

Let v be a place of $\mathbb{Q}$. We say that $(\Pi_v, V) \in \mathrm{Irr}(\mathbb{G}(\mathbb{Q}_v))$ is *θ-stable* if $(\Pi_v, V) \simeq (\Pi_v \circ \theta, V)$ as representations of $\mathbb{G}(\mathbb{Q}_v)$. In that case an easy application of Schur's lemma enables us to choose $A_{\Pi_v} : V \xrightarrow{\sim} V$ which induces $\Pi_v \xrightarrow{\sim} \Pi_v \circ \theta$ and satisfies $A^2_{\Pi_v} = \mathrm{id}$. The last condition pins down A_{Π_v} up to sign. We will say that such A_{Π_v} is *normalized.* In §4.2 and §4.3, a specific normalization $A^0_{\Pi_v}$ will be introduced.

Let S be a finite set of places of $\mathbb{Q}$. Similarly $\Pi^S \in \mathrm{Irr}(\mathbb{G}(\mathbb{A}^S))$ is called θ-stable if $\Pi^S \simeq \Pi^S \circ \theta$, in which case we denote by A_{Π^S} an intertwining operator such that $A^2_{\Pi^S} = \mathrm{id}$. Denote by $\mathrm{Irr}^{\theta\text{-st}}(\mathbb{G}(\mathbb{Q}_v))$ (resp. $\mathrm{Irr}^{\theta\text{-st}}(\mathbb{G}(\mathbb{A}^S))$) the subset of $\mathrm{Irr}(\mathbb{G}(\mathbb{Q}_v))$ (resp. $\mathrm{Irr}(\mathbb{G}(\mathbb{A}^S))$) consisting of θ-stable representations.

Given a θ-stable representation (Π_v, V) and $A_{\Pi_v} : \Pi_v \xrightarrow{\sim} \Pi_v \circ \theta$, we can produce a representation (Π_v^+, V) of $\mathbb{G}^+(\mathbb{Q}_v)$ by setting $\Pi_v^+(g) := \Pi_v(g)$ and $\Pi_v^+(\theta) := A_{\Pi_v}$. Conversely, a representation Π_v^+ of $\mathbb{G}^+(\mathbb{Q}_v)$ yields $\Pi_v := \Pi_v^+|_{\mathbb{G}(\mathbb{Q}_v)} \in \mathrm{Irr}^{\theta\text{-st}}(\mathbb{G}(\mathbb{Q}_v))$ and a normalized operator $A_{\Pi_v} := \Pi_v^+(\theta)$.

We may write $\Pi \in \mathrm{Irr}(\mathbb{G}(\mathbb{A}))$ as $\Pi = \psi \otimes \Pi^1$ for a continuous character $\psi : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$ and $\Pi^1 \in \mathrm{Irr}(GL_{\vec{n}}(\mathbb{A}_F))$, corresponding to the isomorphism $\mathbb{G}(\mathbb{A}) \simeq GL_1(\mathbb{A}_E) \times GL_{\vec{n}}(\mathbb{A}_F)$. Denote by ψ_{Π^1} the central character of Π^1. Corresponding to $\vec{n} = (n_1, \dots, n_r)$, write $\psi_{\Pi^1} = \psi_1 \otimes \cdots \otimes \psi_r$. It is easy to verify that Π is θ-stable if and only if

- $(\Pi^1)^\vee \simeq \Pi^1 \circ c$ and
- $\prod_{i=1}^r \psi_i|_{\mathbb{A}_E^\times} = \psi^c/\psi$.

4.2. **Local base change and BC-matching functions at finite places.** For each finite place v, we say that $f_v \in C_c^\infty(\mathbb{G}_{\vec{n}}(\mathbb{Q}_v))$ and $\phi_v \in C_c^\infty(G_{\vec{n}}(\mathbb{Q}_v))$ are *BC-matching functions*, or ϕ_v is a *BC-transfer* of f_v, if they are "associée" in the sense of [Lab99, 3.2]. (This is non-standard terminology.) Similarly we will define in §4.3 the notion of BC-matching for a pair of functions f_∞ on $\mathbb{G}_{\vec{n}}(\mathbb{R})$ and ϕ_∞ on $G_{\vec{n}}(\mathbb{R})$ which are compactly supported modulo $A_{G,\infty}$. The notion of BC-matching functions obviously extends to the adelic case.

We are going to explain case-by-case how to find a BC-transfer ϕ_v of each f_v and how to define the local base change map $BC_{\vec{n}}$. The BC-transfer and $BC_{\vec{n}}$ are closely related via character identities. We will define normalized intertwining operators $A^0_{\Pi_v}$ for θ-stable representations Π_v in each case.

Case 1. $v \in \mathrm{Unr}_{F/\mathbb{Q}}$ and $v \notin \mathrm{Ram}_{\mathbb{Q}}(\varpi)$.

Let $BC^*_{\vec{n}} : \mathscr{H}^{\mathrm{ur}}(\mathbb{G}(\mathbb{Q}_v)) \to \mathscr{H}^{\mathrm{ur}}(G(\mathbb{Q}_v))$ be the dual map of the L-morphism $BC_{\vec{n}}$ defined in §3.1. Define a map $BC_{\vec{n}} : \mathrm{Irr}^{\mathrm{ur}}(G(\mathbb{Q}_v)) \to \mathrm{Irr}^{\mathrm{ur},\theta\text{-st}}(\mathbb{G}(\mathbb{Q}_v))$ which is uniquely characterized by the following identity: for each $\pi_v \in \mathrm{Irr}^{\mathrm{ur}}(G(\mathbb{Q}_v))$ and $f_v \in \mathscr{H}^{\mathrm{ur}}(\mathbb{G}(\mathbb{Q}_v))$,

$$\chi_{BC_{\vec{n}}(\pi_v)}(f_v) = \chi_{\pi_v}(BC^*_{\vec{n}}(f_v)). \tag{4.4}$$

It is a routine check that $BC_{\vec{n}}(\pi_v)$ is θ-stable, but it is not always true that $BC_{\vec{n}}$ is surjective onto $\mathrm{Irr}^{\mathrm{ur},\theta\text{-st}}(\mathbb{G}(\mathbb{Q}_v))$. (The reason for the latter is essentially the same as in [Min, Rem 4.3], which treats unitary groups.) If $v \in \mathrm{Unr}_{F/\mathbb{Q}} \cap \mathrm{Spl}_{E/\mathbb{Q}}$, the injectivity of $BC_{\vec{n}}$ is easily checked. (For instance, use formula (4.8) for $BC_{\vec{n}}$.) However $BC_{\vec{n}}$ is not injective in general.[4] When $\Pi_v \in \mathrm{Irr}^{\mathrm{ur},\theta\text{-st}}(\mathbb{G}(\mathbb{Q}_v))$, we define $A^0_{\Pi_v} : \Pi_v \xrightarrow{\sim} \Pi_v \circ \theta$ as the one acting on $\Pi_v^{\mathbb{K}_v}$ as $+1$ (rather than -1). (The hyperspecial subgroup $\mathbb{K}_v$ defined in §3.1 is clearly θ-stable.) If $\Pi_v = BC(\pi_v)$ then (4.4) implies

$$\mathrm{tr}\,(\Pi_v(f_v)A^0_{\Pi_v}) = \chi_{\Pi_v}(f_v) = \chi_{\pi_v}(BC^*_{\vec{n}}(f_v)).$$

Suppose a finite set of places S contains $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\varpi) \cup \{\infty\}$. For each $\Pi^S \in \mathrm{Irr}^{\mathrm{ur},\theta\text{-st}}(\mathbb{G}(\mathbb{A}^S))$, denote by $A^0_{\Pi^S} : \Pi^S \xrightarrow{\sim} \Pi^S \circ \theta$ the unique intertwining operator which acts on $(\Pi^S)^{\mathbb{K}^S}$ as $+1$. If $\Pi^S = BC_{\vec{n}}(\pi^S)$ then (4.4) implies that

$$\mathrm{tr}\,(\Pi^S(f^S)A^0_{\Pi^S}) = \chi_{\pi^S}(BC^*_{\vec{n}}(f^S)). \tag{4.5}$$

Case 2. $v \in \mathrm{Spl}_{F/F^+,\mathbb{Q}}$ ($v \in \mathrm{Spl}_{E/\mathbb{Q}}$ or $v \notin \mathrm{Spl}_{E/\mathbb{Q}}$).

There are natural isomorphisms

$$G(\mathbb{Q}_v) \simeq \mathbb{Q}_v^\times \times \prod_{x \in \mathcal{V}_v^+} GL_{\vec{n}}(F_x) \tag{4.6}$$

$$\mathbb{G}(\mathbb{Q}_v) \simeq E_v^\times \times \prod_{x \in \mathcal{V}_v^+} GL_{\vec{n}}(F_x) \times \prod_{x \in \mathcal{V}_v^-} GL_{\vec{n}}(F_x). \tag{4.7}$$

If $v \notin \mathrm{Spl}_{E/\mathbb{Q}}$ then θ acts on $\mathbb{G}(\mathbb{Q}_v)$ as $(\lambda, g_+, g_-) \mapsto (\lambda^c, \lambda^c g_-^\#, \lambda^c g_+^\#)$. If $v \in \mathrm{Spl}_{E/\mathbb{Q}}$ then write $\lambda = (\lambda_+, \lambda_-)$ under $E_v^\times \simeq E_u^\times \times E_{u^c}^\times$ where $u = x|_E$ for any $x \in \mathcal{V}_v^+$. Then θ sends $(\lambda_+, \lambda_-, g_+, g_-)$ to $(\lambda_-, \lambda_+, \lambda_- g_-^\#, \lambda_+ g_+^\#)$.

[4]Suppose that v is not split in E and that the multiplier map $G(\mathbb{Q}_v) \to \mathbb{Q}_v^\times$ is surjective. Then $\pi \not\simeq \pi \otimes \chi_{E_v/\mathbb{Q}_v}$ but $BC_{\vec{n}}(\pi) \simeq BC_{\vec{n}}(\pi \otimes \chi_{E_v/\mathbb{Q}_v})$, where $\chi_{E_v/\mathbb{Q}_v}$ is the quadratic character of $\mathbb{Q}_v^\times$ with kernel $E_v^\times$, viewed as a character of $G(\mathbb{Q}_v)$ via the multiplier map.

We define $BC_{\vec{n}} : \mathrm{Irr}(G(\mathbb{Q}_v)) \to \mathrm{Irr}^{\theta\text{-st}}(\mathbb{G}(\mathbb{Q}_v))$. Write $\pi_v \in \mathrm{Irr}(G(\mathbb{Q}_v))$ as $\pi_{v,0} \otimes \pi_{v,+}$ on the underlying vector space $W_0 \otimes W$. Define $BC_{\vec{n}}(\pi_v)$ on $W_0 \otimes W \otimes W$ by

$$\pi_{v,0} \otimes \pi_{v,0}\psi_{\pi_{v,+}} \otimes \pi_{v,+} \otimes \pi^{\#}_{v,+}, \qquad \text{if } v \in \mathrm{Spl}_{E/\mathbb{Q}} \tag{4.8}$$

$$(\pi_{v,0} \circ N_{E_v/\mathbb{Q}_v})\, \psi^c_{\pi_{v,+}} \otimes \pi_{v,+} \otimes \pi^{\#}_{v,+}, \quad \text{if } v \notin \mathrm{Spl}_{E/\mathbb{Q}} \tag{4.9}$$

where $\pi^{\#}_{v,+}(g) := \pi_{v,+}(g^{\#})$. In particular $\pi^{\#}_{v,+} \simeq \pi^{\vee}_{v,+} \circ c$. Define $A^0_{BC(\pi_v)} : BC(\pi_v) \xrightarrow{\sim} BC(\pi_v) \circ \theta$ by $w_0 \otimes w_+ \otimes w_- \mapsto w_0 \otimes w_- \otimes w_+$.

More generally consider $\Pi_v = \Pi_{v,0} \otimes \Pi_{v,+} \otimes \Pi_{v,-} \in \mathrm{Irr}(\mathbb{G}(\mathbb{Q}_v))$, according to (4.7). If $v \in \mathrm{Spl}_{E/\mathbb{Q}}$, write $\Pi_{v,0} = \Pi_{v,0,+} \otimes \Pi_{v,0,-}$ in view of $E^\times_v \simeq E^\times_u \times E^\times_{u^c}$. We see that Π_v is θ-stable if and only if

$$\Pi_{v,0,+} = \Pi_{v,0,-}\psi_{\Pi_{v,-}},\ \Pi_{v,+} \simeq \Pi^{\#}_{v,-} \quad \text{if } v \in \mathrm{Spl}_{E/\mathbb{Q}} \tag{4.10}$$

$$\Pi_{v,0} = \Pi^c_{v,0}\psi^c_{\Pi_{v,+}}\psi_{\Pi_{v,-}},\ \Pi_{v,+} \simeq \Pi^{\#}_{v,-}, \quad \text{if } v \notin \mathrm{Spl}_{E/\mathbb{Q}}. \tag{4.11}$$

For a θ-stable Π_v, choose $\beta : \Pi_{v,+} \xrightarrow{\sim} \Pi^{\#}_{v,-}$. The same map on the underlying vector spaces induces $\beta^{\#} : \Pi^{\#}_{v,+} \xrightarrow{\sim} \Pi_{v,-}$. Define $A^0_{\Pi_v}$ by $w_0 \otimes w_+ \otimes w_- \mapsto w_0 \otimes (\beta^{\#})^{-1}(w_-) \otimes \beta(w_+)$. It is easy to check that $A^0_{\Pi_v}$ is an isomorphism from Π_v to $\Pi_v \circ \theta$ and that $(A^0_{\Pi_v})^2 = \mathrm{id}$.

Consider $f_v \in C^\infty_c(\mathbb{G}_{\vec{n}}(\mathbb{Q}_v))$ of the form $f_v = f_{v,0} \cdot f_{v,+} \cdot f_{v,-}$ with respect to the decomposition (4.7). If $v \notin \mathrm{Spl}_{E/\mathbb{Q}}$, define $\phi_v = BC^*(f_v)$ by

$$\phi_v(\lambda\lambda^c, g) = \int_{E^\times_v/\mathbb{Q}^\times_v \times \prod_{x \in \mathcal{V}^+_v} GL_{\vec{n}}(F_x)} f_{v,0}(\alpha^{c-1}\lambda) f_{v,+}(\alpha^{c-1}\lambda^{-c} g h^{-1}) f_{v,-}(h^{\#})\, d\alpha dh$$

and $\phi_v(\lambda_0, g) = 0$ if $\lambda_0 \notin N_{E_v/\mathbb{Q}_v}(E^\times_v)$. If v splits in E, define ϕ_v by the same formula except that the integrand is replaced by

$$f_{v,0}(\alpha^{c-1}\lambda) f_{v,+}(\alpha^{c-1}\lambda^{-1}_{-} g h^{-1}) f_{v,-}(h^{\#}).$$

The Haar measure used above is chosen to be compatible with the Haar measures on $G(\mathbb{Q}_v)$ and $\mathbb{G}(\mathbb{Q}_v)$ fixed in §3.1. More concretely, the quotient measure on $E^\times_v/\mathbb{Q}^\times_v$ is given by the Haar measures on $E^\times_v$ and $\mathbb{Q}^\times_v$ for which $\mathcal{O}^\times_{E_v}$ and $\mathbb{Z}^\times_v$ have volume 1, respectively. The measure on each $GL_{\vec{n}}(F_x)$ is such that $GL_{\vec{n}}(\mathcal{O}_{F_x})$ has volume 1.

It is shown by an elementary calculation exactly analogous to the proof of [Rog90, Prop 4.13.2.(a)] (but our case is a little more tedious as it is necessary to take care of similitude), that ϕ_v and f_v are BC-matching functions and that

$$\mathrm{tr}\, \pi_v(\phi_v) = \mathrm{tr}\, \pi_v(BC^*(f_v)) = \mathrm{tr}\,(BC(\pi_v)(f_v) A^0_{BC(\pi_v)}) \tag{4.12}$$

for every $\pi_v \in \mathrm{Irr}(G(\mathbb{Q}_v))$. If v splits in E, it is straightforward to check that BC is injective and that BC^* is surjective.

Remark 4.1. The above discussion is consistent in the following sense. Suppose $v \in \mathrm{Unr}_{F/\mathbb{Q}} \cap \mathrm{Spl}_{F/F^+,\mathbb{Q}}$ and $v \notin \mathrm{Ram}_{\mathbb{Q}}(\varpi)$. For every $\pi_v \in \mathrm{Irr}^{\mathrm{ur}}(G(\mathbb{Q}_v))$, $BC_{\vec{n}}(\pi_v)$ is isomorphic in Case 1 and Case 2. If $\Pi_v \in \mathrm{Irr}^{\mathrm{ur}}(\mathbb{G}(\mathbb{Q}_v))$ is θ-stable, it is easily verified that the two definitions of $A^0_{\Pi_v}$ coincide. Furthermore, for each $f_v \in \mathscr{H}^{\mathrm{ur}}(\mathbb{G}(\mathbb{Q}_v))$ there is no ambiguity about ϕ_v since the two definitions of ϕ_v in Case 1 and Case 2 coincide.

4.3. **Base change of discrete series at infinity.** Recall our convention throughout §4 that we write $G = G_{\vec{n}}$ and $\mathbb{G} = \mathbb{G}_{\vec{n}}$ unless stated otherwise. Let ξ be an irreducible algebraic representation of $G_{\mathbb{C}}$. Consider the natural isomorphism $\mathbb{G}(\mathbb{C}) = G(\mathbb{C} \otimes_{\mathbb{Q}} E) \simeq G(\mathbb{C}) \times G(\mathbb{C})$, induced by $\mathbb{C} \otimes_{\mathbb{Q}} E \xrightarrow{\sim} \mathbb{C} \times \mathbb{C}$ mapping $z \otimes e$ to $(z\tau(e), z\tau^c(e))$. Define a representation Ξ of $\mathbb{G}_{\mathbb{C}}$ by $\Xi := \xi \otimes \xi$. We can extend Ξ to a representation Ξ^+ of $\mathbb{G}^+(\mathbb{C})$ by defining $\Xi^+(\theta)$ as $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$ on the underlying vector space for $\xi \otimes \xi$.

We say that $\Pi_\infty \in \mathrm{Irr}^{\theta\text{-st}}(\mathbb{G}(\mathbb{R}))$ is *θ-discrete* (cf. [AC89, p.17]) if Π_∞ is tempered and not a subquotient of any parabolic induction from a θ-stable tempered representation of a proper θ-stable Levi subgroup of $\mathbb{G}(\mathbb{R})$. For a maximal torus $\mathbb{T}$ of $\mathbb{G}$ contained in $\mathbb{K}_\infty$, define $d(\mathbb{G}_{\mathbb{R}}) := \mathfrak{D}(\mathbb{T}, \mathbb{G}; \mathbb{R})$ in the notation of [Lab99, 1.8]. The value of $d(\mathbb{G}_{\mathbb{R}})$ is independent of the choice of $\mathbb{T}$.

Denote by $A_{\mathbb{G}\theta}$ the split component of the centralizer of θ in $A_{\mathbb{G}}$. So $A_{\mathbb{G}\theta}$ is a $\mathbb{Q}$-torus contained in $A_{\mathbb{G}}$. Set $A_{\mathbb{G}\theta,\infty} := A_{\mathbb{G}\theta}(\mathbb{R})^0$. Note that $A_{\mathbb{G}\theta,\infty} = A_{G,\infty}$ via the inclusion $G(\mathbb{R}) \hookrightarrow \mathbb{G}(\mathbb{R})$. Let

$C_c^\infty(\mathbb{G}(\mathbb{R}),\chi_\xi)$ denote the space of smooth functions $\mathbb{G}(\mathbb{R})\to\mathbb{C}$ which are bi-$\mathbb{K}_\infty$-finite, compactly supported modulo $A_{G,\infty}$ and transforms under $A_{G,\infty}$ by χ_ξ. Let $\mathrm{Irr}(\mathbb{G}(\mathbb{R}),\chi_\xi^{-1})$ denote the subset of $\mathrm{Irr}(\mathbb{G}(\mathbb{R}))$ whose central character is the same as χ_ξ^{-1} on $A_{G,\infty}$.

There exists a function $f_\Xi^{\mathrm{Lef}}\in C_c^\infty(\mathbb{G}(\mathbb{R}),\chi_\xi)$ ([Lab91, Prop 12], cf. [CL99, Thm A.1.1]), which is a twisted analogue of the Euler-Poincaré function in Remark 3.9, characterized by the following property: for each $\Pi_\infty^+\in\mathrm{Irr}(\mathbb{G}^+(\mathbb{R}))$ whose restriction to $A_{G,\infty}$ is χ_ξ^{-1},

$$\mathrm{tr}\,\Pi_\infty^+(f_\Xi^{\mathrm{Lef}})=\sum_k(-1)^k\mathrm{tr}\,(\theta|H^k(\mathrm{Lie}\,(\mathbb{G}(\mathbb{R})/A_{G,\infty}),\mathbb{K}_\infty,\Pi_\infty^+\otimes\Xi^+)\tag{4.13}$$

where $\Pi_\infty^+(f_\Xi^{\mathrm{Lef}}):=\int_{\mathbb{G}(\mathbb{R})/A_{G,\infty}}f_\Xi^{\mathrm{Lef}}(g)\Pi_\infty^+(g\theta)dg$. If the infinitesimal characters of Π_∞ and Ξ do not coincide, then the right hand side of (4.13) is zero since the cohomology vanishes in all degrees. (cf. [Wal88, Prop 9.4.6].) Computing the right hand side of (4.13) as in [Lab, Lem 4.10], we can prove that there exists a unique irreducible θ-stable generic unitary representation $\Pi_\Xi\in\mathrm{Irr}(\mathbb{G}(\mathbb{R}),\chi_\xi^{-1})$ such that $\mathrm{tr}\,\Pi_\Xi^+(f_\Xi^{\mathrm{Lef}})\neq0$ for any extension Π_Ξ^+ of Π_Ξ. We remark that an alternative proof of the existence of the function f_Ξ^{Lef} may be given by the results of Delorme and Mezo ([DM08, Thm 3]). By the computation as in the proof of [Clo91, Prop 3.5], we have

$$\mathrm{tr}\,\Pi_\Xi^+(f_\Xi^{\mathrm{Lef}})=\pm2^{n[F^+:\mathbb{Q}]}\tag{4.14}$$

where the sign depends on the choice of an extension Π_Ξ^+ of Π_Ξ. Let $A^0_{\Pi_\Xi}:\Pi_\Xi\stackrel{\sim}{\to}\Pi_\Xi\circ\theta$ denote the operator $\Pi_\Xi^+(\theta)$ where Π_Ξ^+ is chosen so that the sign in (4.14) is positive. Set

$$f_{\mathbb{G},\Xi}:=f_\Xi^{\mathrm{Lef}}/d(\mathbb{G}_\mathbb{R})=f_\Xi^{\mathrm{Lef}}/2^{n[F^+:\mathbb{Q}]-1}.$$

(A direct computation with Galois cohomology shows $d(\mathbb{G}_\mathbb{R})=2^{n[F^+:\mathbb{Q}]-1}$.) The function $f_{\mathbb{G},\Xi}$ is a stabilizing function in the sense of [Lab99, Def 3.8.2] and a cuspidal function in the sense of [Art88b, p.538] by [CL99, Thm A.1.1].

Remark 4.2. There is a direct product decomposition $\mathbb{G}(\mathbb{A})=\mathbb{G}(\mathbb{A})^1\times A_{\mathbb{G}\theta,\infty}$ ([Art86, §1]). Put $\mathbb{G}(\mathbb{R})^1:=\mathbb{G}(\mathbb{R})\cap\mathbb{G}(\mathbb{A})^1$ and $f^1_{\mathbb{G},\Xi}:=f_{\mathbb{G},\Xi}|_{\mathbb{G}(\mathbb{R})^1}$. Note that the inclusion induces $\mathbb{G}(\mathbb{R})^1\stackrel{\sim}{\to}\mathbb{G}(\mathbb{R})/A_{G,\infty}$ and that $\Pi_\infty^+(f_\Xi^{\mathrm{Lef}})$ in (4.13) is the same as $\Pi_\infty^+(f_\Xi^{\mathrm{Lef}}|_{\mathbb{G}(\mathbb{R})^1}):=\int_{\mathbb{G}(\mathbb{R})^1}f_\Xi^{\mathrm{Lef}}(g)\Pi_\infty^+(g\theta)dg$. Hence

$$\mathrm{tr}\,(\Pi_\Xi(f^1_{\mathbb{G},\Xi})\circ A^0_{\Pi_\Xi})=2\tag{4.15}$$

where the trace is computed with respect to the action of $\mathbb{G}(\mathbb{R})^1$. We also see that $\mathrm{tr}\,(\Pi_\infty^+(f^1_{\mathbb{G},\Xi}))=0$ for any $\Pi_\infty^+\in\mathrm{Irr}(\mathbb{G}^+(\mathbb{R}))$ such that Π_∞ is generic and non-isomorphic to Π_Ξ as a representation of $\mathbb{G}(\mathbb{R})^1$. (Here we need not assume that $\Pi_\infty\in\mathrm{Irr}(\mathbb{G}(\mathbb{R}),\chi_\xi^{-1})$.)

We claim that Π_Ξ is the base change of the L-packet $\Pi_{\mathrm{disc}}(G(\mathbb{R}),\xi^\vee)$ in the following sense. Let $\varphi_\xi:W_\mathbb{R}\to{}^LG$ be the L-parameter (unique up to equivalence) corresponding to $\Pi_{\mathrm{disc}}(G(\mathbb{R}),\xi^\vee)$ ([Lan88, §3]). Let $\varphi_\Xi:=\varphi_\xi|_{W_\mathbb{C}}$. Then it is easy to see that Π_Ξ is the unique generic representation corresponding to φ_Ξ via the local Langlands classification for $G(\mathbb{C})$ (cf. [Kna94]). The fact that φ_ξ is a discrete L-parameter implies that Π_Ξ is θ-discrete. Conversely, φ_ξ is uniquely characterized by Π_Ξ and χ_ξ in the sense that there is a unique φ_ξ (up to equivalence) such that

- $\varphi_\xi|_{W_\mathbb{C}}$ corresponds to Π_Ξ by the local Langlands correspondence and
- $W_\mathbb{R}\stackrel{\varphi_\xi}{\to}{}^LG\to{}^LA_G$ corresponds to a character of $A_G(\mathbb{R})$ which restricts to χ_ξ^{-1} on $A_{G,\infty}$. (The L-morphism ${}^LG\to{}^LA_G$ is induced by the canonical injection $A_G\hookrightarrow G$.)

Recall the notation $\phi_{G,\xi}=\phi_{\varphi_\xi}$ from §3.5. We are about to explain that $f_{\mathbb{G},\Xi}$ and $\phi_{G,\xi}$ are BC-matching functions. Let $\delta\in\mathbb{G}(\mathbb{R})$ be any θ-semisimple element (i.e. $\delta\theta$ is semisimple in $\mathbb{G}^+(\mathbb{R})$) and $\gamma\in G(\mathbb{R})$ be the norm of δ ([Lab99, 2.4]). For such δ and γ, a direct computation shows that

$$\mathrm{tr}\,\Xi^+(\delta\theta)=\mathrm{tr}\,\xi(\gamma).\tag{4.16}$$

Indeed, write $\delta=(\delta_1,\delta_2)\in\mathbb{G}(\mathbb{C})\simeq G(\mathbb{C})\times G(\mathbb{C})$ so that $\gamma=\delta_1\delta_2$. Let $A_1:=\Xi(\delta_1)$, $A_2:=\Xi(\delta_2)$ and $C:=\Xi^+(\theta)$ so that $A_1,A_2\in\mathrm{End}(\xi)$ and $C\in\mathrm{End}(\xi\otimes\xi)$. Recall that C is given by $v_1\otimes v_2\mapsto v_2\otimes v_1$. Then (4.16) boils down to the identity $\mathrm{tr}\,(A_1\otimes A_2)\circ C=\mathrm{tr}\,A_1A_2$, which is an elementary exercise in linear algebra.

Let $I_{\delta\theta}$ (resp. I_γ) denote the neutral component of the centralizer of $\delta\theta$ in $\mathbb{G}(\mathbb{R})$ (resp. γ in $G(\mathbb{R})$). The $\mathbb{R}$-groups $I_{\delta\theta}$ and I_γ are inner forms of each other ([Lab99, Lem 2.4.4]). Let $\overline{I}_{\delta\theta}$ (resp. $\overline{I}_\gamma$) denote an inner form of $I_{\delta\theta}$ (resp. I_γ) which is compact modulo center. Then $\overline{I}_{\delta\theta} \simeq \overline{I}_\gamma$. Choose compatible measures $\mu_{I_{\delta\theta}}$, μ_{I_γ}, $\mu_{\overline{I}_{\delta\theta}}$ and $\mu_{\overline{I}_\gamma}$ on $I_{\delta\theta}$, I_γ, $\overline{I}_{\delta\theta}$ and $\overline{I}_\gamma$, respectively. We fixed a Haar measure $\mu_{A_{G,\infty}}$ on $A_{G,\infty}$ in §3.1. Thus we obtain quotient measures $\mu_{I_{\delta\theta}}/\mu_{A_{G,\infty}}$ and $\mu_{I_\gamma}/\mu_{A_{G,\infty}}$. We can compute stable orbital integrals as in [CL99, Thm A.1.1]. (They consider analogues of $d(G_\mathbb{R})\phi_{G,\xi}$ and $d(\mathbb{G}_\mathbb{R})f_{\mathbb{G},\Xi}$, in case ξ and Ξ are trivial. For the computation of $SO_\gamma^{G(\mathbb{R})}(\phi_{G,\xi})$, one may also use [Kot92a, Lem 3.1].) Since our normalization of Haar measures is different from that of [CL99], we need to include the volume factors in the values of stable orbital integrals.

$$SO_\gamma^{G(\mathbb{R})}(\phi_{G,\xi}) = \mu_{\overline{I}_\gamma}/\mu_{A_{G,\infty}}(\overline{I}_\gamma(\mathbb{R})/A_{G,\infty})^{-1}\mathrm{tr}\,\xi(\gamma)$$

$$SO_{\delta\theta}^{\mathbb{G}(\mathbb{R})}(f_{\mathbb{G},\Xi}) = \mu_{\overline{I}_{\delta\theta}}/\mu_{A_{G,\infty}}(\overline{I}_{\delta\theta}(\mathbb{R})/A_{G,\infty})^{-1}\mathrm{tr}\,\Xi^+(\delta\theta)$$

Here we write $SO_\gamma^{G(\mathbb{R})}(\phi_{G,\xi})$ and $SO_{\delta\theta}^{\mathbb{G}(\mathbb{R})}(f_{\mathbb{G},\Xi})$ for $\Phi^1_{G(\mathbb{R})}(\gamma,\phi_{G,\xi})$ and $\Phi^1_{\mathbb{G}(\mathbb{R})}(\delta, f_{\mathbb{G},\Xi}\theta)$, respectively, in the notation of [CL99]. Observe that for any δ and γ as above, $SO_{\delta\theta}^{\mathbb{G}(\mathbb{R})}(f_{\mathbb{G},\Xi})$ and $SO_\gamma^{G(\mathbb{R})}(\phi_{G,\xi})$ have the same value. Hence the functions $f_{\mathbb{G},\Xi}$ and $\phi_{G,\xi}$ are BC-matching in the sense of §4.2.

4.4. **Transfer for $\widetilde{\zeta}_{n_1,n_2}$ and compatibility of transfers.** Fix n_1 and n_2 with $n_1+n_2=n$, $n_1 \geq n_2 > 0$. Recall that $\widetilde{\zeta}_{n_1,n_2} : {}^L\mathbb{G}_{n_1,n_2} \to {}^L\mathbb{G}_n$ was defined in §3.2. Often $\widetilde{\zeta}_{n_1,n_2}$ will be written as $\widetilde{\zeta}$ in this subsection. We would like to give an explicit recipe for the transfer of functions and representations with respect to $\widetilde{\zeta}_{n_1,n_2}$. Since $\mathbb{G}_{n_1,n_2}$ and $\mathbb{G}_n$ are essentially products of general linear groups, it is easy to work explicitly. Recall that we have given a $\mathbb{Q}$-isomorphism $\mathbb{G}_{\vec{n}} \simeq R_{E/\mathbb{Q}}GL_1 \times R_{F/\mathbb{Q}}GL_{\vec{n}}$ for $\vec{n}=(n)$ and $\vec{n}=(n_1,n_2)$. For each place v of $\mathbb{Q}$,

$$\mathbb{G}_{\vec{n}}(\mathbb{Q}_v) \simeq E_v^\times \times GL_{\vec{n}}(F_v).$$

Let $Q_{n_1,n_2} := R_{E/\mathbb{Q}}GL_1 \times R_{F/\mathbb{Q}}P_{n_1,n_2}$, a parabolic subgroup of $\mathbb{G}_n$. Let $\chi_\varpi : \mathbb{G}_{n_1,n_2}(\mathbb{A}) \to \mathbb{C}^\times$ be a character such that

$$(\lambda, g_1, g_2) \in \mathbb{A}_E^\times \times GL_{n_1}(\mathbb{A}_F) \times GL_{n_2}(\mathbb{A}_F) \quad\mapsto\quad \varpi\left(\lambda^{-N(n_1,n_2)}\prod_{i=1}^{2} N_{F/E}(\det(g_i))^{\epsilon(n-n_i)}\right).$$

For each $f_v \in C_c^\infty(\mathbb{G}_n(\mathbb{Q}_v))$ and $\Pi_{M,v} \in \mathrm{Irr}(\mathbb{G}_{n_1,n_2}(\mathbb{Q}_v))$, define $\widetilde{\zeta}^*(f_v) \in C_c^\infty(\mathbb{G}_{n_1,n_2}(\mathbb{Q}_v))$ and $\widetilde{\zeta}_*(\Pi_{M,v}) \in \mathrm{Irr}(\mathbb{G}_n(\mathbb{Q}_v))$ by

$$\widetilde{\zeta}^*(f_v) := f_v^{Q_{n_1,n_2}} \cdot \chi_{\varpi,v}, \qquad \widetilde{\zeta}_*(\Pi_{M,v}) := \text{n-ind}_{Q_{n_1,n_2}}^{\mathbb{G}_n}(\Pi_{M,v} \otimes \chi_{\varpi,v}).$$

(Here det is the product of the component for $E_v^\times$ with the determinant of the component for $GL_{n_1,n_2}(F_v)$.) If $f_v \in \mathscr{H}^{\mathrm{ur}}(\mathbb{G}_n(\mathbb{Q}_v))$ then $\widetilde{\zeta}^*(f_v)$ is no other than the image of the unramified Hecke algebra morphism which is dual to $\widetilde{\zeta}$. Lemma 3.3 implies that for every v,

$$\mathrm{tr}\,\Pi_{M,v}\left(\widetilde{\zeta}^*(f_v)\right) = \mathrm{tr}\,\left(\widetilde{\zeta}_*(\Pi_{M,v})\right)(f_v). \tag{4.17}$$

Now we check whether the transfers for $\widetilde{\eta}_{n_1,n_2}$, $\widetilde{\zeta}_{n_1,n_2}$, BC_{n_1,n_2} and BC_n are compatible, case by case.

Case 1. $v \in \mathrm{Unr}_{F/\mathbb{Q}}$ and $v \notin \mathrm{Ram}_\mathbb{Q}(\varpi)$.

We have two commutative diagrams as follows. The first one is the dual of the diagram (3.4), thus commutative. Then the commutativity of the second one is easy to deduce from the character relation (3.8), (4.5) and (4.17).

$$\begin{array}{ccc} \mathscr{H}^{\mathrm{ur}}(\mathbb{G}_n(\mathbb{Q}_v)) & \xrightarrow{\widetilde{\zeta}^*} & \mathscr{H}^{\mathrm{ur}}(\mathbb{G}_{n_1,n_2}(\mathbb{Q}_v)) \\ {\scriptstyle BC_n^*}\downarrow & & \downarrow{\scriptstyle BC_{n_1,n_2}^*} \\ \mathscr{H}^{\mathrm{ur}}(G_n(\mathbb{Q}_v)) & \xrightarrow[\widetilde{\eta}^*]{} & \mathscr{H}^{\mathrm{ur}}(G_{n_1,n_2}(\mathbb{Q}_v)) \end{array} \qquad \begin{array}{ccc} \mathrm{Irr}^{\mathrm{ur}}(\mathbb{G}_n(\mathbb{Q}_v)) & \xleftarrow{\widetilde{\zeta}_*} & \mathrm{Irr}^{\mathrm{ur}}(\mathbb{G}_{n_1,n_2}(\mathbb{Q}_v)) \\ {\scriptstyle BC_n}\uparrow & & \uparrow{\scriptstyle BC_{n_1,n_2}} \\ \mathrm{Irr}^{\mathrm{ur}}(G_n(\mathbb{Q}_v)) & \xleftarrow[\widetilde{\eta}_*]{} & \mathrm{Irr}^{\mathrm{ur}}(G_{n_1,n_2}(\mathbb{Q}_v)) \end{array} \tag{4.18}$$

Case 2. $v \in \mathrm{Spl}_{F/F^+,\mathbb{Q}}$. ($v \in \mathrm{Spl}_{E/\mathbb{Q}}$ or $v \notin \mathrm{Spl}_{E/\mathbb{Q}}$)

Here we have similar diagrams as in Case 1. All the maps are previously defined and we are interested in the commutativity. Note that we prefer to use Grothendieck groups rather than the sets of isomorphism classes since parabolic induction (involved in $\widetilde{\zeta}_*$ and $\widetilde{\eta}_*$) can be reducible.

$$\begin{array}{ccc} C_c^\infty(\mathbb{G}_n(\mathbb{Q}_v)) & \xrightarrow{\widetilde{\zeta}^*} & C_c^\infty(\mathbb{G}_{n_1,n_2}(\mathbb{Q}_v)) \\ \downarrow{\scriptstyle BC_n^*} & & \downarrow{\scriptstyle BC_{n_1,n_2}^*} \\ C_c^\infty(G_n(\mathbb{Q}_v)) & \xrightarrow[\widetilde{\eta}^*]{} & C_c^\infty(G_{n_1,n_2}(\mathbb{Q}_v)) \end{array} \qquad \begin{array}{ccc} \mathrm{Groth}(\mathbb{G}_n(\mathbb{Q}_v)) & \xleftarrow{\widetilde{\zeta}_*} & \mathrm{Groth}(\mathbb{G}_{n_1,n_2}(\mathbb{Q}_v)) \\ \uparrow{\scriptstyle BC_n} & & \uparrow{\scriptstyle BC_{n_1,n_2}} \\ \mathrm{Groth}(G_n(\mathbb{Q}_v)) & \xleftarrow[\widetilde{\eta}_*]{} & \mathrm{Groth}(G_{n_1,n_2}(\mathbb{Q}_v)) \end{array} \qquad (4.19)$$

It follows from the definition of maps without difficulty that the second diagram is commutative. We claim that the first diagram is commutative (not as functions but) as invariant distributions, in the following sense: for every $f_v \in C_c^\infty(\mathbb{G}_n(\mathbb{Q}_v))$ and $\pi_v \in \mathrm{Irr}(G_{n_1,n_2}(\mathbb{Q}_v))$,

$$\mathrm{tr}\, \pi_v \left(BC^*_{n_1,n_2}(\widetilde{\zeta}^*(f_v)) \right) = \mathrm{tr}\, \pi_v \left(\widetilde{\eta}^*(BC_n^*(f_v)) \right). \qquad (4.20)$$

To prove this, using earlier character identities, we may instead show that

$$\mathrm{tr}\, \left(BC(\pi_v)(\widetilde{\zeta}^*(f_v)) \circ A^0_{BC(\pi_v)} \right) = \mathrm{tr}\, \left(BC(\widetilde{\eta}_*(\pi_v))(f_v) \circ A^0_{BC(\widetilde{\eta}_*(\pi_v))} \right).$$

This follows from Theorem 2 of [Clo84], noting that $A^0_{BC(\pi_v)}$ gives rise to $A^0_{BC(\widetilde{\eta}_*(\pi_v))}$ as in §6.2 of that article.

Remark 4.3. Again, Case 1 and Case 2 are compatible if $v \in \mathrm{Unr}_{F/\mathbb{Q}} \cap \mathrm{Spl}_{F/F^+,\mathbb{Q}}$, $v \notin \mathrm{Ram}_{\mathbb{Q}}(\varpi)$ and representations are unramified.

Remark 4.4. When $v = \infty$, there is the following analogue of (4.18) and (4.19) on the representation side. Let $\varphi_{H,\infty}$ be a discrete L-parameter of $G_{n_1,n_2}(\mathbb{R})$ and φ_∞ be the L-parameter of $G_n(\mathbb{R})$ given by $\varphi_\infty = \widetilde{\eta}\varphi_{H,\infty}$. Write $BC(\varphi_{H,\infty})$ (resp. $BC(\varphi_\infty)$) for the image of base change, namely the representation of $G_{n_1,n_2}(\mathbb{C})$ (resp. $G_n(\mathbb{C})$) corresponding to $\varphi_{H,\infty}|_{W_\mathbb{C}}$ (resp. $\varphi_\infty|_{W_\mathbb{C}}$). Then we have $\widetilde{\zeta}_*(BC(\varphi_{H,\infty})) = BC(\varphi_\infty)$. This is a simple consequence of the fact that (3.4) is commutative. As for test functions, we do not need an exact analogue of (4.18) or (4.19). (We have a loose analogue.) The discussion of the current remark remains valid if G_n is replaced with any inner form.

4.5. **Simplification of the twisted trace formula.** The twisted trace formula by Arthur ([Art88a], [Art88b]) is unconditional thanks to work of Kottwitz and Rogawski ([KR00]) and recent work of Delorme and Mezo ([DM08]). (The two issues were the trace Paley-Wiener theorem over archimedean fields for non-connected groups and whether the distributions in the invariant trace formula are supported on characters. cf. [Art88a, p.330].) Let $f \in C_c^\infty(\mathbb{G}(\mathbb{A}), \chi_\xi)$. The function $f\theta$ on $\mathbb{G}\theta(\mathbb{A})$ is simply defined as the right translation of f by θ. The twisted trace formula for $\mathbb{G}\theta$ is an equality between

$$I^{\mathbb{G}\theta}_{\mathrm{spec}}(f\theta) = I^{\mathbb{G}\theta}_{\mathrm{geom}}(f\theta) \qquad (4.21)$$

where each side is defined in section 3 and 4 of [Art88b]. Recall from Remark 4.2 that there is a natural isomorphism $\mathbb{G}(\mathbb{A}) \simeq \mathbb{G}(\mathbb{A})^1 \times A_{\mathbb{G}\theta,\infty}$. Let f^1 denote the restriction of f to $\mathbb{G}(\mathbb{A})^1$. Actually both sides of (4.21) can be evaluated in terms of f^1, as remarked on [Art88b, p.504].

Let ξ and Ξ be as in §4.3. Define $ST^G_e(\phi)$ for $\phi \in C_c^\infty(G(\mathbb{A}), \chi_\xi)$ by

$$ST^G_e(\phi) := \sum_\gamma \tau(G) \cdot SO^{G(\mathbb{A})}_\gamma(\phi) \qquad (4.22)$$

where γ runs over a set of representatives for $\mathbb{Q}$-elliptic semisimple stable conjugacy classes in $G(\mathbb{Q})$.

Fix a finite set $S \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$ containing $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\varpi)$. (See (4.1).) From here until the end of §4, we assume that $\phi^S \in \mathscr{H}^{\mathrm{ur}}(G(\mathbb{A}^S))$ (resp. $\phi_{S_{\mathrm{fin}}} \in C_c^\infty(G(\mathbb{A}_{S_{\mathrm{fin}}}))$) is a BC-transfer of f^S (resp. $f_{S_{\mathrm{fin}}}$) according to Case 1 (resp. Case 2) of §4.2. The functions

$$\phi := \phi^S \cdot \phi_{S_{\mathrm{fin}}} \cdot \phi_{G,\xi} \quad \text{and} \quad f := f^S \cdot f_{S_{\mathrm{fin}}} \cdot f_{\mathbb{G},\Xi}$$

are BC-matching functions.

Proposition 4.5. *Suppose that* $[F^+ : \mathbb{Q}] \geq 2$. *Then*

$$I^{\mathbb{G}\theta}_{\mathrm{geom}}(f\theta) = \sum_{\delta} \mathrm{vol}(I_{\delta\theta}(\mathbb{Q})A_{\mathbb{G}\theta,\infty}\backslash I_{\delta\theta}(\mathbb{A}))O^{\mathbb{G}(\mathbb{A})}_{\delta\theta}(f)$$

where δ *runs over a set of representatives for* θ*-elliptic* θ*-conjugacy classes in* $\mathbb{G}(\mathbb{Q})$. *(Here* $I_{\delta\theta} := Z_{\mathbb{G}}(\delta\theta)^0$.*)*

Remark 4.6. In case $F^+ = \mathbb{Q}$, the right hand side has to include more terms. See [Mor10, Prop 8.2.3].

Proof. We know that f is cuspidal at ∞ and that $O^{\mathbb{G}(\mathbb{A})}_{\delta\theta}(f) = 0$ if δ is not θ-elliptic in $\mathbb{G}(\mathbb{R})$ by [CL99, Thm A.1.1]. This is a twisted analogue of the first condition of [Art88b, Cor 7.4].

To prove the proposition, it suffices to show that $I^{\mathbb{G}\theta}_{\mathbf{M}}(\delta\theta, f\theta) = 0$ for every proper Levi subset $\mathbf{M} \subset \mathbb{G}\theta$ and semisimple element $\delta \in \mathbf{M}(\mathbb{Q})$. Once we have done this, we can use the argument in the proof of Theorem 7.1.(b) and Corollary 7.4 of [Art88b] to finish the proof, even though f is not necessarily cuspidal at any other place than ∞. (The assumption that f is cuspidal at two places was imposed by Arthur to guarantee that $I^{\mathbb{G}\theta}_{\mathbf{M}}(\delta\theta, f\theta) = 0$.)

Since $f_{\mathbb{G},\Xi}$ is a cuspidal function, the splitting formula ([Art88a, Prop 9.1]) implies that

$$I^{\mathbb{G}\theta}_{\mathbf{M}}(\delta\theta, f\theta) = \sum_{\mathbf{L}} d^{\mathbb{G}\theta}_{\mathbf{M}}(\mathbf{L}, \mathbb{G})\widehat{I}^{\mathbf{L}}_{\mathbf{M}}(\delta\theta, (f^{\infty}\theta)_{\mathbf{L}})I^{\mathbb{G}\theta}_{\mathbf{M}}(\delta\theta, f_{\mathbb{G},\Xi}\theta)$$

in Arthur's notation, where the sum is taken over the Levi subsets $\mathbf{L}$ of $\mathbb{G}\theta$ containing $\mathbf{M}$. (Note that $I^{\mathbb{G}\theta}_{\mathbf{M}}(\delta\theta, f_{\mathbb{G},\Xi}\theta) = \widehat{I}^{\mathbb{G}\theta}_{\mathbf{M}}(\delta\theta, f_{\mathbb{G},\Xi}\theta)$. cf. [Art88a, Cor 8.3]) The point is that $I^{\mathbb{G}\theta}_{\mathbf{M}}(\delta\theta, f_{\mathbb{G},\Xi}\theta) = 0$ unless $\mathbf{M}$ is a cuspidal Levi subset of $\mathbb{G}\theta$, as shown in the proof of [Mor10, Prop 8.2.3]. So it suffices to prove that $\mathbb{G}\theta$ does not have any proper cuspidal Levi subset. By the very definition of proper cuspidal Levi subsets ([Mor10, §8.2]), we can reduce the proof to showing that G has no proper cuspidal $\mathbb{Q}$-Levi subgroups in the sense of [Mor10, Def 3.1.1]. Recall that a Levi subgroup $M \subset G$ is cuspidal if $M_{\mathbb{R}}$ has no maximal tori which are anisotropic modulo $(A_M)_{\mathbb{R}}$. Suppose that $M \subsetneq G$. Let G_1 denote the kernel of the multiplier map $G \to \mathbb{G}_m$. Consider the Levi subgroup $M_1 := M \cap G_1$ of G_1. For notational convenience we prove that M_1 is not cuspidal, as the same proof will show M is not cuspidal. The fact that $M_1 \subsetneq G_1$ implies that M_1 contains a direct factor of the form $D = R_{F/\mathbb{Q}}GL_a$ for some $a \in \mathbb{Z}_{>0}$. But the center of $D \times_{\mathbb{Q}} \mathbb{R}$ contains a split torus of rank $[F^+ : \mathbb{Q}] > 1$ whereas A_D is a rank 1 torus. So M_1 cannot be cuspidal. □

Corollary 4.7. *(*[Lab, Thm 4.13]*, cf.* [Lab99, Thm 4.3.4]*)*
Let $\tau(G)$ *be the Tamagawa number of* G *(cf. Lemma 3.1). Suppose that* $[F^+ : \mathbb{Q}] \geq 2$. *Then*

$$I^{\mathbb{G}\theta}_{\mathrm{geom}}(f\theta) = \tau(G)^{-1} \cdot ST^G_e(\phi).$$

Proof. We use the notation of [Lab99]. (The only unfortunate difference is that his use of the symbols f and ϕ is opposite to ours.) By théorème 4.3.4 of Labesse,

$$T^{\mathbb{G}\theta}_e(f\theta) = \frac{\tau(\mathbb{G})J_Z(\theta)}{\tau(G)d(G, \mathbb{G})} \cdot ST^G_e(\phi).$$

Comparing the definition of $T^{\mathbb{G}\theta}_e$ ([Lab99, §4.1]) with the formula of Proposition 4.5, we see that $T^{\mathbb{G}\theta}_e(f\theta)$ equals $J_Z(\theta) \cdot I^{\mathbb{G}\theta}_{\mathrm{geom}}(f\theta)$. By Lemma 3.1, $\tau(\mathbb{G}) = 1$. Since $H^1(\mathbb{R}, \mathbb{G})$ is trivial, $d(G, \mathbb{G}) = 1$ (which is defined on [Lab99, p.45]). So the proof is complete. □

In view of (4.2), fix a minimal Levi subgroup

$$M_0 := R_{E/\mathbb{Q}}GL_1 \times R_{F/\mathbb{Q}}(i_{(1,\dots,1)}(\underbrace{GL_1 \times \cdots \times GL_1}_{\sum_i n_i}))$$

of $\mathbb{G} = \mathbb{G}_{\vec{n}}$. Let M be a $\mathbb{Q}$-Levi subgroup of $\mathbb{G}$ containing M_0. (We do not assume that M is θ-stable.) Choose a parabolic subgroup Q containing M as a Levi subgroup. The group $W^{\mathbb{G}\theta}(\mathfrak{a}_M)_{\mathrm{reg}}$ defined on [Art88b, p.517] acts on the set of parabolic subgroups which have M as a Levi component. For each $s \in W^{\mathbb{G}\theta}(\mathfrak{a}_M)_{\mathrm{reg}}$, let Q^s denote $s(Q)$. Choose a representative $w \in \mathbb{G}\theta(\mathbb{Q})$ of s. Note that $\Phi^{-1}\theta$ acts on $\mathbb{G}$ by $(\lambda, g) \mapsto (\lambda^c, \lambda^c\, {}^t g^{-c})$. So $\Phi^{-1}\theta$ preserves any M containing M_0. In particular, $\Phi^{-1}\theta$ defines an element of $W^{\mathbb{G}\theta}(\mathfrak{a}_M)_{\mathrm{reg}}$ for each M.

Consider the regular representation $R_{M,\text{disc}}$ of $M(\mathbb{A})$ on $L^2_{\text{disc}}(M(\mathbb{Q})A_{M,\infty}\backslash M(\mathbb{A}))$. Noting that s acts on M, let $R_{M,\text{disc}}(s)$ denote the action $\phi \mapsto \phi\circ s$ on the underlying space for $R_{M,\text{disc}}$. Let $x\mapsto \rho_Q(s,0,x)$ denote the representation $\text{n-ind}_Q^{\mathbb{G}}(R_{M,\text{disc}})$ of $\mathbb{G}(\mathbb{A})$. Arthur defined the operators $\rho_Q(s,0,x\theta)$ (for $x\in\mathbb{G}(\mathbb{A})$) and

$$\rho_Q(s,0,f^1\theta) : \text{n-ind}_Q^{\mathbb{G}}(R_{M,\text{disc}}) \to \text{n-ind}_{Q^s}^{\mathbb{G}}(R_{M,\text{disc}})$$

on page 516 of [Art88b]. These operators are $\mathbb{G}(\mathbb{A})^1$-equivariant if the $\mathbb{G}(\mathbb{A})^1$-action on the target is twisted by θ. The decomposition $R_{M,\text{disc}} = \oplus_{\Pi_M}\Pi_M$ into irreducible subrepresentations yields a decomposition of operators

$$\rho_Q(s,0,f^1\theta) = \oplus_{\Pi_M}\rho_Q(s,0,f^1\theta;\Pi_M).$$

If Π_M is such that $\Pi_M\simeq\Pi_M^s$ then $\rho_Q(s,0,f^1\theta;\Pi_M)$ can be seen as a composition of the following operations

$$\text{n-ind}_Q^{\mathbb{G}}(\Pi_M) \xrightarrow{\rho_Q(s,0,\theta;\Pi_M)} \text{n-ind}_{Q^s}^{\mathbb{G}}(\Pi_M)^\theta \xrightarrow{\text{n-ind}_{Q^s}^{\mathbb{G}}(\Pi_M)(f^1)} \text{n-ind}_{Q^s}^{\mathbb{G}}(\Pi_M)^\theta \tag{4.23}$$

where the first arrow is described as follows. Let $V(\Pi_M)$ be the underlying vector space for Π_M. Then $\rho_Q(s,0,\theta;\Pi_M)$ is an isomorphism sending $\psi:\mathbb{G}(\mathbb{A})\to V(\Pi_M)$ to ψ' which is defined by $\psi'(g) = R_{M,\text{disc}}(s)(\psi(w^{-1}g\theta))$. This map does not depend on the choice of w. Here $\text{n-ind}_{Q^s}^{\mathbb{G}}(\Pi_M)^\theta$ denotes the representation $\text{n-ind}_{Q^s}^{\mathbb{G}}(\Pi_M)\circ\theta$. It is easy to see from Arthur's description of $\rho_Q(s,0,f^1\theta)$ that the following also holds.

$$\rho_Q(s,0,f^1\theta;\Pi_M) = \rho_Q(s,0,\theta;\Pi_M)\circ \text{n-ind}_Q^{\mathbb{G}}(\Pi_M)^\theta(f^1) \tag{4.24}$$

The intertwining operator $M_{Q|Q^s}(0)$ sends

$$\text{n-ind}_{Q^s}^{\mathbb{G}}(\Pi_M)^\theta \to \text{n-ind}_Q^{\mathbb{G}}(\Pi_M)^\theta. \tag{4.25}$$

As Π_M is unitary, $\text{n-ind}_Q^{\mathbb{G}}(\Pi_M)^\theta$ is irreducible for any choice of $Q\supset M$ and $M_{Q|Q^s}(0)$ is an isomorphism. (cf. [MW89, p.607,(3)])

Proposition 4.8.

$$I^{\mathbb{G}\theta}_{\text{spec}}(f\theta) = \sum_M \frac{|W_M|}{|W_{\mathbb{G}}|}|\det(\Phi^{-1}\theta-1)_{\mathfrak{a}_M^{\mathbb{G}\theta}}|^{-1}\sum_{\Pi_M}\text{tr}\left(M_{Q|Q^{\Phi^{-1}\theta}}(0)\rho_Q(\Phi^{-1}\theta,0,f^1\theta;\Pi_M)\right)$$

where M runs over the Levi subgroups of $\mathbb{G}$ containing M_0 and Π_M runs over the irreducible $\Phi^{-1}\theta$-stable subrepresentations of $R_{M,\text{disc}}$. (By the multiplicity one theorem for general linear groups, each isomorphism class of Π_M contributes to $R_{M,\text{disc}}$ only once.)

Remark 4.9. It is easy to check that $\text{n-ind}_Q^{\mathbb{G}}(\Pi_M)$ is θ-stable if Π_M is $\Phi^{-1}\theta$-stable.

Remark 4.10. Let r be the number of (nonzero) components in $\vec{n}$, where $\mathbb{G}=\mathbb{G}_{\vec{n}}$. If $M=\mathbb{G}$, the term $|\det(\Phi^{-1}\theta-1)_{\mathfrak{a}_M^{\mathbb{G}\theta}}|$ in Proposition 4.8 is equal to 2^r. The same term equals 2^{r+1} if M is a maximal proper Levi subgroup of $\mathbb{G}$.

Proof. By Theorem 7.1.(a) (cf. p.516-517) of [Art88b],

$$I^{\mathbb{G}\theta}_{\text{spec}}(f\theta) = \sum_M\frac{|W_M|}{|W_{\mathbb{G}}|}\sum_s|\det(s-1)_{\mathfrak{a}_M^{\mathbb{G}\theta}}|^{-1}\sum_{\Pi_M}\text{tr}\left(M_{Q|Q^s}(0)\rho_Q(s,0,f^1\theta;\Pi_M)\right)$$

where M is as above, s runs over $W^{\mathbb{G}\theta}(\mathfrak{a}_M)_{\text{reg}}$, and Π_M runs over the irreducible subrepresentations of $R_{M,\text{disc}}$. The Weyl set $W^{\mathbb{G}\theta}(\mathfrak{a}_M)_{\text{reg}}$ is defined on page 517 of [Art88b] by the condition

$$|\det(s-1)_{\mathfrak{a}_M^{\mathbb{G}\theta}}|\neq 0. \tag{4.26}$$

and a description of its elements will be recalled as the proof proceeds. It is easy to see from Arthur's description of $\rho_Q(s,0,f^1\theta)$ that only those Π_M such that $\Pi_M\simeq\Pi_M^s$ contribute to the sum. The proof is complete if we show that

$$\text{tr}\,(M_{Q|Q^s}(0)\rho_Q(s,0,f^1_{\mathbb{G},\Xi}\theta;\Pi_{M,\infty})) = 0 \tag{4.27}$$

for any $s \neq \Phi^{-1}\theta$ (which may occur only when $M \neq \mathbb{G}$). By (4.24), the left side may be rewritten as

$$\mathrm{tr}\,(M_{Q|Q^s}(0) \circ \rho_Q(s,0,\theta;\Pi_{M,\infty}) \circ (\text{n-ind}_Q^{\mathbb{G}_{\vec{n}}}(\Pi_{M,\infty}))^\theta(f^1_{\mathbb{G},\Xi})).$$

Put $\Pi := \text{n-ind}_Q^{\mathbb{G}_{\vec{n}}}(\Pi_{M,\infty})$. Let $A : \Pi \to \Pi^\theta$ denote the operator $M_{Q|Q^s}(0)\rho_Q(s,0,\theta;\Pi_{M,\infty})$. As noted earlier, Π is irreducible and A is an isomorphism. Hence $A \circ A$ is a scalar operator on Π. Let $A_{\Pi^\theta} : \Pi^\theta \xrightarrow{\sim} \Pi$ be a normalized intertwining operator (which can also be viewed as $\Pi \xrightarrow{\sim} \Pi^\theta$). To prove (4.27), we may instead show

$$\mathrm{tr}\,(\Pi^\theta(f^1_{\mathbb{G},\Xi})A_{\Pi^\theta}) = 0. \tag{4.28}$$

We claim that if $\Pi_M \simeq \Pi_M^s$ for $s \neq \Phi^{-1}\theta$ then the infinitesimal character of Π is not regular. For convenience of notation, we prove the claim when $\mathbb{G} = \mathbb{G}_n$ as the proof is identical in the more general case $\mathbb{G} = \mathbb{G}_{\vec{n}}$. In this case M is isomorphic to $\mathbb{G}_{m_1,\dots,m_r}$ with $\sum_{i=1}^r m_i = n$ ($m_i > 0$). There is an $\mathbb{R}$-vector space

$$\mathfrak{a}_M^{\mathbb{G}\theta} \simeq \underbrace{\mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}}_{r}$$

which is a quotient of $\mathrm{Hom}(X^*(A_M),\mathbb{R})$ by $\mathrm{Hom}(X^*(A_{\mathbb{G}\theta}),\mathbb{R})$. An element $s \in W^{\mathbb{G}\theta}(\mathfrak{a}_M)_{\mathrm{reg}}$ can be represented by $s = w(\Phi^{-1}\theta)$ where $\Phi^{-1}\theta$ acts as multiplication by -1 on $\mathfrak{a}_M^{\mathbb{G}\theta}$ and w acts as an element of the symmetric group $\mathcal{S}_r$ which naturally acts on $\mathfrak{a}_M^{\mathbb{G}\theta}$ by permutation. By the assumption $s \neq \Phi^{-1}\theta$, w is nontrivial. Write

$$\Pi_{M,\infty} = \bigotimes_{\sigma \in \Phi_{\mathbb{C}}} (\Pi_{M,\sigma,1} \otimes \cdots \otimes \Pi_{M,\sigma,r})$$

and $w = c_1 \cdots c_k$ ($k \geq 1$) where c_i are mutually disjoint nontrivial cycles in $\mathcal{S}_r$. The condition (4.26) implies that every c_i is an odd cycle. By rearranging m_i's if necessary, let c_1 be the cycle $1 \to 2 \to \cdots \to a \to 1$ for an odd number $a \geq 3$. Then $\Pi_M \simeq \Pi_M^s$ implies that

$$\Pi_{M,1,\sigma} \simeq \Pi^\theta_{M,2,\sigma^c} \simeq \Pi_{M,3,\sigma} \simeq \Pi^\theta_{M,4,\sigma^c} \cdots .$$

This proves the claim since the isomorphism $\Pi_{M,1,\sigma} \simeq \Pi_{M,3,\sigma}$ indicates that the infinitesimal character of Π is not regular.

By the claim, if $s \neq \Phi^{-1}\theta$ and $\Pi_M \simeq \Pi_M^s$ then the infinitesimal character of Π^θ is not equal to that of any irreducible finite dimensional representation of $\mathbb{G}_{\vec{n}}$. In view of Remark 4.2 and the remark below (4.13), we conclude that (4.28) holds.

□

Lemma 4.11. *Suppose that M and Π_M are as in Proposition 4.8. Set*

$$A'_{\text{n-ind}_Q^{\mathbb{G}}(\Pi_M)} := M_{Q|Q^{\Phi^{-1}\theta}}(0) \circ \rho_Q(\Phi^{-1}\theta, 0, \theta; \Pi_M),$$

which is an operator from $\text{n-ind}_Q^{\mathbb{G}}(\Pi_M)$ to $\text{n-ind}_Q^{\mathbb{G}}(\Pi_M)^\theta$. (cf. Remark 4.9.) Then $A'_{\text{n-ind}_Q^{\mathbb{G}}(\Pi_M)}$ is normalized, i.e. $A'_{\text{n-ind}_Q^{\mathbb{G}}(\Pi_M)} \circ A'_{\text{n-ind}_Q^{\mathbb{G}}(\Pi_M)} = \mathrm{id}$.

Remark 4.12. If $M = \mathbb{G}$, things are simpler. Let us write Π for $\text{n-ind}_Q^{\mathbb{G}}(\Pi_M) = \Pi_M$. It is easy to see that A'_Π is given by $R_{M,\mathrm{disc}}(\Phi^{-1}\theta)$, from the paragraph between (4.23) and (4.24).

Remark 4.13. The sign of an analogous intertwining operator in the case of unitary groups is precisely computed in [CHLb] (especially §4.4) by a different method replying on the so-called Whittaker normalization.

Proof. For simplicity we write $s = \Phi^{-1}\theta$ and $\Pi = \text{n-ind}_Q^{\mathbb{G}}(\Pi_M)$. We know A'_Π is an isomorphism since $\rho_Q(s,0,\theta;\Pi_M)^2 = \mathrm{id}$ and $M_{Q|Q^s}(0)$ is an isomorphism. (See the paragraph above Proposition 4.8.)

For ease of reference, we use [Art05] and its notation. Recall that $M_{Q|Q^s}(\lambda)$ for $\lambda \in \mathfrak{a}^*_{Q,\mathbb{C}}$ is defined by a precise global analogue of the first displayed formula of page 135. (Also see page 128 of that article.) If λ lies in a certain chamber then the integral formula for $M_{Q|Q^s}(\lambda)$ absolutely converges ([Art05, Lem 7.1]), and $M_{Q|Q^s}(\lambda)$ is defined by analytic continuation in general. It is a standard fact that the functional equation $M_{Q|Q^s}(\lambda)M_{Q^s|Q}(-\lambda) = \mathrm{id}$ holds for any $\lambda \in \mathfrak{a}^*_{Q,\mathbb{C}}$ (page 129). So the lemma is proved if we show

$$M_{Q|Q^s}(\lambda) \circ \rho_Q(s,0,\theta;\Pi_M) = \rho_Q(s,0,\theta;\Pi_M) \circ M_{Q^s|Q}(-\lambda)$$

for $\lambda = 0$. It suffices to check this equality in the range of absolute convergence. Now this is an easy exercise using our earlier explicit description of $\rho_Q(s, 0, \theta; \Pi_M)$ and the integral formula for $M_{Q|Q^s}(\lambda)$. □

Recall that $A_{G,\infty} = A_{\mathbb{G}\theta,\infty} \subset A_{\mathbb{G},\infty}$. Let $a_{\mathbb{G},\infty} : \mathbb{G}(\mathbb{A}) \twoheadrightarrow A_{G,\infty}$ denote the natural surjection. Define $\widetilde{\chi}_\xi : \mathbb{G}(\mathbb{A}) \to \mathbb{C}^\times$ by $\widetilde{\chi}_\xi := \chi_\xi \circ a_{\mathbb{G},\infty}$. In the notation of the above lemma, set

$$\text{n-ind}_Q^{\mathbb{G}}(\Pi_M)_\xi := \text{n-ind}_Q^{\mathbb{G}}(\Pi_M) \otimes \widetilde{\chi}_\xi^{-1}. \tag{4.29}$$

If Π_M is $\Phi^{-1}\theta$-stable then $\text{n-ind}_Q^{\mathbb{G}}(\Pi_M)_\xi$ is θ-stable. Observe that $A'_{\text{n-ind}_Q^{\mathbb{G}}(\Pi_M)_\xi} := A'_{\text{n-ind}_Q^{\mathbb{G}}(\Pi_M)}$ serves as a normalized intertwining operator for $\text{n-ind}_Q^{\mathbb{G}}(\Pi_M)_\xi$. The following is easily deduced from Lemma 4.11.

Corollary 4.14. *The second summand in Proposition 4.8 is computed as*

$$\operatorname{tr}\left(M_{Q|Q^{\Phi^{-1}\theta}}(0)\rho_Q(\Phi^{-1}\theta, 0, f^1\theta; \Pi_M)\right) = \operatorname{tr}\left(\text{n-ind}_Q^{\mathbb{G}}(\Pi_M)_\xi(f) \circ A'_{\text{n-ind}_Q^{\mathbb{G}}(\Pi_M)_\xi}\right).$$

5. Shimura varieties and Igusa varieties

Throughout §5 and §6, we fix a prime l and an isomorphism $\iota_l : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$.

5.1. **PEL datum for Shimura varieties.** Consider a quintuple $(F, *, V, \langle\cdot,\cdot\rangle, h)$, called a PEL datum, given as follows.

- F is a CM field with an involution $* = c$.
- $V = F^n$ is an F-vector space.
- $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{Q}$ is a nondegenerate Hermitian pairing such that $\langle fv_1, v_2\rangle = \langle v_1, f^c v_2\rangle$ for all $f \in F$, $v_1, v_2 \in V$.
- $h : \mathbb{C} \to \operatorname{End}_F(V) \otimes_{\mathbb{Q}} \mathbb{R}$ is an $\mathbb{R}$-algebra homomorphism such that the bilinear pairing $(v_1, v_2) \mapsto \langle v_1, h(i)v_2\rangle$ is symmetric and positive definite.

Define a $\mathbb{Q}$-group G by

$$G(R) = \{(\lambda, g) \in R^\times \times \operatorname{End}_{F\otimes_{\mathbb{Q}}R}(V \otimes_{\mathbb{Q}} R) \mid \langle gv_1, gv_2\rangle = \lambda(g)\langle v_1, v_2\rangle \text{ for all } v_1, v_2 \in V \otimes_{\mathbb{Q}} R\}$$

for any $\mathbb{Q}$-algebra R. We see that the group G_n defined in §3 is a quasi-split $\mathbb{Q}$-inner form of G.

Fix an embedding $\tau : F \hookrightarrow \mathbb{C}$. Suppose that F contains an imaginary quadratic field E so that $F = EF^+$, where $F^+ := F^{c=1}$. Define $\Phi^+_{\mathbb{C}}$ as in §3.1. Until the end of §6 we further assume that

(i) $n \in \mathbb{Z}_{\geq 3}$ is *odd*,
(ii) $[F^+ : \mathbb{Q}] \geq 2$,
(iii) $\operatorname{Ram}_{F/\mathbb{Q}} \subset \operatorname{Spl}_{F/F^+,\mathbb{Q}}$ (cf. (4.1)),
(iv) $G_{\mathbb{Q}_v}$ is quasi-split at every finite place v, and
(v) For $\sigma \in \Phi^+_{\mathbb{C}}$, (p_σ, q_σ) is $(1, n-1)$ if $\sigma = \tau$ and $(0, n)$ otherwise. (See §3.1 for $\Phi^+_{\mathbb{C}}$.)

We list a few (but not all) implications of the above assumptions to guide readers. The assumptions (ii) and (v) imply that G is anisotropic modulo center over $\mathbb{Q}$ and the reflex field for the PEL datum is F (viewed as a subfield of $\mathbb{C}$ via τ). The assumption (iii) ensures that the local (quadratic) base change is unconditional at every finite place, if ramification is suitably controlled, as it may be defined in an elementary manner as in §4.2. (In general the local base change should involve local L-packets and has not been established yet.) By (iv) there is an isomorphism $G \times_{\mathbb{Q}} \mathbb{A}^\infty \simeq G_n \times_{\mathbb{Q}} \mathbb{A}^\infty$, which we fix.

The following lemma is standard. (cf. [HT01, Lem I.7.1].) All the necessary results in Galois cohomology that go into its proof are found in [Clo91, §2]. The point is that when n is odd, there is no cohomological obstruction for finding a global unitary (similitude) group with prescribed local isomorphism classes.

Lemma 5.1. *As above, let $F = EF^+$ be a CM field. For any $\tau : F \hookrightarrow \mathbb{C}$, there exists a PEL datum $(F, *, V, \langle,\cdot,\cdot,\rangle, h)$ such that the associated group G satisfies (iv) and (v) above.*

More explicitly, we will choose h such that under the natural $\mathbb{R}$-algebra isomorphism $\mathrm{End}_F(V)_{\mathbb{R}} \simeq \prod_{\sigma\in\Phi_{\mathbb{C}}^+} M_n(\mathbb{C})$, the map h sends

$$z \mapsto \left(\begin{pmatrix} zI_{p_\sigma} & 0 \\ 0 & \bar{z}I_{q_\sigma} \end{pmatrix}_{\sigma\in\Phi_{\mathbb{C}}^+} \right) \tag{5.1}$$

for some $p_\sigma, q_\sigma \in \mathbb{Z}_{\geq 0}$ such that $p_\sigma + q_\sigma = n$. There is a standard way to associate a $\mathbb{C}$-morphism $\mu_h : \mathbb{G}_m \to G$ ([Kot92b, Lem 4.1.(2)]). Under the natural isomorphism $G_{\mathbb{C}} \simeq GL_1 \times \prod_{\sigma\in\Phi_{\mathbb{C}}^+} GL_n$, we may describe μ_h as

$$z \mapsto \left(z, \begin{pmatrix} zI_{p_\sigma} & 0 \\ 0 & I_{q_\sigma} \end{pmatrix} \right).$$

Fix a prime $p \in \mathrm{Spl}_{E/\mathbb{Q}}$ such that $p \neq l$. Also fix a place w of F above p. (In fact the case $p = l$ is considered once, only in establishing Proposition 5.3.(v) where we refer to Harris-Taylor for the proof.) Choose $\iota_p : \overline{\mathbb{Q}}_p \overset{\sim}{\to} \mathbb{C}$ such that $\iota_p^{-1}\tau : F \hookrightarrow \overline{\mathbb{Q}}_p$ induces w. We will keep τ, p, w and ι_p fixed until the end of §6.1. Define $\mathcal{V}_p^+$ as in the beginning of §4. For convenience, write $\mathcal{V}_p^+ = \{w_1, \dots, w_r\}$ where $w_1 = w$. Define $\Phi_{w_i} := \mathrm{Hom}_{\mathbb{Q}_p}(F_{w_i}, \overline{\mathbb{Q}}_p)$. Using $\iota_p^{-1}\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ we get

$$W_{F_w} \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}_p/F_w) \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/F).$$

Write $\mu = \mu_{\iota_p}$ for the $\overline{\mathbb{Q}}_p$-morphism $\mu_h \times_{\mathbb{C},\iota_p^{-1}} \overline{\mathbb{Q}}_p$. Let $\mu_0 : \mathbb{G}_m \to \mathbb{G}_m$ denote the identity map. For each w_i define $\mu_{w_i} : \mathbb{G}_m \to (R_{F_{w_i}/\mathbb{Q}_p} GL_n) \times_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \simeq \prod_{\sigma\in\Phi_{w_i}} (GL_n)_{\overline{\mathbb{Q}}_p}$ by

$$z \mapsto \left(z, \begin{pmatrix} zI_{p_\sigma} & 0 \\ 0 & I_{q_\sigma} \end{pmatrix}_{\sigma\in\Phi_{w_i}} \right)$$

so that $\mu = (\mu_0, (\mu_{w_i})_{1\leq i\leq r})$. We have $p_\sigma = 1$ if σ is induced by $\iota_p^{-1}\tau$ and $p_\sigma = 0$ otherwise.

Let us describe the finite set $B(G_{\mathbb{Q}_p}, -\mu)$. Using the isomorphism

$$G_{\mathbb{Q}_p} \simeq GL_1 \times \prod_{1\leq i\leq r} R_{F_{w_i}/\mathbb{Q}_p} GL_n, \tag{5.2}$$

we identify

$$B(G_{\mathbb{Q}_p}, -\mu) = B(GL_1, -\mu_{p,0}) \times \prod_{1\leq i\leq r} B(R_{F_{w_i}/\mathbb{Q}_p} GL_n, -\mu_{w_i})$$

and write $b \in B(G_{\mathbb{Q}_p}, -\mu)$ as $(b_0, (b_{w_i}))$. In view of [Shi09, Ex 4.3], there is a bijection

$$\{h \in \mathbb{Z} : 0 \leq h \leq n-1\} \overset{1-1}{\longleftrightarrow} B(G_{\mathbb{Q}_p}, -\mu) \tag{5.3}$$

where h corresponds to $b(h) = (b_0, (b_{w_i}))$ which is given by $b_0 = b_{1,0}$, $b_w = b_{n-h,h}$ and $b_{w_i} = b_{0,n}$ for $i > 1$ in the notation of §2.4. When $b = b(h)$,

$$J_b(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times (D^\times_{F_w,1/(n-h)} \times GL_h(F_w)) \times \prod_{i>1} GL_n(F_{w_i}). \tag{5.4}$$

Recall from §3.1 that we defined the groups $K_v \subset G_n(\mathbb{Q}_v)$ ($v \neq \infty$) and the measures $\mu_{G_n,v}$ on $G_n(\mathbb{Q}_v)$ for every v as well as $\mu_{A_{G_n,\infty}}$ on $A_{G_n,\infty} = A_{G,\infty}$. For each $v \in \mathrm{Unr}_{F/\mathbb{Q}}$, define a hyperspecial subgroup U_v^{hs} of $G(\mathbb{Q}_v)$ to be the image of K_v under the isomorphism $G(\mathbb{Q}_v) \simeq G_{\vec{n}}(\mathbb{Q}_v)$ which was fixed earlier. We transport $\mu_{G_n,v}$ to a Haar measure $\mu_{G,v}$ on $G(\mathbb{Q}_v)$ for each $v \neq \infty$ via the last isomorphism. To fix a Haar measure on $J_b(\mathbb{Q}_p)$, denote by M_b (cf. §5.5) the quasi-split inner form of J_b over $\mathbb{Q}_p$. We may identify $M_b(\mathbb{Q}_p)$ with $\mathbb{Q}_p^\times \times GL_{n-h,h}(F_w) \times \prod_{i>1} GL_n(F_{w_i})$. Choose a Haar measure on $M_b(\mathbb{Q}_p)$ so that $\mathbb{Z}_p^\times \times GL_{n-h,h}(\mathcal{O}_{F_w}) \times \prod_{i>1} GL_n(\mathcal{O}_{F_{w_i}})$ has volume 1. The measure on $J_b(\mathbb{Q}_p)$ is chosen to be compatible with the one on $M_b(\mathbb{Q}_p)$ in the sense of [Kot88, p.631]. Also choose a Haar measure $\mu_{G,\infty}$ so that $\prod_v \mu_{G,v}/\mu_{A_{G,\infty}}$ is the Tamagawa measure.

5.2. **Shimura varieties and Igusa varieties.** For each open compact subgroup $U \subset G(\mathbb{A}^\infty)$, consider the following moduli problem.

$$\begin{pmatrix} \text{connected locally noetherian} \\ F\text{-schemes} \\ \text{with a geometric point} \end{pmatrix} \longrightarrow \text{(Sets)}$$
$$(S,s) \mapsto \{(A,\lambda,i,\bar{\eta})\}/\sim$$

where the quadruples on the right consist of

- A is an abelian scheme over S
- $\lambda : A \to A^\vee$ is a polarization
- $i : F \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\lambda \circ i(f) = i(f^c)^\vee \circ \lambda$, $\forall f \in F$.
- $\bar{\eta}$ is a $\pi_1(S,s)$-invariant U-orbit of isomorphisms of $F \otimes_{\mathbb{Q}} \mathbb{A}^\infty$-modules $\eta : V \otimes_{\mathbb{Q}} \mathbb{A}^\infty \xrightarrow{\sim} VA_s$ which take the pairing $\langle\cdot,\cdot\rangle$ to the λ-Weil pairing up to $(\mathbb{A}^\infty)^\times$-multiples. (See [Kot92b, §5] for more explanation.)
- An equality of polynomials $\det_{\mathcal{O}_S}(f \,|\, \mathrm{Lie}\, A) = \det_E(f \,|\, V^1)$ holds for all $f \in F$, in the sense of [Kot92b, §5].
- Two quadruples $(A_1, \lambda_1, i_1, \bar{\eta}_1)$ and $(A_2, \lambda_2, i_2, \bar{\eta}_2)$ are equivalent if there is an isogeny $A_1 \to A_2$ taking $\lambda_1, i_1, \bar{\eta}_1$ to $\gamma\lambda_2, i_2, \bar{\eta}_2$ for some $\gamma \in \mathbb{Q}^\times$.

Note that for each S and two geometric points s and s' of S, the values of (S,s) and (S,s') under the above functor are canonically identified. So we can remove the reference to geometric points. And then the above functor can be extended to a functor on the category of all F-schemes in an obvious way. If U is sufficiently small, this functor is representable by a quasi-projective variety over F ([Kot92b, p.391]), which we denote by Sh_U.

Recall that we fixed p and w in §5.1 such that $p \in \mathrm{Spl}_{E/\mathbb{Q}}$ and $w|p$. For each i (including $i = 1$), let Λ_i be a U_p^{hs}-stabilized $\mathcal{O}_{F_{w_i}}$-lattice in $V \otimes_F F_{w_i}$. It can be assumed that Λ_i is self-dual with respect to $\langle\cdot,\cdot\rangle$. For $\vec{m} = (m_1, \ldots, m_r)$, define

$$U^p(\vec{m}) := U^p \times \mathbb{Z}_p^\times \times \prod_i \ker(GL_{\mathcal{O}_{F_{w_i}}}(\Lambda_i) \to GL_{\mathcal{O}_{F_{w_i}}}(\Lambda_i/\mathfrak{m}_{F_{w_i}}^{m_i}\Lambda_i)) \ \subset G(\mathbb{A}^\infty)$$

where $\mathfrak{m}_{F_{w_i}}$ is the maximal ideal of $\mathcal{O}_{F_{w_i}}$. We can construct an integral model of $\mathrm{Sh}_{U^p(\vec{m})}$ over $\mathcal{O}_{F_w}$, via the following analogue of the moduli problem in [HT01, p.108-109]. (The (A,i)-compatibility condition there corresponds to our determinant condition.)

$$\begin{pmatrix} \text{connected locally noetherian} \\ \mathcal{O}_{F_w}\text{-schemes} \\ \text{with a geometric point} \end{pmatrix} \longrightarrow \text{(Sets)}$$
$$(S,s) \mapsto \{(A,\lambda,i,\bar{\eta}^p,\{\alpha_i\}_{i=1}^r)\}/\sim$$

where the tuples on the right consist of

- A is an abelian scheme over S.
- $\lambda : A \to A^\vee$ is a prime-to-p polarization.
- $i : \mathcal{O}_F \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ such that $\lambda \circ i(f) = i(f^c)^\vee \circ \lambda$, $\forall f \in \mathcal{O}_F$.
- $\bar{\eta}$ is a $\pi_1(S,s)$-invariant U^p-orbit of isomorphisms of $F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules $\eta : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{\sim} V^pA_s$ which take the pairing $\langle\cdot,\cdot\rangle$ to the λ-Weil pairing up to $(\mathbb{A}^{\infty,p})^\times$-multiples.
- (Determinant condition) An equality of polynomials $\det_{\mathcal{O}_S}(f \,|\, \mathrm{Lie}\, A) = \det_E(f \,|\, V^1)$ holds for all $f \in \mathcal{O}_F$, in the sense of [Kot92b, §5].
- $\alpha_1 : w^{-m_1}\Lambda_1/\Lambda_1 \to A[w^{m_1}]$ is a Drinfeld w^{m_1}-structure
- for $i > 1$, $\alpha_i : (w_i^{-m_i}\Lambda_i/\Lambda_i) \xrightarrow{\sim} A[w_i^{m_i}]$ is an isomorphism of S-schemes with $\mathcal{O}_{F_{w_i}}$-actions.
- Two tuples $(A_1, \lambda_1, i_1, \bar{\eta}_1^p, \{\alpha_{i,1}\}_{i=1}^r)$ and $(A_2, \lambda_2, i_2, \bar{\eta}_2^p, \{\alpha_{i,2}\}_{i=1}^r)$ are equivalent if there is a prime-to-p isogeny $A_1 \to A_2$ taking $\lambda_1, i_1, \bar{\eta}_1^p, \alpha_{i,1}$ to $\gamma\lambda_2, i_2, \bar{\eta}_2^p, \alpha_{i,2}$ for some $\gamma \in \mathbb{Z}_{(p)}^\times$.

Because of our assumption on (p_τ, q_τ) and the determinant condition, if p is locally nilpotent in S then $A[w^\infty]$ (resp. $A[w_i^\infty]$ for $i > 1$) is a Barsotti-Tate group of dimension 1 (resp. 0) if A is as above. (cf. [HT01, p.108].) This moduli problem is representable by a quasi-projective scheme over $\mathcal{O}_{F_w}$ (using the argument of [Kot92b, p.391]), which will be denoted by $\mathrm{Sh}_{U^p,\vec{m}}$. In fact, $\mathrm{Sh}_{U^p,\vec{m}}$ is projective and flat over $\mathcal{O}_{F,w}$ for all $\vec{m}$ and smooth if $m_1 = 0$. The smoothness and flatness is proved exactly as in [HT01, Lem III.4.1]. The projectivity follows from [Lan08, Thm 5.3.3.1, Rem 5.3.3.2].

The special fiber $\overline{\mathrm{Sh}}_{U^p,\vec{0}} := \mathrm{Sh}_{U^p,\vec{0}} \times_{\mathcal{O}_{F_w}} k(w)$ admits a Newton-polygon stratification into $k(w)$-varieties $\overline{\mathrm{Sh}}^{(h)}_{U^p,\vec{0}}$ where the integer h runs over $0 \le h \le n-1$. The stratification can be described as in [HT01, p.111] or [Man05, p.580]. (Roughly speaking, $\overline{\mathrm{Sh}}^{(h)}_{U^p,\vec{0}}$ is the locus where the Barsotti-Tate $\mathcal{O}_{F_w}$-module $A[w^\infty]$ has étale height h in the sense of [HT01, p.59].) To compare the index sets for strata in two different references, note that each $0 \le h \le n-1$ bijectively corresponds to an element $b \in B(G_{\mathbb{Q}_p}, -\mu)$ under the bijection described in (5.3). When b corresponds to h, we write $\overline{\mathrm{Sh}}^{(b)}_{U^p,\vec{0}}$ for $\overline{\mathrm{Sh}}^{(h)}_{U^p,\vec{0}}$.

We may consider Igusa varieties in the sense of [Man05]. On page 576 of that paper the so-called unramified hypothesis was imposed, which is equivalent to assuming that p is unramified in F in our situation. The unramified hypothesis ensures that Shimura varieties have smooth integral models over $\mathcal{O}_{F_w}$ when no level structure is imposed at p. However the results of that paper carry over to our case (where p may be ramified in F): we substitute $\mathrm{Sh}_{U^p,\vec{0}}$ and $\overline{\mathrm{Sh}}_{U^p,\vec{0}}$ for $\mathfrak{X}_{U^p(0)}$ and $\overline{X}_{U^p(0)}$ in Mantovan's paper. (The same applies to the Newton-polygon strata). As remarked above, $\mathrm{Sh}_{U^p,\vec{0}}$ is smooth over $\mathcal{O}_{F_w}$. We use the results of Drinfeld as in [HT01, Ch II] instead of the Grothendieck-Messing theory. It is worth emphasizing that we can work without the unramified hypothesis since we are in the special case where p splits in E and the condition (v) of §5.1 is satisfied.

Let us briefly recall the definition of Igusa varieties. Choose any Barsotti-Tate group Σ_b over $\overline{\mathbb{F}}_p$ whose associated isocrystal with G-structure corresponds to b in the sense of [Shi09, §4] (cf. [RR96, 3.3-3.5]). Since any two isogenous one-dimensional Barsotti-Tate groups over $\overline{\mathbb{F}}_p$ with $\mathcal{O}_{F_w}$-actions are isomorphic, for each b there is a unique choice of Σ_b up to isomorphism (with additional structure). As a consequence, each central leaf $C_{\Sigma_b} = C_{\Sigma_b,U^p}$ defined in [Man05, §3] coincides with the corresponding stratum $\overline{\mathrm{Sh}}^{(b)}_{U^p,\vec{0}}$. We write $\mathrm{Ig}_{b,U^p,m}$ for the Igusa variety $J_{b,m}$ (which depends on U^p) defined in [Man05, §4]. In general Igusa varieties depend on the choice of Σ_b, but $\mathrm{Ig}_{b,U^p,m}$ only depends on b in our case (up to isomorphism) since Σ_b is unique up to isomorphism. By [Man05, Prop 4], $\mathrm{Ig}_{b,U^p,m}$ are finite étale Galois coverings of $\overline{\mathrm{Sh}}^{(b)}_{U^p,\vec{0}}$ and smooth over $\overline{\mathbb{F}}_p$.

An important point for us is that theorem 22 of [Man05] (also see [Man, Thm 1]), stated as Proposition 5.2 below, works in our case. (We need to make a small change that the Rapoport-Zink spaces should be viewed over the base $\mathcal{O}_{\widehat{F}^{\mathrm{ur}}_w}$ rather than $\widehat{\mathbb{Z}}^{\mathrm{ur}}_p$.) This should not be surprising since Proposition 5.2 is a close analogue (but formulated in a different language) of [HT01, Thm IV.2.9] which works even when p is ramified in F.[5]

Even though the unramified hypothesis mentioned above is imposed in [Shi09] and [Shi10], the results of those papers also carry over to our situation without the hypothesis. Again, this is possible as the conditions (iii) and (v) in §5.1 are satisfied. In fact, the only place where the unramified hypothesis is necessary is the proof of [Shi09, Lem 11.1]. In that proof, in our setting without the unramified hypothesis, we know that $\dim(\mathrm{Lie}\, A[w_i^\infty])$ is 1 if $i=1$ and 0 if $i>1$. Then we can argue as in the proof of [HT01, Lem V.4.1] (in which p may be ramified in F) to prove Lemma 11.1 of [Shi09]. (If the dimensions of the Lie algebras were arbitrary, the argument does not work.) Careful readers may check that the rest of arguments in [Shi09] and [Shi10] goes through and the results of those papers remain true in our situation.

Let ξ be an irreducible algebraic representation of G over $\overline{\mathbb{Q}}_l$. Such a ξ gives rise to a lisse l-adic sheaf on each Sh_U as well as on each $\mathrm{Ig}_{b,U^p,m}$. Let $\mathscr{L}_\xi$ denote those l-adic sheaves by abuse of notation. We write Ig_b and Sh for the projective systems of varieties $\{\mathrm{Ig}_{b,U^p,m}\}$ and $\{\mathrm{Sh}_U\}$, respectively, where m runs over $\mathbb{Z}_{>0}$ and U^p (resp. U) over sufficiently small open compact subgroups of $G(\mathbb{A}^{\infty,p})$ (resp. $G(\mathbb{A}^\infty)$). Define

$$H^k(\mathrm{Sh}, \mathscr{L}_\xi) := \varinjlim_{U} H^k(\mathrm{Sh}_U \times_F \overline{F}, \mathscr{L}_\xi), \qquad H^k_c(\mathrm{Ig}_b, \mathscr{L}_\xi) := \varinjlim_{U^p,m} H^k_c(\mathrm{Ig}_{b,U^p,m}, \mathscr{L}_\xi).$$

[5]In our case, it is appropriate to say that Proposition 5.2 is essentially due to Harris and Taylor. The beauty of Mantovan's work lies in its nice reformulation and generalization of their result.

which are admissible representations of $G(\mathbb{A}^\infty)\times \mathrm{Gal}(\overline{F}/F)$ and $G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p)$, respectively. Define

$$H(\mathrm{Sh},\mathscr{L}_\xi):=\sum_k (-1)^k H^k(\mathrm{Sh},\mathscr{L}_\xi),\qquad H_c(\mathrm{Ig}_b,\mathscr{L}_\xi):=\sum_k (-1)^k H^k_c(\mathrm{Ig}_b,\mathscr{L}_\xi).$$

which belong to $\mathrm{Groth}(G(\mathbb{A}^\infty)\times \mathrm{Gal}(\overline{F}/F))$ and $\mathrm{Groth}(G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p))$, respectively. The space $H^k(\mathrm{Sh},\mathscr{L}_\xi)$ is a semisimple $G(\mathbb{A}^\infty)$-module and admits a decomposition (cf. [HT01, p.103])

$$H^k(\mathrm{Sh},\mathscr{L}_\xi)=\bigoplus_{\pi^\infty}\pi^\infty\otimes R^k_{\xi,l}(\pi^\infty) \tag{5.5}$$

where π^∞ runs over $\mathrm{Irr}(G(\mathbb{A}^\infty))$ and $R^k_{\xi,l}(\pi^\infty)$ is a continuous finite dimensional representation of $\mathrm{Gal}(\overline{F}/F)$. Define $R_{\xi,l}(\pi^\infty):=\sum_k(-1)^kR^k_{\xi,l}(\pi^\infty)$, viewed in $\mathrm{Groth}(\mathrm{Gal}(\overline{F}/F))$.

Let S be a finite set of places of $\mathbb{Q}$ containing p and ∞. Set $S_{\mathrm{fin}}:=S\backslash\{\infty\}$. Let $R\in\mathrm{Groth}(G(\mathbb{A}^S)\times G')$ where G' is a topological group. A typical situation is $R=H(\mathrm{Sh},\mathscr{L}_\xi)$ with $G'=G(\mathbb{A}_{S_{\mathrm{fin}}})\times\mathrm{Gal}(\overline{F}/F)$ or $R=H(\mathrm{Ig}_b,\mathscr{L}_\xi)$ with $G'=G(\mathbb{A}_{S_{\mathrm{fin}}\backslash\{p\}})\times J_b(\mathbb{Q}_p)$. Write $R=\sum_{\pi^S\otimes\rho}n(\pi^S\otimes\rho)\cdot[\pi^S][\rho]$ where $n(\pi^S\otimes\rho)\in\mathbb{Z}$, and π^S and ρ run over $\mathrm{Irr}_l(G(\mathbb{A}^S))$ and $\mathrm{Irr}_l(G')$, respectively. For a given π^S, define $R[\pi^S]\in\mathrm{Groth}(G(\mathbb{A}^S)\times G')$ and $R\{\pi^S\}\in\mathrm{Groth}(G')$ by

$$R[\pi^S]:=\sum_\rho n(\pi^S\otimes\rho)\cdot[\pi^S][\rho],\qquad R\{\pi^S\}:=\sum_\rho n(\pi^S\otimes\rho)\cdot[\rho]$$

where ρ runs over $\mathrm{Irr}_l(G')$. This way we define $H(\mathrm{Sh},\mathscr{L}_\xi)[\pi^S]$, $H(\mathrm{Sh},\mathscr{L}_\xi)\{\pi^S\}$, $H(\mathrm{Ig}_b,\mathscr{L}_\xi)[\pi^S]$ and $H(\mathrm{Ig}_b,\mathscr{L}_\xi)\{\pi^S\}$.

Define a functor $\mathrm{Mant}_{b,\mu}:\mathrm{Groth}(J_b(\mathbb{Q}_p))\to\mathrm{Groth}(G(\mathbb{Q}_p)\times W_{F_w})$ using the notation of [Man05] by (cf. §2.2)

$$\mathrm{Mant}_{b,\mu}(\rho):=\sum_{i,j\ge 0}(-1)^{i+j}\varinjlim_{U_p\subset G(\mathbb{Q}_p)}\mathrm{Ext}^i_{J_b(\mathbb{Q}_p)\text{-smooth}}(H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu,U_p}),\rho))(-D).$$

Here D is the dimension of $\mathcal{M}^{\mathrm{rig}}_{b,\mu,U_p}$, $(-D)$ denotes a Tate twist, and the limit is taken over open compact subgroups U_p of $G(\mathbb{Q}_p)$. The following proposition is the theorem 22 of [Man05] ([Man, Thm 1]), which holds in our case as explained above.

Proposition 5.2. *With the notation as above, there is an equality in* $\mathrm{Groth}(G(\mathbb{A}^\infty)\times W_{F_w})$

$$H(\mathrm{Sh},\mathscr{L}_\xi)=\sum_{b\in B(G_{\mathbb{Q}_p},-\mu)}\mathrm{Mant}_{b,\mu}(H_c(\mathrm{Ig}_b,\mathscr{L}_\xi)).$$

The Rapoport-Zink spaces $\mathcal{M}^{\mathrm{rig}}_{b,\mu,U_p}$ admit product decompositions into Rapoport-Zink spaces of EL-types, corresponding to the decompositions (5.2), $b=(b_0,(b_{w_i}))$ and $\mu=(\mu_0,(\mu_{w_i}))$. (cf. [Far04, 2.3.7.1, Ex. 2.3.21].) This induces a corresponding decomposition of $\mathrm{Mant}_{b,\mu}$. Namely, if we write each $\rho\in\mathrm{Irr}(J_b(\mathbb{Q}_p))$ as $\rho_0\otimes(\otimes_i\rho_{w_i})$ according to (5.4), then

$$\mathrm{Mant}_{b,\mu}(\rho)=\mathrm{Mant}_{b_0,\mu_0}(\rho_0)\otimes\left(\otimes_i\,\mathrm{Mant}_{b_{w_i},\mu_{w_i}}(\rho_{w_i})\right). \tag{5.6}$$

To the irreducible representation ξ, there is a way to attach $a_0(\xi)\in\mathbb{Z}$, $\vec{a}(\xi)_\sigma\in\mathbb{Z}^n$ and $w(\xi)\in\mathbb{Z}$ for each $\sigma\in\Phi^+_{\mathbb{C}}$ as in (3.18) and the paragraph preceding (3.18). The following proposition is an analogue of [HT01, Prop III.2.1], except the last assertion comes from [HT01, Lem III.4.2] (for which we allow $p=l$). The proof of Harris and Taylor works in our case and will be omitted.

Proposition 5.3. *Recall that we fixed $\tau:F\hookrightarrow\mathbb{C}$ and $\iota_l:\overline{\mathbb{Q}}_l\xrightarrow{\sim}\mathbb{C}$. Let U_∞ be the centralizer of h in $G(\mathbb{R})$.*

(i) *The following holds where π_∞ runs over $\Pi_{\mathrm{unit}}(G(\mathbb{R}),\iota_l\xi^\vee)$. We denote the (discrete) automorphic multiplicity by $m(\cdot)$.*

$$\dim R^k_{\xi,l}(\pi^\infty)=|\ker^1(\mathbb{Q},G)|\sum_\pi m(\iota_l(\pi^\infty)\otimes\pi_\infty)\dim H^k(\mathrm{Lie}\,G(\mathbb{R}),U_\infty,\pi_\infty\otimes\iota_l\xi)$$

(ii) *Let y be a prime of F not dividing l. For any $\sigma\in W_{F_y}$, each eigenvalue α of $R^k_{\xi,l}(\pi^\infty)(\sigma)$ satisfies $\alpha\in\overline{\mathbb{Q}}$ and $|\alpha|^2\in|k(y)|^{\mathbb{Z}}$ under any embedding $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$.*

(iii) *For almost all primes y of F, for all eigenvalues α of $R^k_{\xi,l}$ and for all embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, we have $|\alpha|^2 = |k(y)|^{k+w(\xi)}$.*

(iv) *$R^k_{\xi,l}(\pi^\infty)$ is potentially semistable at every $y|l$.*

(v) *Suppose that a prime q splits in E and that $q \in \{p, l\}$. Write $\pi_q = \pi_{q,0} \otimes (\otimes_{v\in\mathcal{V}_q^+} \pi_v)$ and let $y \in \mathcal{V}_q^+$ be the place determined by $\iota_q^{-1}\tau : F \hookrightarrow \overline{\mathbb{Q}}_q$. If $\pi_{q,0}$ and π_y are unramified, then $R^k_{\xi,l}(\pi^\infty)$ is crystalline at y if $q = l$ and unramified at y if $q = p$.*

5.3. **Stable trace formula for Igusa varieties.** Recall that the Haar measures on $G(\mathbb{A}^\infty)$, $J_b(\mathbb{Q}_p)$ and $G_{\vec{n}}(\mathbb{A})/A_{G_{\vec{n}},\infty}$ are fixed (§3.1, §5.1), where $G_{\vec{n}}$ denotes elliptic endoscopic groups for G. The goal of this subsection is to state the stable trace formula for Igusa varieties, which was the main result of [Shi10].

We need to pin down transfer factors. For each $G_{\vec{n}}$, fix $\Delta^0_v(\cdot,\cdot)^{G_n}_{G_{\vec{n}}}$ as in §3.4 at each $v \neq \infty$, where we take $\Delta^0_v \equiv 1$ for every $v \neq \infty$ if $\vec{n} = (n)$. Choose the transfer factor $\Delta_v(\cdot,\cdot)^G_{G_{\vec{n}}}$ ($v \neq \infty$) so that

$$\Delta_v(\cdot,\cdot)^G_{G_{\vec{n}}} = \Delta^0_v(\cdot,\cdot)^{G_n}_{G_{\vec{n}}} \tag{5.7}$$

via the isomorphism $G \times_{\mathbb{Q}} \mathbb{A}^\infty \simeq G_n \times_{\mathbb{Q}} \mathbb{A}^\infty$ that was fixed in §5.2. We choose the unique $\Delta_\infty(\cdot,\cdot)^G_{G_{\vec{n}}}$ such that the product formula

$$\prod_v \Delta_v(\gamma_H, \gamma)^G_{G_{\vec{n}}} = 1$$

holds ([LS87, (6.4)]) for any matching pair (γ_H, γ) with $\gamma \in G(\mathbb{Q})$, i.e. for any semisimple $\gamma \in G(\mathbb{Q})$ and any (G, H)-regular semisimple $\gamma_H \in G_{\vec{n}}(\mathbb{A})$ with matching stable conjugacy classes.

Fix (j, B) as in §4.3, once and for all. Recall that $\Delta_{j,B}$ was defined in §3.5. Let $e_{\vec{n}}(\Delta_\infty) \in \mathbb{C}^\times$ denote the constant such that

$$\Delta_\infty(\gamma_H, \gamma)^G_{G_{\vec{n}}} = e_{\vec{n}}(\Delta_\infty)\Delta_{j,B}(\gamma_H, \gamma) \tag{5.8}$$

for any matching pair $(\gamma_H, \gamma) \in G_{\vec{n}}(\mathbb{R}) \times G(\mathbb{R})$. Note that $e_{\vec{n}}(\Delta_\infty) = 1$ for $\vec{n} = (n)$. We claim that for each $\vec{n} = (n_1, n_2)$,

$$e_{\vec{n}}(\Delta_\infty) \in (\mathbb{C}^\times)^1, \tag{5.9}$$

namely that $|e_{\vec{n}}(\Delta_\infty)| = 1$. The argument is as follows. It is not hard to see from the definition (§3.4) that for every $v \neq \infty$, $\Delta_v(\gamma_H, \gamma)^G_{G_{\vec{n}}}$ is equal to $\Delta_{IV,v}(\gamma_H, \gamma)$ up to $(\mathbb{C}^\times)^1$, the latter being the ratio of Weyl discriminants at v defined in [LS87, §3.6]. By the product formula (5.7), the same is true for $v = \infty$. On the other hand, $\Delta_{j,B}(\gamma_H, \gamma)$ is also equal to $\Delta_{IV,\infty}(\gamma_H, \gamma)$ up to $(\mathbb{C}^\times)^1$, as can be seen from the definition of [Kot90, p.184]. (Note that $\chi_{G,H}$ in that article is a unitary character in our case.) Hence the claim is proved.

Remark 5.4. Although a more careful analysis of transfer factors would show that $e_{\vec{n}}(\Delta_\infty) \in \{\pm 1\}$, we have not attempted to do it here. Instead, we prove the same fact with an ad hoc argument later in the proof of Theorem 6.1. There $e_{\vec{n}}(\Delta_\infty)$ shows up in the coefficient of a spectral identity, which must be a real number, hence $+1$ or -1.

Let $\phi^{\infty,p} \cdot \phi'_p \in C^\infty_c(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$ be a complex-valued function. Assume that $\phi^{\infty,p} \cdot \phi'_p$ is an acceptable function ([Shi09, Def 6.2]). For each elliptic endoscopic group $G_{\vec{n}}$ for G, we recall the construction of the function $\phi^{\vec{n}}_{\mathrm{Ig}}$ on $G_{\vec{n}}(\mathbb{A})$. We may assume that $\phi^{\infty,p}$ has the form $\phi^{\infty,p} = \prod_{v\neq p,\infty} \phi_v$ as the general case follows via finite linear combination.

For each place $v \neq p, \infty$, let $\phi^{\vec{n}}_{\mathrm{Ig},v} \in C^\infty_c(G_{\vec{n}}(\mathbb{Q}_v))$ be a $\Delta_v(\cdot,\cdot)^G_{G_{\vec{n}}}$-transfer of ϕ_v (§3.4). Set $H := G_{\vec{n}}$ in order to make the notation compatible with some references. Put

$$\phi^{\vec{n}}_{\mathrm{Ig},p} := h^H_p \tag{5.10}$$

where h^H_p is the function constructed from ϕ'_p in §6.3 of [Shi10], with the convention of §8.1 of that paper. (The construction of h^H_p is briefly recalled in (5.32).) Set

$$\phi^{\vec{n}}_{\mathrm{Ig},\infty} := e_{\vec{n}}(\Delta_\infty) \cdot (-1)^{q(G)} \langle \mu_h, s\rangle \sum_{\varphi_H} \det(\omega_*(\varphi_H)) \cdot \phi_{\varphi_H} \tag{5.11}$$

in the notation of §3.5, where φ_H runs over the equivalence classes of L-parameters such that $\widetilde{\eta}\varphi_H \sim \phi_\xi$. Observe that $\phi^{\vec{n}}_{\mathrm{Ig},\infty}$ is the function h_∞ of [Kot90, p.186] multiplied by $e_{\vec{n}}(\Delta_\infty)$.

The latter constant is multiplied to make up for the difference between Δ_∞ and $\Delta_{j,B}$.

The following stable trace formula is proved in [Shi10, Thm 7.2]. It is worth noting that the proof uses the fundamental lemma in an essential way.

Proposition 5.5. *If $\phi^{\infty,p}\cdot\phi'_p\in C^\infty_c(G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p))$ is acceptable,*

$$\mathrm{tr}\,(\phi^{\infty,p}\times\phi'_p|\iota_l H_c(\mathrm{Ig}_b,\mathscr{L}_\xi)) \;=\; |\ker^1(\mathbb{Q},G)|\sum_{G_{\vec{n}}}\iota(G,G_{\vec{n}})ST_e^{G_{\vec{n}}}(\phi^{\vec{n}}_{\mathrm{Ig}}) \qquad (5.12)$$

where the sum runs over the set $\mathcal{E}^{\mathrm{ell}}(G)$ of elliptic endoscopic triples $(G_{\vec{n}},s_{\vec{n}},\eta_{\vec{n}})$.

Let us explain the constants $\iota(G,G_{\vec{n}})$. By definition $\iota(G,G_{\vec{n}})=\tau(G)\tau(G_{\vec{n}})^{-1}|\mathrm{Out}(G_{\vec{n}},s_{\vec{n}},\eta_{\vec{n}})|^{-1}$. Recalling that n is odd, $|\mathrm{Out}(G_{\vec{n}},s_{\vec{n}},\eta_{\vec{n}})|$ equals 1 for any $\vec{n}=(n)$ or (n_1,n_2). (It equals 2 if n were even and $\vec{n}=(n/2,n/2)$. cf. [Rog90, Prop 4.6.1] in the case of unitary groups.) Now it is easy to compute, by using (3.3),

$$\iota(G,G_{\vec{n}})=\begin{cases}1, & \text{if}\quad \vec{n}=(n)\\ 1/2, & \text{if}\quad \vec{n}=(n_1,n_2)\end{cases}$$

5.4. **Stable trace formula for L^2-automorphic spectrum of $G_{\vec{n}}(\mathbb{A})$.** Keep the convention from the last subsection. In particular we use the same Haar measures and the same transfer factors as in §5.3.

Denote by $R_{G,\iota_l\xi}$ the regular representation of $G(\mathbb{A})$ on the space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}),\chi^{-1}_{\iota_l\xi})$ consisting of those functions $G(\mathbb{Q})\backslash G(\mathbb{A})\to\mathbb{C}$ which transform under $A_{G,\infty}$ by $\chi^{-1}_{\iota_l\xi}$ and are square integrable modulo $A_{G,\infty}$. Let $\pi^0_\infty\in\Pi_{\mathrm{disc}}(G(\mathbb{R}),\xi^\vee)$. For any $\phi^\infty\in C^\infty_c(G(\mathbb{A}^\infty))$, let $(\phi^{\vec{n}})^\infty$ be a $\Delta(\cdot,\cdot)^G_{G_{\vec{n}}}$-transfer of ϕ^∞. Denote by $\phi^{\vec{n}}_{\pi^0_\infty}$ the product of $e_{\vec{n}}(\Delta_\infty)$ with $\phi^{\vec{n}}_{\pi^0_\infty}$ given by (3.13). Then $\phi^{\vec{n}}_{\pi^0_\infty}$ is a $\Delta(\cdot,\cdot)^G_{G_{\vec{n}}}$-transfer of $\phi_{\pi^0_\infty}$. (We have to multiply $e_{\vec{n}}(\Delta_\infty)$ due to the difference of transfer factors. See formula (5.8).) The following proposition is an analogue of Proposition 5.5, which is derived from the trace formula for compact quotients by stabilizing geometric terms after Langlands and Kottwitz ([Lan83], [Kot86]; especially theorem 9.6 of the latter). Note that Proposition 5.6 is unconditional, as is Proposition 5.5. Although the stabilization of Langlands and Kottwitz relies on the fundamental lemma and the transfer conjecture, these were settled by a recent proof of Ngô ([Ngo]), building on work of Waldspurger and others.

Proposition 5.6. *The following equality holds, where the first sum is taken over the set of isomorphism classes of $\pi\in\mathrm{Irr}(G(\mathbb{A}))$ and the second is over the set $\mathcal{E}^{\mathrm{ell}}(G)$ of elliptic endoscopic triples $(G_{\vec{n}},s_{\vec{n}},\eta_{\vec{n}})$.*

$$\mathrm{tr}\,R_{G,\iota_l\xi}\left(\phi^\infty\cdot\phi_{\pi^0_\infty}\right)=\sum_\pi m(\pi)\cdot\mathrm{tr}\,\pi(\phi^\infty\cdot\phi_{\pi^0_\infty})=\sum_{G_{\vec{n}}}\iota(G,G_{\vec{n}})ST_e^{G_{\vec{n}}}((\phi^{\vec{n}})^\infty\cdot\phi^{\vec{n}}_{\pi^0_\infty}) \qquad (5.13)$$

Remark 5.7. The number $|\ker^1(\mathbb{Q},G)|$ shows up in the formula (5.12) but not in (5.13). This comes from the fact that our moduli varieties Sh_U over F are $|\ker^1(\mathbb{Q},G)|$-copies of the usual canonical models of Shimura varieties. See [Kot92b, §8] for explanation.

Remark 5.8. Proposition 5.6 will not be used in this paper until the proof of Corollary 6.5.

5.5. **Definition of $\text{n-Red}^b_{\vec{n}}$.** In §5.5 we will freely use notation and terminology from [Shi10], especially section 6 there.

For each $(H,s,\eta)=(G_{\vec{n}},s_{\vec{n}},\eta_{\vec{n}})$ in $\mathcal{E}^{\mathrm{ell}}(G)$, recall that there is a finite set $\mathcal{E}^{\mathrm{eff}}_p(J_b,G;H)$ consisting of (isomorphism classes of) triples (M_H,s_H,η_H). Such (M_H,s_H,η_H) is a G-endoscopic triple for J_b. The $\mathbb{Q}_p$-group M_H is equipped with a $\mathbb{Q}_p$-morphism $\nu_{M_H}:\mathbb{D}\to M_H$ and a finite set $\mathcal{I}(M_H,H)$ consisting of certain $\mathbb{Q}_p$-embeddings $M_H\hookrightarrow H$ whose images are Levi subgroups of H. We will use the normalization of transfer factors $\Delta(\cdot,\cdot)^{M_b}_{M_H}$ and $\Delta(\cdot,\cdot)^{J_b}_{M_H}$ as in [Shi10, Eqn (8.6)]. The constant $c_{M_H}\in\{\pm1\}$, assigned to each (M_H,s_H,η_H), may be evaluated as in section 8.1 of the same paper. As the numbers c_{M_H} intervene in the definition (5.32) of $\phi^{\vec{n}}_{\mathrm{Ig},p}$, they will be included in the definition of $\text{n-Red}^b_{\vec{n}}$ (thus also $\mathrm{Red}^b_{\vec{n}}$), which is motivated by Lemma 5.10 below.

Define n-$\mathrm{Red}^b_{\vec{n}}$ to be the composition of the following maps.

$$\mathrm{Groth}(H(\mathbb{Q}_p)) \longrightarrow \bigoplus_{(M_H,s_H,\eta_H)} \mathrm{Groth}(M_H(\mathbb{Q}_p)) \stackrel{\oplus\widetilde{\eta}_{H,*}}{\longrightarrow} \mathrm{Groth}(M_b(\mathbb{Q}_p)) \stackrel{LJ^{M_b}_{J_b}}{\longrightarrow} \mathrm{Groth}(J_b(\mathbb{Q}_p)) \quad (5.14)$$

The first map is the direct sum of the maps from $\mathrm{Groth}(H(\mathbb{Q}_p))$ to $\mathrm{Groth}(M_H(\mathbb{Q}_p))$ for all $(M_H, s_H, \eta_H) \in \mathcal{E}^{\mathrm{eff}}_p(J_b, G; H)$, where the map for each (M_H, s_H, η_H) is given by $\oplus_i c_{M_H} \cdot J^H_{P(i\nu_{M_H})^{\mathrm{op}}}$ as i runs over $\mathcal{I}(M_H, H)$. In fact, $\mathcal{I}(M_H, H)$ is always a singleton in our case, so we will simply write P_{M_H} for $P(i\nu_{M_H})$. (See Case 1 and 2 below.) As for $\widetilde{\eta}_{H,*}$, an explicit definition is given below case by case. This map $\widetilde{\eta}_{H,*}$ should be seen as the functorial transfer with respect to the L-morphism $\widetilde{\eta}_H$. Noting that $M_b(\mathbb{Q}_p)$ is a product of general linear groups, $LJ^{M_b}_{J_b}$ is the "Jacquet-Langlands" map on Grothendieck groups defined by [Bad07] (cf. §2.4).

Define $\mathrm{Red}^b_{\vec{n}}$ by

$$\mathrm{Red}^b_{\vec{n}}(\pi_{H,p}) := \text{n-}\mathrm{Red}^b_{\vec{n}}(\pi_{H,p}) \otimes \bar{\delta}^{1/2}_{P(\nu_b)}.$$

<u>Case 1</u>. $\vec{n} = (n)$, i.e. $(H, s, \eta) = (G_n, 1, \mathrm{id})$.

In this case $\mathcal{E}^{\mathrm{eff}}_p(J_b, G; H)$ has a unique isomorphism class represented by $(M_H, s_H, \eta_H) = (M_b, 1, \mathrm{id})$. So we may take $\widetilde{\eta}_H = \mathrm{id}$ and $\widetilde{\eta}_{H,*} = \mathrm{id}$. In that case $c_{M_H} = e_p(J_b)$ ([Shi10, Rem 6.4]). There are isomorphisms

$$\begin{array}{ccccccc} G(\mathbb{Q}_p) & \simeq & \mathbb{Q}_p^\times & \times & GL_n(F_w) & \times & \prod_{i>1} GL_n(F_{w_i}) \\ M_b(\mathbb{Q}_p) & \simeq & \mathbb{Q}_p^\times & \times & GL_{n-h}(F_w) \times GL_h(F_w) & \times & \prod_{i>1} GL_n(F_{w_i}). \end{array} \quad (5.15)$$

An analogous decomposition for $J_b(\mathbb{Q}_p)$ was given in (5.4). The set $\mathcal{I}(M_H, H)$ contains a unique element, which may be represented by the Levi embedding $i_{M_b} : M_b \hookrightarrow G$ which is the obvious block diagonal embedding on the F_w-component with respect to (5.15). (The $G(\mathbb{Q}_p)$-conjugacy class of i_{M_b} is canonical.) Let $h \in [0, n-1]$ be the integer corresponding to b as in (5.3). We see that $e_p(J_b) = (-1)^{n-h-1}$ in view of (5.4). If $\pi_p = \pi_{p,0} \otimes (\otimes_i \pi_{w_i}) \in \mathrm{Irr}_l(G(\mathbb{Q}_p))$ then it is clear that

$$\text{n-}\mathrm{Red}^b_n(\pi_p) = (-1)^{n-h-1}\pi_{p,0} \otimes \text{n-}\mathrm{Red}^{n-h,h}(\pi_w) \otimes (\otimes_{i>1}\pi_{w_i}). \quad (5.16)$$

where n-$\mathrm{Red}^{n-h,h}$ is defined in §2.4. An analogue of (5.16) holds for $\mathrm{Red}^b_n(\pi_p)$ if n-$\mathrm{Red}^{n-h,h}$ is replaced by $\mathrm{Red}^{n-h,h}$ on the right hand side.

<u>Case 2</u>. $\vec{n} = (n_1, n_2)$. i.e. $(H, s, \eta) = (G_{n_1,n_2}, s_{n_1,n_2}, \eta_{n_1,n_2})$.

In this case we have the following isomorphisms over $\mathbb{Q}_p$.

$$\begin{array}{ccccccc} G & \simeq & GL_1 & \times & \prod_{i\geq 1} R_{F_{w_i}/\mathbb{Q}_p} GL_n & & \\ H & \simeq & GL_1 & \times & \prod_{i\geq 1} R_{F_{w_i}/\mathbb{Q}_p} GL_{n_1,n_2} & & \\ M_b & \simeq & GL_1 & \times & R_{F_w/\mathbb{Q}_p} GL_{n-h,h} & \times & \prod_{i>1} R_{F_{w_i}/\mathbb{Q}_p} GL_n \\ J_b & \simeq & GL_1 & \times & R_{F_w/\mathbb{Q}_p}\left(D^\times_{F_w,1/(n-h)} \times GL_h\right) & \times & \prod_{i>1} R_{F_{w_i}/\mathbb{Q}_p} GL_n \end{array} \quad (5.17)$$

Consider the following two groups which will be viewed as Levi subgroups of H via the natural block diagonal embeddings, which are to be denoted by $i_{M_{H,1}}$ and $i_{M_{H,2}}$.

$$\begin{array}{lllll} M_{H,1} & = & GL_1 & \times & R_{F_w/\mathbb{Q}_p} GL_{n-h,h-n_2,n_2} \times \prod_{i>1} R_{F_{w_i}/\mathbb{Q}_p} GL_{n_1,n_2} \quad (\text{if } h \geq n_2) \\ M_{H,2} & = & GL_1 & \times & R_{F_w/\mathbb{Q}_p} GL_{n-h,h-n_1,n_1} \times \prod_{i>1} R_{F_{w_i}/\mathbb{Q}_p} GL_{n_1,n_2} \quad (\text{if } h \geq n_1) \end{array} \quad (5.18)$$

The dual groups are described as follows. The L-groups are given by an obvious action of $W_{\mathbb{Q}_p}$ on the dual groups. Namely $W_{\mathbb{Q}_p}$ permutes the index sets $\mathrm{Hom}(F_{w_i}, \overline{\mathbb{Q}}_p)$.

$$\begin{array}{ccccccc} \widehat{G} & = & \mathbb{C}^\times & \times & \prod_{i\geq 1} GL_n(\mathbb{C})^{\mathrm{Hom}(F_{w_i},\overline{\mathbb{Q}}_p)} & & \\ \widehat{H} & = & \mathbb{C}^\times & \times & \prod_{i\geq 1} GL_{n_1,n_2}(\mathbb{C})^{\mathrm{Hom}(F_{w_i},\overline{\mathbb{Q}}_p)} & & \\ \widehat{M_b} & = & \mathbb{C}^\times & \times & GL_{n-h,h}(\mathbb{C})^{\mathrm{Hom}(F_w,\overline{\mathbb{Q}}_p)} & \times & \prod_{i>1} GL_n(\mathbb{C})^{\mathrm{Hom}(F_{w_i},\overline{\mathbb{Q}}_p)} \\ \widehat{M}_{H,1} & = & \mathbb{C}^\times & \times & GL_{n-h,h-n_2,n_2}(\mathbb{C})^{\mathrm{Hom}(F_w,\overline{\mathbb{Q}}_p)} & \times & \prod_{i>1} GL_{n_1,n_2}(\mathbb{C})^{\mathrm{Hom}(F_{w_i},\overline{\mathbb{Q}}_p)} \\ \widehat{M}_{H,2} & = & \mathbb{C}^\times & \times & GL_{n-h,h-n_1,n_1}(\mathbb{C})^{\mathrm{Hom}(F_w,\overline{\mathbb{Q}}_p)} & \times & \prod_{i>1} GL_{n_1,n_2}(\mathbb{C})^{\mathrm{Hom}(F_{w_i},\overline{\mathbb{Q}}_p)} \end{array} \quad (5.19)$$

We give the maps $\eta_{H,j} : \widehat{M}_{H,j} \to \widehat{M}_b$ $(j = 1, 2)$ so that $\eta_{H,j}$ is the identity on $\mathbb{C}^\times$ and the obvious block diagonal embedding on the F_{w_i}-component $(i \geq 1)$. Extend $\eta_{H,1}$ to $\widetilde{\eta}_{H,1} : {}^L M_{H,1} \to {}^L M_b$ by sending $z \in W_{\mathbb{Q}_p}$ to

$$\Big(\varpi(z)^{-N(n_1,n_2)}, (\varpi(z)^{\epsilon(n-n_1)}, \varpi(z)^{\epsilon(n-n_1)}, \varpi(z)^{\epsilon(n-n_2)}), (\varpi(z)^{\epsilon(n-n_1)}, \varpi(z)^{\epsilon(n-n_2)})\Big) \rtimes z.$$

Similarly define $\widetilde{\eta}_{H,2} : {}^L M_{H,2} \to {}^L M_b$, which maps $z \in W_{\mathbb{Q}_p}$ to

$$\Big(\varpi(z)^{-N(n_1,n_2)}, (\varpi(z)^{\epsilon(n-n_2)}, \varpi(z)^{\epsilon(n-n_2)}, \varpi(z)^{\epsilon(n-n_1)}), (\varpi(z)^{\epsilon(n-n_1)}, \varpi(z)^{\epsilon(n-n_2)})\Big) \rtimes z.$$

With respect to (5.19), let

$$s_{M_{H,1}} := (1, (1, 1, -1), (1, 1)) \in Z(\widehat{M}_{H,1}), \qquad s_{M_{H,2}} := (1, (-1, -1, 1), (1, 1)) \in Z(\widehat{M}_{H,2}).$$

Recall that the sets $\mathcal{E}^{\mathrm{ef}}(M_b, G; H)$ and $\mathcal{E}^{\mathrm{eff}}(J_b, G; H)$ are defined in [Shi10, §6.2]. Certainly $(M_{H,j}, s_{M_{H,j}}, \eta_{H,j})$ $(j = 1, 2)$ belong to $\mathcal{E}^{\mathrm{ef}}(M_b, G; H)$. (In general $\mathcal{E}^{\mathrm{ef}}(M_b, G; H)$ has other elements, but they do not concern us since they are not contained in $\mathcal{E}^{\mathrm{eff}}(J_b, G; H)$.) Using the fact that J_b has $D^\times_{F_w,1/(n-h)} \times GL_h(F_w)$ in its product decomposition, it is easy to check that

$$\mathcal{E}^{\mathrm{eff}}(J_b, G; H) = \begin{cases} \emptyset, & \text{if } \quad h < n_2, \\ \{(M_{H,1}, s_{M_{H,1}}, \eta_{H,1})\}, & \text{if } \quad n_2 \leq h < n_1, \\ \{(M_{H,j}, s_{M_{H,j}}, \eta_{H,j}),\ j = 1, 2\}, & \text{if } \quad h \geq n_1. \end{cases} \tag{5.20}$$

(In order that (M_H, s_{M_H}, η_H) lies in $\mathcal{E}^{\mathrm{eff}}(J_b, G; H)$, the element s_{M_H} should transfer to $\widehat{J}_b = \widehat{M}_b$ via η_H so that it is either $+1$ or -1 in the $GL_{n-h}(\mathbb{C})$ block of the F_w-component, since $D_{F_w,1/(n-h)}$ is a division algebra.) From now on, whenever we consider $(M_{H,j}, s_{M_{H,j}}, \eta_{H,j})$, we assume the condition on h of (5.20) so that the triple belongs to $\mathcal{E}^{\mathrm{eff}}(J_b, G; H)$.

Let $\widetilde{l}_{M_{H,j}} : {}^L M_{H,j} \to {}^L H$ $(j = 1, 2)$ be the obvious embedding, except that $\widetilde{l}_{M_{H,2}}$ on the F_w-component is given by

$$(A_1, A_2, A_3) \in GL_{n-h,h-n_1,n_1} \mapsto \left(A_3, \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\right) \in GL_{n_1,n_2}.$$

Then one can directly check that $\widetilde{l}_{M_b} : {}^L M_b \to {}^L G$ can be chosen to be a $\widehat{G}$-conjugate of the obvious embedding so that the following commutes.

$$\begin{array}{ccc} {}^L M_b & \xrightarrow{\widetilde{l}_{M_b}} & {}^L G \\ \uparrow {\scriptstyle \widetilde{\eta}_H} & & \uparrow {\scriptstyle \widetilde{\eta}} \\ {}^L M_{H,j} & \xrightarrow{\widetilde{l}_{M_{H,j}}} & {}^L H \end{array} \tag{5.21}$$

For each $j \in \{1, 2\}$, the set $\mathcal{I}(M_{H,j}, H)$ has a single element, which may be represented by the Levi embedding $i_{M_{H,j}} : M_{H,j} \hookrightarrow H$. The parabolic subgroup $P_{M_{H,j}} \subset H$ is generated by $M_{H,j}$ and upper triangular matrices of GL_{n_1,n_2} at the F_w-component.

We are about to define $\widetilde{\eta}_{H,j,*} : \mathrm{Groth}(M_{H,j}(\mathbb{Q}_p)) \to \mathrm{Groth}(M_b(\mathbb{Q}_p))$ and give a relevant trace identity, in a way similar to Case 2 of §3.4. Let $u := w|_E$. Define a unitary character $\chi^+_{u,j} : M_{H,j}(\mathbb{Q}_p) \to \mathbb{C}^\times$ such that

$$\begin{array}{rcl} \chi^+_{u,j}(\lambda) & = & \varpi_u(\lambda)^{-N(n_1,n_2)}, \\ \chi^+_{u,j}(g_{w,1}, g_{w,2}, g_{w,3}) & = & \begin{cases} \varpi_u\left(N_{F_w/E_u}\left(\det((g_{w,1}g_{w,2})^{\epsilon(n-n_1)} g_{w,3}^{\epsilon(n-n_2)})\right)\right), & j = 1, \\ \varpi_u\left(N_{F_w/E_u}\left(\det((g_{w,1}g_{w,2})^{\epsilon(n-n_2)} g_{w,2}^{\epsilon(n-n_1)})\right)\right), & j = 2, \end{cases} \\ \chi^+_{u,j}(g_{w_i,1}, g_{w_i,2}) & = & 1, \end{array} \tag{5.22}$$

where $(\lambda, (g_{w,1}, g_{w,2}, g_{w,3}), (g_{w_i,1}, g_{w_i,2}))$ denotes an element of $M_{H,j}(\mathbb{Q}_p)$ with respect to (5.18). For each $\phi^*_p \in C^\infty_c(M_b(\mathbb{Q}_p))$ and $\pi_{M_{H,j}} \in \mathrm{Irr}_l(M_{H,j}(\mathbb{Q}_p))$, define

$$\phi_p^{M_{H,j}} := (\phi^*_p)^{Q_j} \cdot \chi^+_{u,j}, \quad \text{and} \quad \widetilde{\eta}_{H,j,*}(\pi_{M_{H,j}}) := \text{n-ind}^{M_b}_{Q_j}(\pi_{M_{H,j}} \otimes \chi^+_{u,j}) \tag{5.23}$$

where Q_j is any parabolic subgroup of M_b which has $M_{H,j}$ as a Levi subgroup. As in §3.4, we can normalize $\Delta_p(\cdot,\cdot)^{M_b}_{M_{H,j}}$ with respect to $\widetilde{\eta}_{H,j}$ so that $\phi_p^{M_{H,j}}$ is a $\Delta_p(\cdot,\cdot)^{M_b}_{M_{H,j}}$-transfer of ϕ_p^*. Note that $\widetilde{\eta}_{H,j,*}$ is independent of the choice of Q_j. We have the following identity analogous to (3.10). The first equality holds by definition and the second by Lemma 3.3.(ii).

$$\mathrm{tr}\,\pi_{M_{H,j}}\left(\phi_p^{M_{H,j}}\right) = \mathrm{tr}\,(\pi_{M_H,j}\otimes\chi^+_{u,j})((\phi_p^*)^{Q_j}) = \mathrm{tr}\,\left(\widetilde{\eta}_{H,j,*}(\pi_{M_{H,j}})\right)\left(\phi_p^*\right) \tag{5.24}$$

The next job is to compute $c_{M_{H,j}}\in\{\pm1\}$. We use the result and notation from [Shi10, §8.1]. Note that our s_{n_1,n_2} is the element $s\in Z(\widehat{H})$ of that article. We may take the decomposition $s=s_1s_2$ with $s_1\in Z(\widehat{H})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ and $s_2=1$. It is easy to compute ν_b as in [Shi09, Ex 4.3]. From this we see that $\widehat{\nu}_b^{M_{H,j}}: Z(\widehat{M}_{H,j})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}\to\mathbb{C}^\times$ can be described as

$$\begin{array}{lllll} \mathbb{C}^\times\times & ((\mathbb{C}^\times)^3)^{\mathrm{Hom}(F_w,\overline{\mathbb{Q}}_p)}\times & \prod_{i>1}((\mathbb{C}^\times)^2)^{\mathrm{Hom}(F_{w_i},\overline{\mathbb{Q}}_p)} & \longrightarrow & \mathbb{C}^\times \\ (z, & (z_{w,1},z_{w,2},z_{w,3}), & (z_{w_i,1},z_{w_i,2})) & \mapsto & \left\{\begin{array}{ll} zz_{w,1}, & \text{if } j=1,\\ zz_{w,2}, & \text{if } j=2.\end{array}\right.\end{array} \tag{5.25}$$

(Note that the number of copies of $\mathbb{C}^\times$ may be smaller in (5.25). Namely in case $h=n-n_j$ for $j\in\{1,2\}$, (5.25) is correct after erasing the corresponding copy of $\mathbb{C}^\times$ from the F_w-component.) Now [Shi10, Eqn (8.7)] tells us that

$$c_{M_{H,j}} = e_p(J_b)\mu_1(s_2)\langle\widehat{\nu}_b^{M_{H,j}},s_1\rangle^{-1} = \left\{\begin{array}{ll} e_p(J_b), & \text{if } j=1,\\ -e_p(J_b), & \text{if } j=2.\end{array}\right. \tag{5.26}$$

Of course we know that $e_p(J_b)=(-1)^{n-h-1}$.

Recall the definition of n-$\mathrm{Red}^b_{n_1,n_2}$ from (5.14). In the current case, we see from (5.20) and (5.26) that n-$\mathrm{Red}^b_{n_1,n_2}$ is equal to

$$\left\{\begin{array}{cc} 0, & h<n_2,\\ e_p(J_b)\cdot LJ^{M_b}_{J_b}\circ\widetilde{\eta}_{M_{H,1},*}\circ J^H_{P^{\mathrm{op}}_{M_{H,1}}}, & n_2\le h<n_1,\\ e_p(J_b)\sum_{j=1}^2(-1)^{j-1}LJ^{M_b}_{J_b}\circ\widetilde{\eta}_{M_{H,j},*}\circ J^H_{P^{\mathrm{op}}_{M_{H,j}}}, & h\ge n_1.\end{array}\right. \tag{5.27}$$

We set up notation for Lemma 5.9. Let $\pi_{H,p}$ be any representation of $\mathrm{Irr}_l(H(\mathbb{Q}_p))$ and set $\pi_{M,p}:=\pi_{H,p}\otimes\chi^+_{\varpi,u}$, where $\chi^+_{\varpi,u}$ is defined in Case 2 of §3.4.[6] Put $\pi_p:=\widetilde{\eta}_*(\pi_{H,p})$, or equivalently

$$\pi_p := \text{n-ind}^G_H(\pi_{M,p}).$$

Here H is viewed as a Levi subgroup of G (over $\mathbb{Q}_p$). Write

$$\pi_{M,p}=\pi_{p,0}\otimes\bigotimes_{i\ge1}\left(\pi_{M,w_i,1}\otimes\pi_{M,w_i,2}\right),\qquad \pi_p=\pi_{p,0}\otimes\bigotimes_{i\ge1}\pi_{w_i}$$

where $\pi_{p,0}\in\mathrm{Irr}_l(\mathbb{Q}_p^\times)$, $\pi_{M,w_i,j}\in\mathrm{Irr}_l(GL_{n_j}(F_{w_i}))$ and $\pi_{w_i}\in\mathrm{Groth}(GL_n(F_{w_i}))$. (As a parabolic induction, π_{w_i} may be reducible.) Let us write the following Jacquet modules as finite sums of irreducible representations.

$$J^{GL_{n_1}}_{P^{\mathrm{op}}_{n-h,h-n_2}}(\pi_{M,w,1})=\sum_k\alpha_{k,1}\otimes\alpha_{k,2},\qquad J^{GL_{n_2}}_{P^{\mathrm{op}}_{n-h,h-n_1}}(\pi_{M,w,2})=\sum_k\beta_{k,1}\otimes\beta_{k,2}. \tag{5.28}$$

Define $X_1(h,\pi_{H,p}),X_2(h,\pi_{H,p})\in\mathrm{Groth}(D^\times_{F_w,1/(n-h)}\times GL_h(F_w))$ as follows.

$$\begin{array}{rcl} X_1(h,\pi_{H,p}) &=& \left\{\begin{array}{ll}\sum_k LJ_{n-h}(\alpha_{k,1})\otimes\text{n-ind}(\alpha_{k,2}\otimes\pi_{M,w,2}), & \text{if } h\ge n_2,\\ 0, & \text{if } h<n_2,\end{array}\right.\\ X_2(h,\pi_{H,p}) &=& \left\{\begin{array}{ll}\sum_k LJ_{n-h}(\beta_{k,1})\otimes\text{n-ind}(\beta_{k,2}\otimes\pi_{M,w,1}), & \text{if } h\ge n_1,\\ 0, & \text{if } h<n_1.\end{array}\right.\end{array}$$

[6]There is no Levi subgroup M in this subsection. The notation $\pi_{M,p}$ is justified by the fact that $BC(\pi_{M,p})$ should appear as the p-component of Π_M of §6.1. (The same holds for $BC(\pi_{H,p})$ and Π_H.) The use of M in the subscript is intended to reflect the fact that π_p is parabolically induced from $\pi_{M,p}$. (In contrast, π_p is viewed as an endoscopic transfer of $\pi_{H,p}$.)

It is immediately checked that (5.29) below provides an equivalent definition for $X_1(h,\pi_{H,p})$ when $h\ge n_2$ and $X_2(h,\pi_{H,p})$ when $h\ge n_1$.

$$\begin{aligned} X_1(h,\pi_{H,p}) &= \text{n-ind}^{GL_h}_{GL_{h-n_2,n_2}}\left(\text{n-Red}^{n-h,h-n_2}(\pi_{M,w,1})\otimes\pi_{M,w,2}\right)\\ X_2(h,\pi_{H,p}) &= \text{n-ind}^{GL_h}_{GL_{h-n_1,n_1}}\left(\text{n-Red}^{n-h,h-n_1}(\pi_{M,w,2})\otimes\pi_{M,w,1}\right) \end{aligned}\tag{5.29}$$

Lemma 5.9. *Put ourselves in Case 2 as above. The following hold in* $\mathrm{Groth}(J_b(\mathbb{Q}_p))$.

(i) $\text{n-Red}^b_n(\pi_p)=e_p(J_b)\cdot\pi_{p,0}\otimes(X_1(h,\pi_{H,p})+X_2(h,\pi_{H,p}))\otimes(\otimes_{i>1}\pi_{w_i})$.

(ii) $\text{n-Red}^b_{n_1,n_2}(\pi_{H,p})=e_p(J_b)\cdot\pi_{p,0}\otimes(X_1(h,\pi_{H,p})-X_2(h,\pi_{H,p}))\otimes(\otimes_{i>1}\pi_{w_i})$.

Proof. We will present a proof when $h\ge n_1$. The same proof works in the other cases if the terms involving $h-n_1$ (resp. $h-n_1$ and $h-n_2$) are disregarded in case $n_2\le h<n_1$ (resp. $h<n_2$).

The proof of (i) goes as follows. Recall from (5.16) that

$$\text{n-Red}^b_n(\pi_p)=e_p(J_b)\cdot\pi_{p,0}\otimes X\otimes(\otimes_{i>1}\pi_{w_i})$$

where $X\in\mathrm{Groth}(D^\times_{F_w,1/(n-h)}\times GL_h(F_w))$ is described as

$$\begin{aligned} X &= \text{n-Red}^{n-h,h}(\pi_w)=LJ_{n-h}\left(J^{GL_n}_{P^{\mathrm{op}}_{n-h,h}}(\text{n-ind}(\pi_{M,w,1}\otimes\pi_{M,w,2}))\right)\\ &= LJ_{n-h}\left(\text{n-ind}^{GL_h}_{GL_{h-n_2,n_2}}\left(J^{GL_{n_1}}_{P^{\mathrm{op}}_{n-h,h-n_2}}(\pi_{M,w,1})\right)+\text{n-ind}^{GL_h}_{GL_{h-n_1,n_1}}\left(J^{GL_{n_2}}_{P^{\mathrm{op}}_{n-h,h-n_1}}(\pi_{M,w,2})\right)+Y\right) \end{aligned}$$

The last identity is implied by the geometrical lemma ([BZ77, p.448]), where Y is a certain linear combination of irreducible representations of $GL_{n-h}(F_w)\times GL_h(F_w)$ of which each $GL_{n-h}(F_w)$-component is a full parabolic induction from a proper Levi subgroup. It follows from [Bad07, Prop 3.3] that $LJ_{n-h}(Y)=0$. Therefore $X=X_1(h,\pi_{H,p})+X_2(h,\pi_{H,p})$ and the proof of (i) is complete.

To demonstrate (ii), we use the identity

$$\widetilde{\eta}_{M_{H,j},*}\circ J^H_{P^{\mathrm{op}}_{M_{H,j}}}(\pi_{H,p})=\text{n-ind}^{M_b}_{M_H}\circ J^H_{P^{\mathrm{op}}_{M_{H,j}}}(\pi_{M,p})\tag{5.30}$$

which is verified from the definition of $\widetilde{\eta}_{M_{H,j},*}$. By (5.27) and (5.30),

$$\begin{aligned} \text{n-Red}^b_{n_1,n_2}(\pi_{H,p}) &= e_p(J_b)\sum_{j=1}^{2}(-1)^{j-1}LJ^{M_b}_{J_b}\circ\text{n-ind}^{M_b}_{M_H}\circ J^H_{P^{\mathrm{op}}_H}(\pi_{M,p})\\ &= e_p(J_b)\cdot\pi_{p,0}\otimes X\otimes\left(\bigotimes_{i>1}(\text{n-ind}(\pi_{M,w_i,1}\otimes\pi_{M,w_i,2}))\right) \end{aligned}$$

where $X\in\mathrm{Groth}(D^\times_{F_w,1/(n-h)}\times GL_h(F_w))$ is given by

$$LJ_{n-h}\circ\left(\text{n-ind}^h_{h-n_2,n_2}\circ J^{GL_{n_1,n_2}}_{P^{\mathrm{op}}_{n-h,h-n_2,n_2}}-\text{n-ind}^h_{n_1,h-n_1}\circ J^{GL_{n_1,n_2}}_{P^{\mathrm{op}}_{n_1,n-h,h-n_1}}\right)(\pi_{M,w,1}\otimes\pi_{M,w,2}).\tag{5.31}$$

Plugging in (5.28), we obtain

$$X=\sum_k LJ(\alpha_{k,1})\otimes\text{n-ind}(\alpha_{k,2}\otimes\pi_{M,w,2}))-\sum_k LJ(\beta_{k,1})\otimes\text{n-ind}(\pi_{M,w,1}\otimes\beta_{k,2})$$

which is nothing but $X_1(h,\pi_{H,p})-X_2(h,\pi_{H,p})$.

□

5.6. $\text{n-Red}^b_{\vec{n}}$ **and** $\phi^{\vec{n}}_{\mathrm{Ig},p}$**.** The following lemma shows that the construction of $\phi^{\vec{n}}_{\mathrm{Ig},p}$ is "dual" to the representation-theoretic operation $\mathrm{Red}^b_{\vec{n}}$. Lemma 5.10 is a key input in the analysis of the p-part of representations in the proof of Theorem 6.1.

Lemma 5.10. *Let* $(H,s,\eta)=(G_{\vec{n}},s_{\vec{n}},\eta_{\vec{n}})\in\mathcal{E}^{\mathrm{ell}}(G)$. *For any* $\pi_{H,p}\in\mathrm{Groth}(H(\mathbb{Q}_p))$,

$$\mathrm{tr}\,\pi_{H,p}(\phi^{\vec{n}}_{\mathrm{Ig},p})=\mathrm{tr}\,(\mathrm{Red}^b_{\vec{n}}(\pi_{H,p}))(\phi'_p).$$

(Here test functions are $\overline{\mathbb{Q}}_l$*-valued.)*

Proof. We freely use the results and notation of [Shi10, §6.3]. Recall that by definition (see the formula above Lemma 6.6 of [Shi10])

$$\phi^{\vec{n}}_{\mathrm{Ig},p} = \sum_{(M_H,s_H,\eta_H)} c_{M_H} \cdot \widetilde{\phi}^{M_H}_p \tag{5.32}$$

as functions on $H(\mathbb{Q}_p)$, where the sum is taken over $\mathcal{E}^{\mathrm{eff}}_p(J_b,G;H)$. As we noted earlier, $\mathcal{I}(M_H,H)$ is a singleton, so we chose to write $\widetilde{\phi}^{M_H}_p$ rather than $\widetilde{\phi}^{M_H,i}_p$ with $i \in \mathcal{I}(M_H,H)$. By [Shi10, Lem 3.8],

$$\mathrm{tr}\,\pi_{H,p}(\widetilde{\phi}^{M_H}_p) = \mathrm{tr}\,\left(J^H_{P^{\mathrm{op}}_{M_H}}(\pi_{H,p})\right)(\phi^{M_H}_p). \tag{5.33}$$

Here $\phi^{M_H}_p \in C^\infty_c(M_H(\mathbb{Q}_p))$ is a $\Delta_p(\cdot,\cdot)^{J_b}_{M_H}$-transfer of $\phi^0_p := \phi'_p \cdot \bar{\delta}^{1/2}_{P(\nu_b)} \in C^\infty_c(J_b(\mathbb{Q}_p))$. The normalization of [Shi10, (8.6)] is adopted for transfer factors, namely

$$\Delta_p(\gamma_{M_H},\delta)^{J_b}_{M_H} = e_p(J_b)\cdot \Delta_p(\gamma_{M_H},\gamma_0)^{M_b}_{M_H} \tag{5.34}$$

if δ and γ_0 are transfers of $\gamma_{M_H} \in M_H(\mathbb{Q}_p)$.

We claim that the transfer from ϕ^0_p to $\phi^{M_H}_p$ factors through as

$$\phi^0_p \in C^\infty_c(J_b(\mathbb{Q}_p)) \quad \rightsquigarrow \quad \phi^*_p \in C^\infty_c(M_b(\mathbb{Q}_p)) \quad \rightsquigarrow \quad \phi^{M_H}_p \in C^\infty_c(M_H(\mathbb{Q}_p))$$

in the sense that if ϕ^*_p is a transfer of ϕ^0_p via $\Delta_p(\cdot,\cdot)^{J_b}_{M_b} \equiv e_p(J_b)$ then $\phi^{M_H}_p$ is a $\Delta_p(\cdot,\cdot)^{M_b}_{M_H}$-transfer of ϕ^*_p. To prove the claim, we check the transfer identity for orbital integrals on regular semisimple elements. Since $\phi^{M_H}_p$ is a $\Delta_p(\cdot,\cdot)^{J_b}_{M_H}$-transfer of ϕ^0_p,

$$O^{M_H(\mathbb{Q}_p)}_{\gamma_{M_H}}(\phi^{M_H}_p) = \Delta_p(\gamma_{M_H},\delta)^{J_b}_{M_H}\cdot O^{J_b(\mathbb{Q}_p)}_{\delta}(\phi^0_p) \tag{5.35}$$

for any (J_b,M_H)-regular γ_{M_H} and its transfer δ. (Recall that a stable conjugacy class is the same as a conjugacy class in the groups $J_b(\mathbb{Q}_p)$ and $M_H(\mathbb{Q}_p)$ as well as $M_b(\mathbb{Q}_p)$.) On the other hand, as ϕ^*_p is a transfer of ϕ^0_p, Lemma 2.18.(i) of [Shi10] tells us that $O^{M_b(\mathbb{Q}_p)}_{\gamma_0}(\phi^*_p) = e_p(J_b)\cdot O^{J_b(\mathbb{Q}_p)}_{\delta}(\phi^0_p)$ if there exists $\delta \in J_b(\mathbb{Q}_p)$ matching γ_0 and $O^{M_b(\mathbb{Q}_p)}_{\gamma_0}(\phi^*_p) = 0$ if else. Together with (5.34) and (5.35), the last fact implies that $\phi^{M_H}_p$ is a $\Delta_p(\cdot,\cdot)^{M_b}_{M_H}$-transfer of ϕ^*_p as claimed.

It follows from (5.24), Lemma 3.3 and [Shi10, Lem 2.18.(ii)] that for $\pi_{M_H,p} \in \mathrm{Irr}(M_H(\mathbb{Q}_p))$,

$$\begin{aligned} \mathrm{tr}\,\pi_{M_H,p}(\phi^{M_H}_p) &= \mathrm{tr}\,(\widetilde{\eta}_{H,*}(\pi_{M_H,p}))\,(\phi^*_p) = \mathrm{tr}\,(LJ\,(\widetilde{\eta}_{H,*}(\pi_{M_H,p})))\,(\phi^0_p) \\ &= \mathrm{tr}\,\left(LJ(\widetilde{\eta}_{H,*}(\pi_{M_H,p}))\otimes \bar{\delta}^{1/2}_{P(\nu_b)}\right)(\phi'_p). \end{aligned} \tag{5.36}$$

(When $H = G_n$, the first identity holds trivially since $\widetilde{\eta}_{H,*} = \mathrm{id}$ and we may take $\phi^{M_H}_p = \phi^*_p$.) The identities (5.32), (5.33) and (5.36) complete the proof. □

6. Computation of cohomology

We keep the notation and assumptions from §5. The prime p, the place w of F and the isomorphism ι_p are fixed in §6.1 (they have been fixed soon after Lemma 5.1), but allowed to vary in §6.2 under certain constraints.

6.1. **Cohomology of Igusa varieties.** The main goal of this subsection is to compute part of the cohomology of Igusa varieties after quadratic base change. The main ingredients are the stable trace formula for Igusa varieties and the twisted trace formula.

Choose the character $\varpi : W_E \to \mathbb{C}^\times$ (introduced in §3.1) such that $\mathrm{Ram}_{\mathbb{Q}}(\varpi) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$. (This is possible by Lemma 7.1 that will be proved later but does not depend on this section. Recall that the last condition on ϖ is assumed throughout §4.) Let Ξ be the algebraic representation of $(\mathbb{G}_n)_{\mathbb{C}}$ given by $\iota_l\xi$ as in §4.3. (Put $\iota_l\xi$ in place of ξ there.) Let $\Pi = \psi\otimes\Pi^1$ be an automorphic representation of $\mathbb{G}_n(\mathbb{A}) \simeq GL_1(\mathbb{A}_E)\times GL_n(\mathbb{A}_F)$. Assume that

- $\Pi \simeq \Pi\circ\theta$,
- Π_∞ is generic and Ξ-cohomological (in particular, the central characters of Π and $\Xi^\vee$ coincide on $A_{\mathbb{G}_n\theta,\infty}$),
- $\mathrm{Ram}_{\mathbb{Q}}(\Pi) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$

where $\mathrm{Ram}_{\mathbb{Q}}(\Pi)$ denotes the set of finite primes p where Π is ramified. By Ξ-cohomological we mean that there exists k such that $H^k(\mathrm{Lie}\,\mathbb{G}_n(\mathbb{R}), \mathbb{K}'_\infty, \Pi_\infty \otimes \Xi) \neq 0$, where $\mathbb{K}'_\infty$ is as in §4.3. In particular Π_∞ is isomorphic to Π_Ξ as in §4.3.

Recall that $\mathrm{Ram}_{F/\mathbb{Q}}$ is contained in $\mathrm{Spl}_{F/F^+,\mathbb{Q}}$ by our previous assumption in §5.1. Let S_{fin} be a finite set of places of $\mathbb{Q}$ such that

$$\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi) \cup \mathrm{Ram}_{\mathbb{Q}}(\varpi) \cup \{p\} \subset S_{\mathrm{fin}} \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}} \tag{6.1}$$

and put $S := S_{\mathrm{fin}} \cup \{\infty\}$.

We will consider two cases for Π.

<u>Case ST</u>. ("stable")

Assume that Π is cuspidal.

<u>Case END</u>. ("endoscopic")

Let $m_1, m_2 \in \mathbb{Z}_{>0}$ be such that $m_1 > m_2$ and $m_1 + m_2 = n$. (Recall that $n \in \mathbb{Z}_{\geq 3}$ is odd.) Let Π_i $(i = 1, 2)$ be a cuspidal automorphic representation of $GL_{m_i}(\mathbb{A}_F)$ and Ξ_i be an irreducible algebraic representation of $GL_{m_i}(F \otimes_{\mathbb{Q}} \mathbb{C})$. We will set $\psi_H := \psi \otimes \varpi^{N(m_1,m_2)}$ and

$$\Pi_H := \psi_H \otimes \Pi_1 \otimes \Pi_2, \qquad \Pi := \widetilde{\zeta}_{m_1,m_2,*}(\Pi_H),$$

$$\Pi_{M,i} := \Pi_i \otimes (\varpi \circ N_{F/E} \circ \det)^{\epsilon(n-m_i)}, \qquad \Pi_M := \psi \otimes \Pi_{M,1} \otimes \Pi_{M,2}.$$

In addition to the previous assumptions on Π, suppose that (for $i = 1, 2$)

(i) $\Pi_i^\vee \simeq \Pi_i \circ c$,
(ii) $\psi_{\Pi_1}\psi_{\Pi_2} = \psi_H^c/\psi_H$,
(iii) $\Pi_{i,\infty}$ is cohomological for an irreducible algebraic representation Ξ_i and

By (i) and (ii), Π_H is a θ-stable cuspidal representation of $\mathbb{G}_{m_1,m_2}(\mathbb{A})$. Denote by $\pi_{H,p} \in \mathrm{Irr}_l(G_{m_1,m_2}(\mathbb{Q}_p))$ the unique representation (up to isomorphism) such that $BC(\iota_l \pi_{H,p}) \simeq \Pi_{H,p}$. Denote by φ_H the discrete parameter for $G_{m_1,m_2}(\mathbb{R})$ such that $BC(\varphi_H) \simeq \Pi_{H,\infty}$ (with the notation $BC(\varphi_H)$ as in Remark 4.4).

Observe that $\Pi \simeq \text{n-ind}^{\mathbb{G}_n}_{\mathbb{G}_{m_1,m_2}}(\Pi_M)$. (The last parabolic induction is irreducible; for general linear groups, any parabolic induction of a unitary representation is irreducible.) Let Π_M^0 denote the twist of Π_M by a character of $A_{\mathbb{G}_{m_1,m_2},\infty}$ (via the canonical surjection $\mathbb{G}_{m_1,m_2}(\mathbb{A}) \to A_{\mathbb{G}_{m_1,m_2},\infty}$) such that Π_M^0 is trivial on $A_{\mathbb{G}_{m_1,m_2},\infty}$. Then it is easy to see that

$$\Pi \simeq \text{n-ind}^{\mathbb{G}_n}_{\mathbb{G}_{m_1,m_2}}(\Pi_M^0) \otimes \widetilde{\chi}_{\iota_l \xi}^{-1}. \tag{6.2}$$

Let us define certain parameters in (Case END). For $i \in \{1, 2\}$, let $b^i_{\sigma,1} \geq \cdots \geq b^i_{\sigma,m_i}$ ($\sigma \in \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{C})$) be the integers parametrizing the highest weight attached to Ξ_i and put

$$\beta^i_{\sigma,j} := -b^i_{\sigma,m_i+1-j} + \frac{m_i + 1 - 2j}{2}, \qquad \gamma^i_{\sigma,j} := \beta^i_{\sigma,j} + \epsilon(n - m_i) \cdot \frac{\delta}{2} \tag{6.3}$$

where δ is the odd integer such that $\varpi_\infty(z) = (z/\overline{z})^{\delta/2}$. (The numbers $\gamma^i_{\sigma,j}$ should be thought of as parameters for $\Pi_{M,i}$.) Recall that we defined $\alpha(\iota_l\xi)_{\sigma,j}$ ($\sigma \in \Phi^+_{\mathbb{C}}$, $1 \leq j \leq n$) from $\iota_l\xi$ in (3.18). For any σ and $j < n$, we have $\alpha(\iota_l\xi)_{\sigma,j} > \alpha(\iota_l\xi)_{\sigma,j+1}$. Since $\Pi_\infty \simeq \text{n-ind}(\Pi_{M,\infty})$, it is easy to see that for each $\sigma \in \Phi^+_{\mathbb{C}}$,

$$\{\alpha(\iota_l\xi)_{\sigma,j} : 1 \leq j \leq n\} = \{\gamma^1_{\sigma,j} : 1 \leq j \leq m_1\} \coprod \{\gamma^2_{\sigma,j} : 1 \leq j \leq m_2\}.$$

Thus there is a unique partition $\{1, \ldots, n\} = W^1_\sigma \coprod W^2_\sigma$ with the following property for each $i \in \{1, 2\}$: $\alpha(\iota_l\xi)_{\sigma,k} = \gamma^i_{\sigma,j}$ for some $j \in [1, m_i]$ if and only if $k \in W^i_\sigma$.

We are done with describing the two cases for Π. Let us set up more notations before stating the main result of §6.1. For any $R \in \mathrm{Groth}(G(\mathbb{A}^S) \times G')$ (over $\overline{\mathbb{Q}}_l$), where $G' = G(\mathbb{A}_{S_{\mathrm{fin}}}) \times \mathrm{Gal}(\overline{F}/F)$ or $G' = G(\mathbb{A}_{S_{\mathrm{fin}} \backslash \{p\}}) \times J_b(\mathbb{Q}_p)$. Define

$$R\{\Pi^S\} := \sum_{\pi^S} R\{\pi^S\} \quad \text{and} \quad R[\Pi^S] := \sum_{\pi^S} R[\pi^S] \tag{6.4}$$

where each sum runs over $\pi^S \in \mathrm{Irr}_l^{\mathrm{ur}}(G(\mathbb{A}^S))$ such that $BC(\iota_l \pi^S) \simeq \Pi^S$. (The right hand sides of (6.4) are defined as in §5.2.) An easy observation is that $H(\mathrm{Sh}, \mathscr{L}_\xi)\{\Pi^S\}$ and $H_c(\mathrm{Ig}_b, \mathscr{L}_\xi)\{\Pi^S\}$ are virtual admissible representations of the corresponding G'. Let $BC_T : \mathrm{Groth}(G(\mathbb{A}_T)) \to \mathrm{Groth}(\mathbb{G}_n(\mathbb{A}_T))$ denote the $\mathbb{Z}$-linear extension of the base change map defined in §4.2, where $T \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$ is a finite set. (A priori BC_T is defined on virtual $\mathbb{C}$-representations but also defined on virtual $\overline{\mathbb{Q}}_l$-representations via $\iota_l : \overline{\mathbb{Q}}_l \simeq \mathbb{C}$.)

Theorem 6.1. *Define an integer $C_G := |\ker^1(\mathbb{Q}, G)| \cdot \tau(G)$. Denote by $\pi_p \in \mathrm{Irr}_l(G(\mathbb{Q}_p))$ a representation such that $BC(\iota_l \pi_p) \simeq \Pi_p$. (Such a π_p is unique up to isomorphism as p splits in E. cf. §4.2.) For each $b \in B(G_{\mathbb{Q}_p}, -\mu)$, the following equalities hold in $\mathrm{Groth}(\mathbb{G}_n(\mathbb{A}_{S_{\mathrm{fin}}\backslash\{p\}}) \times J_b(\mathbb{Q}_p))$.*

(i) *(Case ST) There is a constant $e_0 \in \{\pm 1\}$, independent of b, such that*

$$BC_{S_{\mathrm{fin}}\backslash\{p\}}(H_c(\mathrm{Ig}_b, \mathscr{L}_\xi)\{\Pi^S\}) = C_G \cdot e_0 \cdot [\iota_l^{-1}\Pi_{S_{\mathrm{fin}}\backslash\{p\}}][\mathrm{Red}_n^b(\pi_p)]. \tag{6.5}$$

(ii) *(Case END) There are constants $e_1, e_2 \in \{\pm 1\}$, independent of b, such that*

$$BC_{S_{\mathrm{fin}}\backslash\{p\}}(H_c(\mathrm{Ig}_b, \mathscr{L}_\xi)\{\Pi^S\}) = C_G \left([\iota_l^{-1}\Pi_{S_{\mathrm{fin}}\backslash\{p\}}] \left[\frac{1}{2}(e_1 \mathrm{Red}_n^b(\pi_p) + e_2 \mathrm{Red}^b_{m_1,m_2}(\pi_{H,p})) \right] \right). \tag{6.6}$$

Remark 6.2. A priori the sign e_0 depends on Π. The signs e_1 and e_2 depend not only on Π but also on Π_H and other data, at least a priori. However it turns out that e_0 and e_1 always have the same value, as we will see later in Corollary 6.5.(ii). As for e_2, refer to Remark 6.3 in case $m_2 = 1$.

Proof. In the first three paragraphs, we explain the choice of test functions to be used in the trace formula. Choose $(f^n)^S$ and $f^n_{S_{\mathrm{fin}}\backslash\{p\}}$ as any functions in $\mathscr{H}^{\mathrm{ur}}(\mathbb{G}_n(\mathbb{A}^S))$ and $C_c^\infty(\mathbb{G}_n(\mathbb{A}_{S_{\mathrm{fin}}\backslash\{p\}}))$, respectively. Let $\phi^S := BC_n^*((f^n)^S)$ (resp. $\phi_{S_{\mathrm{fin}}\backslash\{p\}} := BC_n^*(f^n_{S_{\mathrm{fin}}\backslash\{p\}})$) as in Case 1 (resp. Case 2) of §4.2. Set $\phi^{\infty,p} := \phi^S \phi_{S_{\mathrm{fin}}\backslash\{p\}}$. Choose any $\phi'_p \in C_c^\infty(J_b(\mathbb{Q}_p))$ such that $\phi^{\infty,p}\phi'_p$ is an acceptable function. We construct other test functions from these.

For each elliptic endoscopic groups $G_{\vec{n}}$ for G, let $(\phi^{\vec{n}}_{\mathrm{Ig}})^S$ (resp. $\phi^{\vec{n}}_{\mathrm{Ig},S_{\mathrm{fin}}\backslash\{p\}}$) be the $\Delta(\cdot,\cdot)^G_{G_{\vec{n}}}$-transfer of ϕ^S (resp. $\phi_{S_{\mathrm{fin}}\backslash\{p\}}$) defined in §3.4. Define $(f^{n_1,n_2})^S := \widetilde{\zeta}^*((f^n)^S)$ and $f^{n_1,n_2}_{S_{\mathrm{fin}}\backslash\{p\}} = \widetilde{\zeta}^*(f^n_{S_{\mathrm{fin}}\backslash\{p\}})$ as in Case 1 and Case 2 of §4.4. Recall from (4.18) and (4.19) that $BC^*_{n_1,n_2}((f^{n_1,n_2})^S) = (\phi^{n_1,n_2})^S$ and that $BC^*_{n_1,n_2}(f^{n_1,n_2}_{S_{\mathrm{fin}}\backslash\{p\}})$ and $\phi^{n_1,n_2}_{\mathrm{Ig},S_{\mathrm{fin}}\backslash\{p\}}$ have the same trace against every admissible representation of $\mathbb{G}_{n_1,n_2}(\mathbb{A}_{S_{\mathrm{fin}}\backslash\{p\}})$.

Let $\phi^{\vec{n}}_{\mathrm{Ig},p}$ (resp. $\phi^{\vec{n}}_{\mathrm{Ig},\infty}$) be the function arising from ϕ'_p (resp. ξ) in (5.10) (resp. (5.11)). Choose $f^{\vec{n}}_p$ so that $BC_n^*(f^{\vec{n}}_p) = \phi^{\vec{n}}_{\mathrm{Ig},p}$. (This is possible because BC_n^* is surjective at p. See §4.2.) Define

$$f^{\vec{n}}_\infty := e_{\vec{n}}(\Delta_\infty) \cdot (-1)^{q(G)} \langle \mu_h, s \rangle \sum_{\varphi_{\vec{n}}} \det(\omega_*(\varphi_{\vec{n}})) \cdot f_{\mathbb{G}_{\vec{n}},\Xi(\varphi_{\vec{n}})} \tag{6.7}$$

where the sum runs over $\varphi_{\vec{n}} : W_\mathbb{R} \to {}^L G_{\vec{n}}$ (up to equivalence) such that $\widetilde{\eta}\varphi_{\vec{n}} \sim \varphi_\xi$. Observe that $q(G) = n-1$ (cf. (3.11)). Here $\Xi(\varphi_{\vec{n}})$ denotes the algebraic representation of $\mathbb{G}_{\vec{n}}$ arising from $\xi(\varphi_{\vec{n}})$ (defined in Remark 3.9) as in the beginning of §4.3. Recall that $f_{\mathbb{G}_{\vec{n}},\Xi(\varphi_{\vec{n}})}$ was defined in §4.3. By §4.3 and the comparison of (5.11) and (6.7), it is verified that $f^{\vec{n}}_\infty$ and $\phi^{\vec{n}}_{\mathrm{Ig},\infty}$ are BC-matching functions. Put $f^{\vec{n}} := (f^{\vec{n}})^S \cdot f^{\vec{n}}_{S_{\mathrm{fin}}\backslash\{p\}} \cdot f^{\vec{n}}_p \cdot f^{\vec{n}}_\infty$.

Consider the formula of Proposition 5.5. By Corollary 4.7, the formula (4.21), Proposition 4.8 and Corollary 4.14, we see that (recalling the notation $A'_{(\cdot)}$ from Lemma 4.11)

$$\begin{aligned} &\mathrm{tr}\,(\phi^{\infty,p}\phi'_p | \iota_l H(\mathrm{Ig}_b, \mathscr{L}_\xi)) \qquad\qquad (6.8)\\ = \;& C_G \Bigg(\frac{1}{2} \sum_{\Pi'} \mathrm{tr}\,(\Pi'_\xi(f^n) A'_{\Pi'_\xi}) + \frac{1}{2} \sum_{\mathbb{G}_{n_1,n_2}} I^{\mathbb{G}_{n_1,n_2}\theta}_{\mathrm{spec}}(f^{n_1,n_2}) \\ &+ \sum_{M \subsetneq \mathbb{G}_n} \frac{|W_M|}{|W_{\mathbb{G}_n}|} |\det(\Phi^{-1}\theta - 1)_{\mathfrak{a}_M^{\mathbb{G}_n \theta}}|^{-1} \sum_{\Pi'_M} \mathrm{tr}\, \left(\text{n-ind}^{\mathbb{G}_n}_M (\Pi'_M)_\xi (f^n) \circ A'_{\text{n-ind}^{\mathbb{G}_n}_M (\Pi'_M)_\xi} \right) \Bigg) \end{aligned}$$

where the first sum runs over θ-stable (equivalently, $\Phi^{-1}\theta$-stable) subrepresentations Π' of $R_{\mathbb{G}_n,\mathrm{disc}}$, the second over the groups $\mathbb{G}_{n_1,n_2}$ coming from elliptic endoscopic groups G_{n_1,n_2} for G (with $n_1 >$

$n_2 > 0$), the third over proper Levi subgroups M of $\mathbb{G}_n$ containing M_0 and the fourth over $\Phi^{-1}\theta$-stable subrepresentations Π'_M of $R_{M,\mathrm{disc}}$. Keep in mind that Proposition 5.5 works on the condition that $\phi^{\infty,p}\phi'_p$ is acceptable. So the same condition is imposed on (6.8). However, we claim that (6.8) holds without such a condition.

Let us prove the claim. Fix test functions outside the p-component. Fix any ϕ'_p, without assuming $\phi^{\infty,p}\phi'_p$ is acceptable. As shown in [Shi09, Lem 6.3], there is a certain element fr^s in the center of $J_b(\mathbb{Q}_p)$ such that $\phi^{\infty,p}(\phi'_p)^{(N)}$ is acceptable for any $N \gg 0$, where $(\phi'_p)^{(N)}(g) = \phi'_p(g(fr^s)^N)$. So (6.8) is true if ϕ'_p is replaced by $(\phi'_p)^{(N)}$ (and if at the same time $f_p^{\vec{n}}$ and $\phi^{\vec{n}}_{\mathrm{Ig},p}$ are constructed from $(\phi'_p)^{(N)}$ rather than ϕ'_p), for any $N \gg 0$. In other words, by (4.12), Corollary 4.14 and Lemma 5.10, both sides of (6.8) are finite linear combinations of the terms which have the form $\mathrm{tr}\,\rho((\phi'_p)^{(N)})$ for some $\rho \in \mathrm{Irr}(J_b(\mathbb{Q}_p))$. Now the argument in the proof of [Shi09, Lem 6.4] shows that the equality (6.8) holds for $\phi^{\infty,p}(\phi'_p)^{(N)}$ for every integer N, in particular for $N = 0$. Hence the claim is proved.

Now that (6.8) is known to be true without acceptability assumption, we may work with arbitrary test functions $\phi^{\infty,p}\phi'_p$. To proceed, we divide into two cases.

(Case ST). Choose a decomposition $A'_\Pi = A'_{(\Pi)^S} A'_{\Pi_{S_{\mathrm{fin}}}} A'_{\Pi_\infty}$ as a product of normalized intertwining operators. Set

$$\frac{A'_\Pi}{A^0_\Pi} := \frac{A'_{\Pi^S}}{A^0_{\Pi^S}} \cdot \frac{A'_{\Pi_{S_{\mathrm{fin}}}}}{A^0_{\Pi_{S_{\mathrm{fin}}}}} \cdot \frac{A'_{\Pi_\infty}}{A^0_{\Pi_\infty}} \in \{\pm 1\}. \tag{6.9}$$

(For the definition of the denominators on the right side, see §4.2 and §4.3. By definition $A^0_{\Pi_{S_{\mathrm{fin}}}} = \prod_{v\in S_{\mathrm{fin}}} A^0_{\Pi_v}$.) In the formula (4.14), any term involving f^{n_1,n_2} may be rewritten as the trace of an induced representation against f^n, by using (4.17). This fact together with Corollary 4.14 guarantees that $\mathrm{tr}\,\Pi^S((f^n)^S)$ appears only in the first sum of (6.8), according to the multiplicity one result of Jacquet and Shalika ([JS81a], [JS81b]; see [AC89, p.200] for summary), which implies that the string of Satake parameters outside a finite set S of a cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$ unramified outside S does not occur as that of automorphic representations of $GL_n(\mathbb{A}_F)$ which are subquotients of induced representations from proper Levi subgroups of $GL_n(\mathbb{A}_F)$. Thus the right side of (6.8) has the following form

$$C_G\left(\frac{1}{2}\frac{A'_\Pi}{A^0_\Pi}\chi_{\Pi^S}((f^n)^S)\mathrm{tr}\,(\Pi_S(f^n_S)A^0_{\Pi_S}) + \sum_{(\Pi')^S\not\simeq\Pi^S}\chi_{(\Pi')^S}((f^n)^S)\times\left(\begin{array}{c}\text{expression in}\\ \text{terms of } f^{\vec{n}}_S\end{array}\right)\right) \tag{6.10}$$

where $(\Pi')^S$ runs over a set of unramified representations of $\mathbb{G}(\mathbb{A}^S)$ not isomorphic to Π^S. (Note that $(\Pi')^S = \Pi^S$ implies that $\Pi'_\xi = \Pi' \otimes \widetilde{\chi}^{-1}_{\iota_l\xi}$ is isomorphic to Π by the strong multiplicity one and the fact that Π'_ξ and Π transform by the same character on $A_{\mathbb{G}_n,\infty}$. Hence the first summand in (6.8) for $(\Pi')^S = \Pi^S$ equals $\mathrm{tr}\,(\Pi(f^n)A_\Pi)$, which is the first term in (6.10).)

On the other hand, we can write $\mathrm{tr}\,(\phi^{\infty,p}\phi'_p|\iota_l H(\mathrm{Ig}_b,\mathscr{L}_\xi))$ in the following form using (4.5).

$$\begin{aligned} &\mathrm{tr}\,\Pi^S((f^n)^S)\,\mathrm{tr}\,\left(\phi_{S_{\mathrm{fin}}\backslash\{p\}}\phi'_p|\iota_l H(\mathrm{Ig}_b,\mathscr{L}_\xi)\{\Pi^S\}\right)\\ +\ &\sum_{(\pi')^S}\mathrm{tr}\,BC((\pi')^S)((f^n)^S)\,\mathrm{tr}\,\left(\phi_{S_{\mathrm{fin}}\backslash\{p\}}\phi'_p|\iota_l H(\mathrm{Ig}_b,\mathscr{L}_\xi)\{(\pi')^S\}\right)\end{aligned} \tag{6.11}$$

The above sum runs over $(\pi')^S \in \mathrm{Irr}^{\mathrm{ur}}(G(\mathbb{A}^S))$ such that $BC((\pi')^S) \not\simeq \Pi^S$. If the test functions on S are fixed, both (6.10) and (6.11) are finite sums (as $(f^n)^S$ varies in $\mathscr{H}^{\mathrm{ur}}(\mathbb{G}_n(\mathbb{A}^S))$). We deduce from linear independence of characters that

$$\mathrm{tr}\,\left(\phi_{S_{\mathrm{fin}}\backslash\{p\}}\phi'_p|\iota_l H(\mathrm{Ig}_b,\mathscr{L}_\xi)\{\Pi^S\}\right) = \frac{C_G}{2}\frac{A'_\Pi}{A^0_\Pi}\cdot\mathrm{tr}\,(\Pi_S(f^n_S)A_{\Pi^0_S}) \tag{6.12}$$

Recall that $\Pi_\infty \simeq \Pi_\Xi$. In view of (4.15), the construction of f^n_∞ implies that

$$\mathrm{tr}\,(\Pi_\infty(f^n_\infty)A^0_{\Pi_\infty}) = 2(-1)^{q(G)}. \tag{6.13}$$

On the other hand, by Lemma 5.10 and (4.12),

$$\mathrm{tr}\,(\Pi_p(f^n_p)A^0_{\Pi_p}) = \mathrm{tr}\,\iota_l\pi_p(\phi^n_{\mathrm{Ig},p}) = \mathrm{tr}\,\iota_l\mathrm{Red}^b_n(\pi_p)(\phi'_p) \tag{6.14}$$

Therefore if we set $e_0 := (-1)^{q(G)} A'_\Pi/A^0_\Pi$, then $\mathrm{tr}\ \left(\phi_{S_{\mathrm{fin}}\backslash\{p\}}\phi'_p|\iota_l H(\mathrm{Ig}_b,\mathscr{L}_\xi)\{\Pi^S\}\right)$ equals

$$C_G\cdot e_0\cdot \mathrm{tr}\ \left(\Pi_{S_{\mathrm{fin}}\backslash\{p\}}(f^n_{S_{\mathrm{fin}}\backslash\{p\}})A^0_{\Pi_{S_{\mathrm{fin}}\backslash\{p\}}}\right)\cdot \mathrm{tr}\ \iota_l\mathrm{Red}^b_n(\pi_p)(\phi'_p). \tag{6.15}$$

Applying (4.12) to the places in $S_{\mathrm{fin}}\backslash\{p\}$, we finish the proof of the assertion (i). (Use the fact that the twisted characters of non-isomorphic θ-stable representations are linearly independent. cf. [AC89, Lem 6.3, p.52].) Obviously e_0 is independent of b.

(Case END). We imitate the previous argument for (Case ST). By the multiplicity one principle for Satake parameters by Jacquet and Shalika, (6.8) may be rewritten as

$$\mathrm{tr}\ \left(\phi^{\infty,p}\phi'_p|\iota_l H(\mathrm{Ig}_b,\mathscr{L}_\xi)\right)=\frac{C_G}{4}(X_1+X_2+X_3), \tag{6.16}$$

$$\begin{aligned}\text{where}\quad X_1 &= \mathrm{tr}\ \left(\text{n-ind}^{\mathbb{G}_n}_{\mathbb{G}_{m_1,m_2}}(\Pi^0_M)_\xi(f^n)\circ A'_{\text{n-ind}^{\mathbb{G}_n}_{\mathbb{G}_{m_1,m_2}}(\Pi^0_M)_\xi}\right)\\ X_2 &= \mathrm{tr}\ \left(\Pi_H(f^{m_1,m_2})\circ A'_{\Pi_H}\right)\end{aligned}$$

and X_3 is a linear combination of evaluation against f^S of unramified Hecke characters of $\mathscr{H}^{\mathrm{ur}}(\mathbb{G}_n(\mathbb{A}^S))$ different from χ_{Π^S}. Note that X_1 comes from the last term in (6.8) in the case where the standard Levi subgroups M of $\mathbb{G}_n$ is conjugate to $\mathbb{G}_{m_1,m_2}$. (There are $|W_{\mathbb{G}_n}|/|W_M|$ such Levi subgroups.) The term X_2 appears in the second summation on the right side of (6.8), namely those terms in the expansion of $I^{\mathbb{G}_{m_1,m_2}\theta}_{\mathrm{spec}}(f^{m_1,m_2})$ where the Levi subgroup of $\mathbb{G}_{m_1,m_2}$ is $\mathbb{G}_{m_1,m_2}$ itself. As there is no danger of confusion, let us agree to write n-ind(Π^0_M) instead of $\text{n-ind}^{\mathbb{G}_n}_{\mathbb{G}_{m_1,m_2}}(\Pi^0_M)$. Define the signs (+1 or −1)

$$A'_{\text{n-ind}(\Pi^0_M)_\xi}/A^0_{\text{n-ind}(\Pi^0_M)_\xi}\quad\text{and}\quad A'_{\Pi_H}/A^0_{\Pi_H}$$

as in (6.9). Define $e_1 := (-1)^{q(G)}A'_{\text{n-ind}(\Pi^0_M)_\xi}/A^0_{\text{n-ind}(\Pi^0_M)_\xi}$. Recall from (6.2) and (4.29) that there is an isomorphism $\Pi\simeq \text{n-ind}(\Pi^0_M)_\xi$, under which we transport $A'_{\text{n-ind}(\Pi^0_M)_\xi}$ to A'_Π. So we may rewrite X_1 as

$$X_1=e_1(-1)^{q(G)}\cdot\mathrm{tr}\,(\Pi(f^n)A^0_\Pi)=e_1(-1)^{q(G)}\cdot\chi_{\Pi^S}((f^n)^S)\cdot\mathrm{tr}\,(\Pi_S(f^n_S)A^0_{\Pi_S}) \tag{6.17}$$

whereas (4.17) implies that

$$\begin{aligned}X_2 &= \frac{A'_{\Pi_H}}{A^0_{\Pi_H}}\cdot\chi_{\Pi^S}((f^n)^S)\cdot\mathrm{tr}\,(\Pi_{S_{\mathrm{fin}}\backslash\{p\}}(f^n_{S_{\mathrm{fin}}\backslash\{p\}})A^0_{\Pi_{S_{\mathrm{fin}}\backslash\{p\}}})\\ &\quad\times\ \mathrm{tr}\,(\Pi_{H,p}(f^{m_1,m_2}_p)A^0_{\Pi_{H,p}})\cdot\mathrm{tr}\,(\Pi_{H,\infty}(f^{m_1,m_2}_\infty)A^0_{\Pi_{H,\infty}}).\end{aligned} \tag{6.18}$$

By Lemma 5.10 and (4.12), we have the following analogue of (6.14).

$$\mathrm{tr}\,(\Pi_p(f^{m_1,m_2}_p)A^0_{\Pi_p})=\mathrm{tr}\ \left(\iota_l\mathrm{Red}^b_{m_1,m_2}(\pi_{H,p})\right)(\phi'_p) \tag{6.19}$$

Moreover the expression (6.7) along with (4.15) implies that, since only $\varphi_{\vec{n}}=\varphi_H$ in the sum of (6.7) contributes nontrivially (where φ_H was defined in §6.1),

$$\mathrm{tr}\,(\Pi_\infty(f^{m_1,m_2}_\infty)A^0_{\Pi_\infty})=2e_2\cdot(A'_{\Pi_H}/A^0_{\Pi_H}) \tag{6.20}$$

if we set

$$e_2:=(-1)^{q(G)}e_{m_1,m_2}(\Delta_\infty)\langle\mu_h,s\rangle\det(\omega_*(\varphi_H))\cdot(A'_{\Pi_H}/A^0_{\Pi_H}). \tag{6.21}$$

By linear independence of unramified Hecke characters outside S, the identity (6.16) becomes, in view of (6.13)-(6.21),

$$\mathrm{tr}\ \left(\phi_{S_{\mathrm{fin}}\backslash\{p\}}\phi'_p|\iota_l H(\mathrm{Ig}_b,\mathscr{L}_\xi)\{\Pi^S\}\right)=\frac{C_G}{2}(Y_1+Y_2),\qquad\text{where} \tag{6.22}$$

$$\begin{aligned}Y_1 &= e_1\cdot\mathrm{tr}\ \left(\Pi_{S_{\mathrm{fin}}\backslash\{p\}}(f^n_{S_{\mathrm{fin}}\backslash\{p\}})A_{\Pi^0_{S_{\mathrm{fin}}\backslash\{p\}}}\right)\cdot\mathrm{tr}\ \left(\iota_l\mathrm{Red}^b_n(\pi_p)\right)(\phi'_p)\\ Y_2 &= e_2\cdot\mathrm{tr}\ \left(\Pi_{S_{\mathrm{fin}}\backslash\{p\}}(f^n_{S_{\mathrm{fin}}\backslash\{p\}})A^0_{\Pi_{S_{\mathrm{fin}}\backslash\{p\}}}\right)\cdot\mathrm{tr}\ \left(\iota_l\mathrm{Red}^b_{m_1,m_2}(\pi_{H,p})\right)(\phi'_p).\end{aligned}$$

In view of (4.12), the above identities imply the assertion (ii).

Clearly e_1 belongs to $\{\pm1\}$ by definition but we only know $e_2\in(\mathbb{C}^\times)^1$ a priori. The fact that $e_2\in\{\pm1\}$ can be proved as follows. Observe that the definition of e_2 (as well as e_1) does not depend on b. If $\mathrm{Red}^b_{m_1,m_2}(\pi_{H,p})=0$ for all $b\in B(G_{\mathbb{Q}_p},-\mu)$, then (6.6) remains valid for all b with any choice of e_2, in particular we may choose e_2 in $\{\pm 1\}$. Otherwise there exists b such that $\mathrm{Red}^b_{m_1,m_2}(\pi_{H,p})$ is not trivial. We see from (6.6), which was proved a priori with $\mathbb{C}$-coefficients, that $e_2\in\{\pm1\}$ since the multiplicities of representations on the left side of (6.6) are certainly integers.

□

Remark 6.3. Recall that the sign e_2 is defined in (6.21). We remark on the dependence of e_2 on the $\Phi^{-1}\theta$-stable representation $\Pi_H\in\mathrm{Irr}(\mathbb{G}_{m_1,m_2}(\mathbb{A}))$ when $m_2=1$. Note that $e_{m_1,m_2}(\Delta_\infty)$ depends only on the choice of transfer factors and not on Π_H. The same is true for $\langle\mu_h,s\rangle$. In fact, according to (3.20), $\langle\mu_h,s\rangle=1$ with the convention of §3.6. Recall that $\Pi_H=\psi_H\otimes\Pi_1\otimes\Pi_2$. Using the fact that both ψ_H and Π_2 are one-dimensional characters, it is easy to prove that $A_{\Pi_H}/A^0_{\Pi_H}$ depends only on Π_1, and not on ψ_H and Π_2. Therefore if Π_1 remains the same, it is only $\det(\omega_*(\varphi_H))$, a factor coming from real endoscopy, which may vary on the right side of (6.21).

6.2. **Galois representations in the cohomology of Shimura varieties.** We remind the reader that we keep assuming (i)-(v) of §5.1 and that Π is as in the beginning of §6.1. All results of this subsection relies on these assumptions. (Some of them can be strengthened by the results of §7.)

In the last subsection we fixed p, w and S. Here we want to allow p, w and S to vary. (For each $p\in\mathrm{Spl}_{E/\mathbb{Q}}\backslash\{l\}$, we freely change the choice of $\iota_p:\overline{\mathbb{Q}}_p\simeq\mathbb{C}$ to consider all the places w above p. Recall from §5.1 how ι_p was chosen.) Define $\mathscr{R}_l(\Pi)$ to be the set of $\pi^\infty=\pi^S\otimes\pi_{S_{\mathrm{fin}}}\in\mathrm{Irr}_l(G(\mathbb{A}^\infty))$ such that

- $R^k_{\xi,l}(\pi^\infty)\neq0$ for some k.
- π^S is unramified,
- $BC(\iota_l\pi^S)\simeq\Pi^S$ and
- $BC(\iota_l\pi_{S_{\mathrm{fin}}})\simeq\Pi_{S_{\mathrm{fin}}}$.

Note that the definition of $\mathscr{R}_l(\Pi)$ does not involve the choice of the prime p. It is easy to see that the definition of $\mathscr{R}_l(\Pi)$ is independent of S, as long as S satisfies (6.1). (We use the following fact about $\pi_v\in\mathrm{Irr}_l(G(\mathbb{Q}_v))$: Suppose Π_v is unramified. If $v\in\mathrm{Unr}_{F/\mathbb{Q}}\cap\mathrm{Spl}_{F/F^+,\mathbb{Q}}$ then $BC(\iota_l\pi_v)\simeq\Pi_v$ implies that π_v is also unramified.)

Define a representation $\widetilde{R}^k_l(\Pi)$ of $\mathrm{Gal}(\overline{F}/F)$ and $\widetilde{R}_l(\Pi)\in\mathrm{Groth}(\mathrm{Gal}(\overline{F}/F))$ by

$$\widetilde{R}^k_l(\Pi):=\sum_{\pi^\infty\in\mathscr{R}_l(\Pi)}R^k_{\xi,l}(\pi^\infty),\qquad \widetilde{R}_l(\Pi):=\sum_k(-1)^k\widetilde{R}^k_l(\Pi).\tag{6.23}$$

Theorem 6.4. *Let $p\in\mathrm{Spl}_{E/\mathbb{Q}}$ be a prime different from l and w be any place dividing p. Let π_p be as in Theorem 6.1 and write $\pi_p=\pi_0\otimes(\otimes_i\pi_{w_i})$ as usual. Then the following holds in* $\mathrm{Groth}(W_{F_w})$.

(i) *(Case ST)*

$$\widetilde{R}_l(\Pi)|_{W_{F_w}}=C_G\cdot e_0\cdot\left[(\pi^{-1}_{p,0}\circ\mathrm{Art}^{-1}_{\mathbb{Q}_p})|_{W_{F_w}}\otimes\iota_l^{-1}\mathscr{L}_{F_w,n}(\Pi^1_w)\right].$$

(ii) *(Case END)*

$$\widetilde{R}_l(\Pi)|_{W_{F_w}}=\begin{cases} C_G\cdot e_1\cdot\left[(\pi^{-1}_{p,0}\circ\mathrm{Art}^{-1}_{\mathbb{Q}_p})|_{W_{F_w}}\otimes\iota_l^{-1}\mathscr{L}_{F_w,m_1}(\Pi_{M,1,w})|\cdot|^{-m_2/2}_{W_{F_w}}\right], & \text{if } e_1=e_2,\\ C_G\cdot e_1\cdot\left[(\pi^{-1}_{p,0}\circ\mathrm{Art}^{-1}_{\mathbb{Q}_p})|_{W_{F_w}}\otimes\iota_l^{-1}\mathscr{L}_{F_w,m_2}(\Pi_{M,2,w})|\cdot|^{-m_1/2}_{W_{F_w}}\right], & \text{if } e_1=-e_2.\end{cases}$$

Proof. For the proof we may fix p and $w|p$ as in the theorem. Choose $\iota_p:\overline{\mathbb{Q}}_p\simeq\mathbb{C}$ such that $\iota_p^{-1}\tau$ induces w. (Note that $\widetilde{R}^k_l(\Pi)$ for each k is defined independently of ι_p.)

Consider (Case ST). Let us take the $\{\Pi^S\}$-parts of the identity in Proposition 5.2 and apply $BC_{S_{\mathrm{fin}}\backslash\{p\}}$. In view of Theorem 6.1 the following holds in $\mathrm{Groth}(\mathbb{G}_n(\mathbb{A}_{S_{\mathrm{fin}}\backslash\{p\}})\times G(\mathbb{Q}_p)\times W_{F_w})$,

$$\sum_{(\pi')^\infty}[BC(\pi'_{S_{\mathrm{fin}}\backslash\{p\}})][\pi'_p][R_{\xi,l}((\pi')^\infty)]=C_G\cdot[\iota_l^{-1}\Pi_{S_{\mathrm{fin}}\backslash\{p\}}]\left(\sum_{b\in B(G_{\mathbb{Q}_p},-\mu)}[\mathrm{Mant}_{b,\mu}(\mathrm{Red}^b_n(\pi_p))]\right)\tag{6.24}$$

where the first sum runs over $(\pi')^\infty\in\mathrm{Irr}_l(G(\mathbb{A}^\infty))$ such that $(\pi')^S$ is unramified and $BC(\iota_l(\pi')^S)\simeq\Pi^S$. Of course we are using the same π_p as in Theorem 6.1. Observe that $[\mathrm{Mant}_{b,\mu}(\mathrm{Red}^b_n(\pi_p))]$ equals

$$\text{n-Mant}_{1,0}(\pi_{p,0})\otimes\text{n-Mant}_{n-h,h}(\text{n-Red}^{n-h,h}(\pi_w))\otimes(\otimes_{i>1}\text{n-Mant}_{0,n}(\pi_{w_i}))$$

by (2.2) and (5.6), if $h = h(b)$. Proposition 2.3 implies that the right hand side of (6.24) is

$$C_G \cdot [\iota_l^{-1}\Pi_{S_{\mathrm{fin}}\backslash\{p\}}][\pi_p]\left[(\pi_{p,0}^{-1} \circ \mathrm{Art}_{\mathbb{Q}_p}^{-1})|_{W_{F_w}} \otimes \iota_l^{-1}\mathscr{L}_{F_w,n}(\Pi_w^1)\right]. \quad (6.25)$$

By comparing the left side of (6.24) with (6.25), we see that the summands in the left side of (6.24) which do not satisfy $BC(\pi'_{S_{\mathrm{fin}}\backslash\{p\}}) \simeq \iota_l^{-1}\Pi_{S_{\mathrm{fin}}\backslash\{p\}}$ must be canceled out. Hence the first sum in (6.24) can be replaced by a sum over $(\pi')^\infty \in \mathscr{R}_l(\Pi)$ without disturbing the equality.

In (Case END) a similar argument works, so we only indicate changes. The same identity as (6.24) holds if we replace $[\mathrm{Mant}_{b,\mu}(\mathrm{Red}_n^b(\pi_p))]$ by

$$\left[\frac{1}{2}\mathrm{Mant}_{b,\mu}(e_1\mathrm{Red}_n^b(\pi_p) + e_2\mathrm{Red}^b_{m_1,m_2}(\pi_{H,p}))\right]. \quad (6.26)$$

Consider the case $e_1 = e_2$. By Lemma 5.9, the formula (6.26) equals

$$e_1 \cdot \text{n-Mant}_{1,0}(\pi_{p,0}) \otimes \text{n-Mant}_{n-h,h}(X_1(h,\pi_{H,p})) \otimes (\otimes_{i>1}\text{n-Mant}_{0,n}(\pi_{w_i}))$$

for $h = h(b)$. By (5.29), $\text{n-Mant}_{n-h,h}(X_1(h,\pi_{H,p}))$ vanishes if $h < m_2$. If $h \geq m_2$, it equals

$$\text{n-ind}_{GL_{h-m_2,m_2}}^{GL_h}\left((\text{n-Mant}_{n-h,h-m_2}(\text{n-Red}^{n-h,h-m_2}(\Pi_{M,1,w})) \otimes \Pi_{M,2,w})\right) \otimes |\cdot|_{W_{F_w}}^{-m_2/2}$$

by Proposition 2.2.(iii). Proposition 2.3 implies that

$$\sum_{0\leq h\leq n-1} \text{n-Mant}_{n-h,h}(X_1(h,\pi_{H,p})) = [\iota_l^{-1}\Pi_w][\iota_l^{-1}\mathscr{L}_{F_w,n}(\Pi_{M,1,w})] \otimes |\cdot|_{W_{F_w}}^{-m_2/2}.$$

From this the conclusion easily follows in (Case END) with $e_1 = e_2$. The case $e_1 = -e_2$ is proved in the same way.

□

Corollary 6.5. *(cf.* [HT01, Cor VI.2.7]*) Recall the assumptions made at the start of §6.2. For each $\pi^\infty \in \mathscr{R}_l(\Pi)$ the following are true.*

(i) *$R^k_{\xi,l}(\pi^\infty) \neq 0$ if and only if $k = n-1$. Similarly $\widetilde{R}^k_l(\Pi) \neq 0$ if and only if $k = n-1$.*
(ii) *$e_0 = (-1)^{n-1}$ in (Case ST) and $e_1 = (-1)^{n-1}$ in (Case END).*
(iii) *Every $\pi_\infty \in \Pi_{\mathrm{unit}}(G(\mathbb{R}),\iota_l\xi^\vee)$ is $\iota_l\xi$-cohomological. If $\pi_\infty \in \Pi_{\mathrm{unit}}(G(\mathbb{R}),\iota_l\xi^\vee)$ is such that $m(\iota_l(\pi^\infty)\otimes\pi_\infty) > 0$, then $\pi_\infty \in \Pi_{\mathrm{disc}}(G(\mathbb{R}),\iota_l\xi^\vee)$.*
(iv) *Write $\Pi_{\mathrm{disc}}(G(\mathbb{R}),\iota_l\xi^\vee) = \{\pi^1_\infty,\ldots,\pi^n_\infty\}$ as in §3.6. Then*

$$\sum_{\pi^\infty\in\mathscr{R}_l(\Pi)} m(\iota_l(\pi^\infty)\otimes\pi^i_\infty) = \begin{cases} \tau(G), & \text{for all } i, \quad \text{in (Case ST)},\\ \tau(G), & \text{if } i\leq m_1,\ e_1=e_2, \text{ or } i>m_1,\ e_1=-e_2, \text{ in (Case END)},\\ 0, & \text{if } i>m_1,\ e_1=e_2, \text{ or } i\leq m_1,\ e_1=-e_2, \text{ in (Case END)}.\end{cases}$$

(Recall that $\tau(G) = \tau(G_n)$ equals 1 or 2 by Lemma 3.1. In some cases we computed this number in Remark 3.2.)

Remark 6.6. In the proofs of Corollary 6.5 and Corollary 6.7 we largely borrow argument from Harris and Taylor, who attribute their result to Clozel. (Especially the second assertion of (iii) is due to Clozel.) In doing so, it is worth remarking that the two conditions in [HT01, Cor VI.2.7, Cor VI.2.8] are not necessary in our situation. For instance we do not assume that π^∞ is generic at a finite prime split in E. In the setting of Clozel and Harris-Taylor, the base change of π^∞ is an automorphic representation of a non quasi-split inner form of $\mathbb{G}_n$ and the genericity condition ensures that the image of the base change transfers to a cuspidal automorphic representation of $\mathbb{G}_n$. However, we work directly with $\mathbb{G}_n$ and a cuspidal representation Π is given at the outset in (Case ST), so no such assumption is necessary. (In (Case END), use the cuspidality of Π_i.) We also note that we use the strength of the stable trace formula and the twisted trace formula in order to prove (iv) of Corollary 6.5. The proof of its counterpart in corollary VI.2.7 of Harris-Taylor was simpler.

Proof. The first assertion of (i) follows from the second at once. To prove the second assertion of (i), we argue exactly as in [HT01, p.207], appealing to our Theorem 6.4 instead of their corollary V.6.3. (The part (iii) of Proposition 5.3 is also used.)

Note that (ii) is an immediate consequence of (i).

Let us prove (iii). The first part of (iii) follows from [SR99, Thm 1.8] (which identifies every $\pi_\infty \in \Pi_{\text{unit}}(G(\mathbb{R}), \iota_l\xi^\vee)$ with a unitary representation studied in [VZ84]) and the computation of the Lie algebra cohomology in [VZ84]. Observe that (i) and Proposition 5.3 imply that if $m(\iota_l(\pi^\infty)\otimes\pi_\infty) > 0$ then

$$H^k(\mathrm{Lie}\, G(\mathbb{R}), U_\infty, \pi_\infty \otimes \iota_l(\xi)) \neq 0$$

if and only if $k = n-1$. The second part of (iii) can be deduced from this and the results of [VZ84]. (See [HT01, p.207-208] for detailed argument.)

Finally we prove (iv). The argument goes in a similar way as in the proof of Theorem 6.1. Let $(f^n)^\infty = (f^n)^S \cdot f^n_{S_{\text{fin}}} \in C_c^\infty(\mathbb{G}_n(\mathbb{A}^\infty))$ be any function such that $(f^n)^S \in \mathscr{H}^{\text{ur}}(\mathbb{G}_n(\mathbb{A}^S))$. Obtain $(f^{n_1,n_2})^S$, $(\phi^n)^S$, $(\phi^{n_1,n_2})^S$, $f^{n_1,n_2}_{S_{\text{fin}}}$, $\phi^n_{S_{\text{fin}}}$ and $\phi^{n_1,n_2}_{S_{\text{fin}}}$ from $(f^n)^\infty$, as in the beginning of the proof of Theorem 6.1, except that $S_{\text{fin}}\backslash\{p\}$ should be now replaced by S_{fin}. Define

$$f^{\vec{n}}_{\pi^i_\infty} := e_{\vec{n}}(\Delta_\infty)\cdot(-1)^{q(G)}\sum_{\varphi_{\vec{n}}}\langle a_{\omega_*(\varphi_H)\omega_{\pi^i_\infty}}, s\rangle \det(\omega_*(\varphi_{\vec{n}}))\cdot f_{\mathbb{G}_{\vec{n}},\Xi(\varphi_{\vec{n}})} \tag{6.27}$$

where the sum runs over $\varphi_{\vec{n}}$ such that $\widetilde{\eta}\varphi_{\vec{n}}$ is equivalent to $\varphi_{\iota_l\xi}$. Then $f^{\vec{n}}_{\pi^i_\infty}$ and $\phi^{\vec{n}}_{\pi^i_\infty}$ are BC-matching. (See (3.13) and the last paragraph of §4.3. Refer to the paragraph above Proposition 5.6 for the definition of $\phi^{\vec{n}}_{\pi^i_\infty}$ and for the reason why $e_{\vec{n}}(\Delta_\infty)$ appears.) Applying the results of §4.5 to Proposition 5.6, we see that

$$\mathrm{tr}\, R_{G,\iota_l\xi}\left(\phi^\infty\cdot\phi_{\pi^i_\infty}\right) = \sum_\pi m(\pi)\cdot \mathrm{tr}\,\pi(\phi^\infty\cdot\phi_{\pi^i_\infty}) = \sum_{G_{\vec{n}}}\iota(G,G_{\vec{n}})I^{\mathbb{G}_{\vec{n}}\theta}_{\text{spec}}((f^{\vec{n}})^\infty\cdot f^{\vec{n}}_{\pi^i_\infty}). \tag{6.28}$$

By construction of $f^n_{\pi^i_\infty}$,

$$\mathrm{tr}\,(\Pi_\infty(f^n_{\pi^i_\infty})A^0_{\Pi_\infty}) = 2(-1)^{q(G)},$$

whereas

$$\mathrm{tr}\,(\Pi_\infty(f^{n_1,n_2}_{\pi^i_\infty})A^0_{\Pi_\infty}) = (-1)^{q(G)}e_{n_1,n_2}(\Delta_\infty)\langle a_{\omega_*(\varphi_H)\omega_{\pi^i_\infty}}, s\rangle\det(\omega_*(\varphi_H)).$$

Let

$$e(i) := \frac{\mathrm{tr}\,(\Pi_\infty(f^{m_1,m_2}_{\pi^i_\infty})A^0_{\Pi_\infty})}{\mathrm{tr}\,(\Pi_\infty(f^{m_1,m_2}_{\infty})A^0_{\Pi_\infty})} = \frac{\langle a_{\omega_*(\varphi_H)\omega_{\pi^i_\infty}}, s\rangle}{\langle \mu_h, s\rangle}$$

where f^{m_1,m_2}_∞ is as in §6.1. Using the convention of §3.6 we can compute that $e(i)=1$ if $i\le m_1$ and $e(i)=-1$ if $i>m_1$. (See (3.20) and (3.21).)

Arguing as in the proof of Theorem 6.1, we obtain in (Case ST)

$$\mathrm{tr}\, R_{G,\iota_l\xi}\{\Pi^S\}(\phi_{S_{\text{fin}}}\cdot\phi_{\pi^0_\infty}) = \tau(G)\cdot e_0\cdot \mathrm{tr}\,(\Pi_{S_{\text{fin}}}(f^n_{S_{\text{fin}}})A^0_{\Pi_{S_{\text{fin}}}}). \tag{6.29}$$

In (Case END),

$$\mathrm{tr}\, R_{G,\iota_l\xi}\{\Pi^S\}(\phi_{S_{\text{fin}}}\cdot\phi_{\pi^0_\infty}) = \tau(G)\cdot(e_1+e(i)\cdot e_2)\cdot \mathrm{tr}\,(\Pi_{S_{\text{fin}}}(f^n_{S_{\text{fin}}})A^0_{\Pi_{S_{\text{fin}}}}). \tag{6.30}$$

The formula (6.29) along with (iii) of the corollary implies that

$$\sum_{\pi^\infty} m(\iota_l(\pi^\infty)\otimes\pi_\infty) = \tau(G)$$

where the sum runs over $\pi^\infty \in \mathrm{Irr}_l(G(\mathbb{A}^\infty))$ such that $BC(\iota_l\pi^S)\simeq\Pi^S$, $BC(\iota_l\pi_{S_{\text{fin}}})\simeq\Pi_{S_{\text{fin}}}$ and $R_{\xi,l}(\pi^\infty)\neq(0)$. This proves (iv) in (Case ST). Similarly, the assertion (iv) in (Case END) easily follows from (6.30).

□

Recall that we defined integers $a_0(\iota_l\xi)$ and $a(\iota_l\xi)_{\sigma,i}$ for $\sigma\in\Phi^+_{\mathbb{C}}$ and $1\le i\le n$ in the paragraph preceding (3.18). Let $\kappa: F\hookrightarrow\overline{\mathbb{Q}}_l$ be a $\mathbb{Q}$-algebra embedding. For each integer $k\in[1,n]$, set

$$j_\kappa(k) := k-1-a(\iota_l\xi)_{\iota_l\kappa,k} - a_0(\iota_l\xi).$$

(Note that $j_\kappa(k_1)\neq j_\kappa(k_2)$ if $k_1\neq k_2$.) Let $\mathcal{W}_\kappa$ (resp. $\mathcal{W}^0_\kappa$) be the set of $j_\kappa(k)$ (resp. $k-1-a(\iota_l\xi)_{\iota_l\kappa,k}$) for those $k\in[1,n]$ such that

- (Case ST) any k is allowed.
- (Case END) $k\in W^1_{\iota_l\kappa}$ if $e_1=e_2$; $k\in W^2_{\iota_l\kappa}$ if $e_1=-e_2$. (The sets $W^1_{\iota_l\kappa}$ and $W^2_{\iota_l\kappa}$ were defined in §6.1.)

Corollary 6.7. *(*[HT01, Cor VI.2.8]*) Let $\kappa = \iota_l^{-1}\tau$. Then*

$$\dim \mathrm{gr}^w D_{\mathrm{DR},\kappa}(\widetilde{R}_l^{n-1}(\Pi)) = \begin{cases} C_G, & \text{if } w \in \mathcal{W}_\kappa, \\ 0, & \text{if } w \notin \mathcal{W}_\kappa. \end{cases}$$

Proof. The proof of [HT01, Cor VI.2.8] works almost verbatim in our case, if we use the results of Corollary 6.5 instead of [HT01, Cor VI.2.7]. We only need to consistently work with the sum over all $\pi^\infty \in \mathcal{R}_l(\Pi)$, rather than with a single π^∞. For instance, the last two identities of [HT01, p.209] become in our case

$$\dim \mathrm{gr}^{j_\kappa(k)} D_{\mathrm{DR},\sigma}(\widetilde{R}_l^{n-1}(\Pi)) = |\ker^1(\mathbb{Q}, G)| \sum_{\pi^\infty \in \mathcal{R}_l(\Pi)} m(\iota_l(\pi^\infty) \otimes \pi_\infty^k) = C_G.$$

In the course of proof, we use an analogue of the part 6 of [HT01, Prop III.2.1], which is also true in our case. Note that our $j_\kappa(k)$ is different from j_k of Harris-Taylor since we have put $\{a(\iota_l\xi)_{\sigma,i}\}_{1\le i\le n}$ in decreasing order.

□

Corollary 6.8. *There exists a (true) continuous semisimple representation $R'_l(\Pi)$ of $\mathrm{Gal}(\overline{F}/F)$ on a $\overline{\mathbb{Q}}_l$-vector space which is*

- *(Case ST) n-dimensional,*
- *(Case END) m_1-dimensional if $e_1 = e_2$; m_2-dimensional if $e_1 = -e_2$,*

such that for any place w of F satisfying $w|_{\mathbb{Q}} \in \mathrm{Spl}_{E/\mathbb{Q}}$ and $w|_{\mathbb{Q}} \neq l$,

$$R'_l(\Pi)|_{W_{F_w}} = \begin{cases} \iota_l^{-1}\mathscr{L}_{n,F_w}(\Pi_w^1), & & \textit{(Case ST)}, \\ \iota_l^{-1}(\mathscr{L}_{m_1,F_w}(\Pi_{M,1,w}) \otimes |\cdot|_{W_{F_w}}^{-m_2/2}), & e_1 = e_2, & \textit{(Case END)}, \\ \iota_l^{-1}(\mathscr{L}_{m_2,F_w}(\Pi_{M,2,w}) \otimes |\cdot|_{W_{F_w}}^{-m_1/2}), & e_1 = -e_2, & \textit{(Case END)}. \end{cases} \quad (6.31)$$

in $\mathrm{Groth}(W_{F_w})$. In particular, $R'_l(\Pi)$ is independent of τ and ψ. Moreover, for every $\kappa : F \hookrightarrow \overline{\mathbb{Q}}_l$,

$$\dim \mathrm{gr}^w D_{\mathrm{DR},\kappa}(R'_l(\Pi)) = \begin{cases} 1, & \text{if } w \in \mathcal{W}^0_\kappa, \\ 0, & \text{if } w \notin \mathcal{W}^0_\kappa. \end{cases} \quad (6.32)$$

Proof. Consider the semisimplification $\widetilde{R}$ of $(-1)^{n-1}\widetilde{R}_l^{n-1}(\Pi)$. Then $\widetilde{R}$ is a true representation of $\mathrm{Gal}(\overline{F}/F)$ whose dimension is C_G times the expected dimension of $R'_l(\Pi)$ in the corollary. We deduce from Theorem 6.4 and the Cebotarev density theorem that $\widetilde{R}$ is independent of the choice of τ. (A priori the construction of $\widetilde{R}_l^{n-1}(\Pi)$ depends on τ as the PEL datum does.) Thus an obvious analogue of (6.32) for $\widetilde{R}$ is true for every $\kappa : F \hookrightarrow \overline{\mathbb{Q}}_l$ by Corollary 6.7. The proof of [HT01, Prop VII.1.8] (see Remark 6.9 below) shows that there exists a semisimple representation $\widetilde{R}'_l(\Pi)$ such that

$$\widetilde{R} = C_G \cdot \widetilde{R}'_l(\Pi).$$

Define

$$R'_l(\Pi) := \widetilde{R}'_l(\Pi) \otimes \mathrm{rec}_{l,\iota_l}(\psi)|_{\mathrm{Gal}(\overline{F}/F)}$$

where $\mathrm{rec}_{l,\iota_l}(\psi^c)$ denotes the continuous l-adic character $\mathrm{Gal}(\overline{E}/E) \to GL_1(\overline{\mathbb{Q}}_l)$ corresponding to ψ^c via class field theory. (See [HT01, p.20].) The identity (6.31) for each w follows from Theorem 6.4. The last two assertions are easy to see. □

Remark 6.9. When importing argument from the proof of [HT01, Prop VII.1.8], the two conditions in that proposition are not necessary for the same reason as in Remark 6.6. In the proof of proposition VII.1.8, the use of corollaries VI.2.7 and VI.2.8 of Harris-Taylor can simply be replaced by the use of their counterparts, namely Corollaries 6.5 and 6.7.

In (Case END), define

$$\mathcal{W}^i_\kappa := \{k - 1 - b^i_{\iota_l\kappa,k} : 1 \le k \le m_i\}.$$

Corollary 6.10. *In (Case END), there exists a continuous semisimple representation*

$$R''_l(\Pi) : \mathrm{Gal}(\overline{F}/F) \to GL_{m_i}(\overline{\mathbb{Q}}_l),$$

where $i = 1$ if $e_1 = e_2$ and $i = 2$ if $e_1 = -e_2$, such that for any place w of F satisfying $w|_{\mathbb{Q}} \in \mathrm{Spl}_{E/\mathbb{Q}}$ and $w|_{\mathbb{Q}} \neq l$,

$$[R''_l(\Pi)|_{W_{F_w}}] = [\iota_l^{-1}\mathscr{L}_{m_i,F_w}(\Pi_{i,w})]$$

and for every $\kappa : F \hookrightarrow \overline{\mathbb{Q}}_l$,

$$\dim \mathrm{gr}^w D_{\mathrm{DR},\kappa}(R''_l(\Pi)) = \begin{cases} 1, & \text{if } w \in \mathcal{W}^i_\kappa, \\ 0, & \text{if } w \notin \mathcal{W}^i_\kappa. \end{cases}$$

Proof. Define $R''_l(\Pi) := R'_l(\Pi) \otimes \mathrm{rec}_{l,\iota_l}\left((\varpi \circ N_{F/E})^{\epsilon(n-m_i)} \otimes |\cdot|^{(n-m_i)/2}\right)$, where $|\cdot| : \mathbb{A}_F^\times/F^\times \to \mathbb{R}^\times_{>0}$ is the modulus character. With this definition, the current corollary is easily deduced from the previous one. As for the Hodge-Tate numbers, we use the fact that

$$\mathcal{W}^i_\kappa = \{w + \frac{\epsilon(n-m_i)\cdot\delta - (n-m_i)}{2} : w \in \mathcal{W}^0_\kappa\},$$

which is easily seen from the discussion in the paragraph preceding (6.2). □

Remark 6.11. We end this section with a remark on generalization. Regarding the results of this section, it is natural to ask whether one can work with more general Π than those considered in (Case ST) or (Case END). (We restricted ourselves to these two cases since they are enough for the purpose of proving our main results in §7. We have not discovered a promising way to strengthen the results in §7 by considering more general Π.)

The method of this paper mostly works if Π is induced from a cuspidal automorphic representation $\psi \otimes (\otimes_{i=1}^r \Pi_i)$ of $\mathbb{G}_{\vec{n}}(\mathbb{A})$ for $\vec{n}$ of any length r where each Π_i is θ-stable. For instance, we can define $\widetilde{R}_l(\Pi)$ in the same manner and prove analogues of most results of §6.2, including Theorem 6.4. A drawback is that we have less control over the sign factors such as e_0, e_1 and e_2, which show up in the twisted trace formula. (Compare with Corollary 6.5.(ii) and Lemma 7.3. It is expected that the sign factors would be precisely computed by means of the Whittaker normalization of intertwining operators as in [CHLb]. For instance, one would be able to compute the sign of the intertwining operator of our Lemma 4.11.) Apart from that, there is no new difficulty other than complication in book-keeping. We have pursued only the case $r \leq 2$ mainly because that is enough for our application to the construction of Galois representations.

We have not tried to deal with the case where Π is induced from a discrete but not cuspidal representation of $\mathbb{G}_{\vec{n}}(\mathbb{A})$. This case may present new difficulties and the computation would be more complicated. We merely remark that Corollary 6.5.(i) is not expected to be true in that case.

7. Construction of Galois representations

In this section we establish some instances of the global Langlands correspondence and prove the local-global compatibility as an application of our computation of the cohomology of Shimura varieties in §6.2.

Let L be a number field, L' a finite soluble extension over L and Π^1 an automorphic representation of $GL_n(\mathbb{A}_L)$. We frequently write $BC_{L'/L}(\Pi^1)$ for the base change lifting of Π^1 in the sense of [AC89, Ch 3].

7.1. **Constructing Galois representations under technical assumptions.** Let E be an imaginary quadratic field, F be a CM field, and F^+ be the maximal totally real subfield of F. Let $m \in \mathbb{Z}_{\geq 2}$. Let Π^0 be a cuspidal automorphic representation of $GL_m(\mathbb{A}_F)$. Consider the following assumptions on (E, F, Π^0).

- $F = EF^+$
- $[F^+ : \mathbb{Q}] \geq 2$
- $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi^0) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$
- $(\Pi^0)^\vee \simeq \Pi^0 \circ c$
- Π^0_∞ is cohomological for an irreducible algebraic representation Ξ^0 of $GL_m(F \otimes_{\mathbb{Q}} \mathbb{C})$.

Let us associate highest weight integers $(a_{\sigma,1} \geq \cdots \geq a_{\sigma,m})$ to Ξ^0, where σ runs over $\mathrm{Hom}_{\mathbb{Q}}(F,\mathbb{C})$. For $1 \leq k \leq m$, let

$$j_\sigma(k) := k - 1 - a_{\sigma,k}. \tag{7.1}$$

If m is even, assume in addition that

- there exist $\sigma_0 \in \mathrm{Hom}_{\mathbb{Q}}(F,\mathbb{C})$ and an odd number k such that $a_{\sigma_0,k} > a_{\sigma_0,k+1}$.

If the above assumption is satisfied, we will say that Ξ^0 is *slightly regular* (at σ_0). If Ξ^0 is slightly regular at σ_0 then it is also slightly regular at σ_0^c since $(\Pi^0)^\vee \simeq \Pi^0 \circ c$.

If m is odd, set

$$n := m, \quad \Pi^1 := \Pi^0 \quad \text{and} \quad \Xi^1 := \Xi^0.$$

If m is even, set

$$n := m+1, \quad \Pi_1 := \Pi^0 \quad \text{and} \quad \Pi_{M,1} := \Pi^0 \otimes (\varpi \circ N_{F/E} \circ \det)$$

and choose any algebraic Hecke character $\Pi_2 = \Pi_{M,2} : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ which satisfies the following.

- $\mathrm{Ram}_{\mathbb{Q}}(\Pi_{M,2}) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$,
- $\Pi_{M,2}\Pi_{M,2}^c = 1$ and
- $\Pi^1 := \text{n-ind}(\Pi_{M,1} \otimes \Pi_{M,2})$ is such that Π^1_∞ is cohomological for an irreducible algebraic representation Ξ^1.

Allow m to be odd or even. Let us fix an embedding $\tau : F \hookrightarrow \mathbb{C}$. Choose a PEL datum $(F, *, V, \langle\cdot,\cdot\rangle, h)$ as in Lemma 5.1 (in particular $\dim_F V = n$) and write G for the associated group. Observe that the assumptions (i)-(v) in §5.1 are verified. Choose a character $\varpi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ as in §3.1, namely ϖ has the property that $\varpi|_{\mathbb{A}^\times}$ is the quadratic character for $E/\mathbb{Q}$ coming from class field theory. Let δ denote the odd integer such that $\varpi_\infty(z) = (z/\overline{z})^{\delta/2}$ (using the identification $(E \otimes_{\mathbb{Q}} \mathbb{R})^\times \simeq \mathbb{C}^\times$ via $\tau|_E$). In fact we choose ϖ as in the following lemma.

Lemma 7.1. *The Hecke character $\varpi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ can be chosen so that*

- $\varpi|_{\mathbb{A}^\times}$ *is as described above,*
- δ *is sufficiently large,*[7] *and*
- $\mathrm{Ram}_{\mathbb{Q}}(\varpi) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$.

Proof. It is standard that ϖ can be chosen to satisfy the first two conditions. If $\mathrm{Ram}_{\mathbb{Q}}(\varpi) \not\subseteq \mathrm{Spl}_{F/F^+,\mathbb{Q}}$, let R be the set of primes $q \in \mathrm{Ram}_{\mathbb{Q}}(\varpi)$ which are not contained in $\mathrm{Spl}_{F/F^+,\mathbb{Q}}$. By our initial assumption, such q must be inert in E. Suppose that there exists a continuous character $\varpi^0 : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ such that

- ϖ^0 is unramified outside $\mathrm{Spl}_{F/F^+,\mathbb{Q}} \cup R \cup \{\infty\}$,
- $\varpi^0_q|_{\mathcal{O}_{E_q}^\times} = \varpi_q|_{\mathcal{O}_{E_q}^\times}$ and $\varpi^0_q(q) = 1$ for each $q \in R$,
- $\varpi^0_\infty = 1$ and $\varpi^0|_{\mathbb{A}^\times} = 1$.

Then ϖ/ϖ^0 is the desired character of the lemma.

It remains to prove that ϖ^0 as above exists. Let T (resp. S) denote the set of places v of E such that $v|_{\mathbb{Q}} \in \mathrm{Spl}_{F/F^+,\mathbb{Q}} \cup R$ (resp. $v|_{\mathbb{Q}} \in R$). Define $U^{T\cup\{\infty\}} := \prod_{v\notin T,\ v\nmid\infty} \mathcal{O}_{E_v}^\times$ and $U_S := \prod_{v\in S} \mathcal{O}_{E_v}^\times$. Choose a sufficiently small open compact subgroup $U_{T\backslash S} \subset \mathbb{A}_{E,T\backslash S}^\times$ so that $(U^{T\cup\{\infty\}}U_{T\backslash S}U_S) \cap E^\times = (1)$. (This is possible since $|\mathcal{O}_E^\times| < \infty$.) Define a finite character ϖ' on $(U^{T\cup\{\infty\}}U_{T\backslash S}U_S E^\times)/E^\times$ so that $\varpi'|_{U_S} = \varpi|_{U_S}$ and ϖ' is trivial on $U^{T\cup\{\infty\}}U_{T\backslash S}$. It is elementary to check that ϖ' extends (uniquely) to a finite continuous character on

$$(U^{T\cup\{\infty\}}U_{T\backslash S}\mathbb{A}_{E,S}^\times E_\infty^\times \mathbb{A}^\times E^\times)/E^\times,$$

which is an open subgroup of $\mathbb{A}_E^\times/E^\times$, so that $\varpi'|_{E_\infty^\times \mathbb{A}^\times} = 1$ and $\varpi'_q(q) = 1$ for every prime $q \in R$. Finally we extend this character to $\mathbb{A}_E^\times/E^\times$ to obtain a desired ϖ^0.

□

[7] We included this condition on δ just in case, but later realized that it was not used in the later argument.

The following lemma is an exact analogue of [HT01, Lem VI.2.10] except that the condition (iv) is new. This additional condition is guaranteed by an argument which is very similar to the proof of Lemma 7.1. Thus we omit the proof of Lemma 7.2.

Lemma 7.2. *Let Π^1 and Ξ^1 be as above. (Allow m to be either odd or even.) We can find a character $\psi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ and an algebraic representation $\xi_\mathbb{C}$ of G over $\mathbb{C}$ satisfying (i), (ii), (iii) and (iv) below.*

(i) $\psi_{\Pi^1} = \psi^c/\psi$,

(ii) Ξ^1 *is isomorphic to the restriction of* Ξ *to* $(R_{F/\mathbb{Q}}\mathrm{GL}_n) \times_\mathbb{Q} \mathbb{C}$*, where* Ξ *is constructed from* $\xi_\mathbb{C}$ *as in §4.3,*

(iii) $\xi_\mathbb{C}|_{E_\infty^\times}^{-1} = \psi_\infty^c$*, and*

(iv) $\mathrm{Ram}_\mathbb{Q}(\psi) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$.

Moreover if l splits in E then (for any choice of ι_l) we may require that ψ satisfy the following as well as (i)-(iv).

(v) $\psi_{\mathcal{O}_{E_u}^\times} = 1$ *where u is the place above l induced by $\iota_l^{-1}\tau|_E$.*

Suppose that a prime l and $\iota_l : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ are fixed. Choose $\xi_\mathbb{C}$ and ψ as in Lemma 7.2 and put ourselves in the situation of (Case ST) or (Case END) of §6.1, according as m is odd or even, by setting $\xi := \iota_l^{-1}\xi_\mathbb{C}$ and $\Pi := \psi \otimes \Pi^1$. These data prepare us to run the argument of §5 and §6.

We need another lemma before stating results on Galois representations. If m is even, consider the numbers $b^1_{\sigma,j}$ and $\gamma^1_{\sigma,j}$ defined in §6.1. Thus the numbers $\{b^1_{\sigma,j}\}$ correspond to the highest weight for $\Xi^1 = \Xi^0$ and $a_{\sigma,j} = b^1_{\sigma,j}$ for all $\sigma \in \mathrm{Hom}_\mathbb{Q}(F,\mathbb{C})$ and $1 \le j \le m$. Moreover,

$$\gamma^1_{\sigma,j} - \gamma^1_{\sigma,j+1} = (b^1_{\sigma,j} - b^1_{\sigma,j+1}) + 1 = (a_{\sigma,j} - a_{\sigma,j+1}) + 1$$

where the first equality follows from (6.3). As Ξ^0 is slightly regular, there exist $\sigma_0 : F \hookrightarrow \mathbb{C}$ and an odd k such that $a_{\sigma_0,k} - a_{\sigma_0,k+1} \ge 1$, which implies

$$\gamma^1_{\sigma_0,k} - \gamma^1_{\sigma_0,k+1} \ge 2.$$

Since Ξ^0 is also slightly regular at σ_0^c as we observed before, it may be assumed that $\sigma_0 \in \Phi^+_\mathbb{C}$ without loss of generality. (Recall that $\sigma_0 \in \Phi^+_\mathbb{C}$ is equivalent to $\sigma_0|_E = \tau|_E$.) Let χ and χ' be algebraic Hecke characters of $GL_1(\mathbb{A}_F)$ such that $\chi\chi^c = 1$ and $\chi'(\chi')^c = 1$. Denote by $c_\sigma, c'_\sigma \in \mathbb{Z}$ the integers such that $\chi_\sigma(z) = (z/\overline{z})^{c_\sigma}$ and $\chi'_\sigma(z) = (z/\overline{z})^{c'_\sigma}$ for each $\sigma \in \mathrm{Hom}_\mathbb{Q}(F,\mathbb{C})$. We are always able to choose χ and χ' such that

$$\gamma^1_{\sigma_0,m} > c_{\sigma_0}, \quad \gamma^1_{\sigma_0,m-k} > c'_{\sigma_0} > \gamma^1_{\sigma_0,m-k+1}$$

and for all $\sigma \in \Phi^+_\mathbb{C}$ different from σ_0,

$$\gamma^1_{\sigma,m} > c_\sigma, \quad \gamma^1_{\sigma,m} > c'_\sigma.$$

Lemma 7.3. *If m is even, suppose that χ and χ' are chosen as above. Then in Theorem 6.1, we have $e_2 = e_1$ for either $\Pi_2 = \chi$ or $\Pi_2 = \chi'$. It is independent of the choice of $\tau : F \hookrightarrow \mathbb{C}$ (which was fixed in §5.1 and remained to be fixed in §5 and §6) whether $\Pi_2 = \chi$ or $\Pi_2 = \chi'$ works.*

Proof. By Corollary 6.5.(i), $e_1 = (-1)^{n-1}$. By Remark 6.3, the sign e_2 depends only on the factor $\det(\omega_*(\varphi_H)) \in \{\pm 1\}$, where φ_H is by definition the discrete L-parameter such that

$$BC(\varphi_H) \simeq \psi_{H,\infty} \otimes \Pi_{1,\infty} \otimes \Pi_{2,\infty}.$$

To prove the first assertion, it suffices to show that $\det(\omega_*(\varphi_H))$ has different signs for $\Pi_2 = \chi$ and $\Pi_2 = \chi'$. Let us compute $\det(\omega_*(\varphi_H))$ using the explicit description of $\omega_*(\varphi_H)$ in (3.19). We adopt the notation of §3.6 so that for each $\sigma \in \Phi^+_\mathbb{C}$, $\gamma^1_{\sigma,j} = \gamma(\xi)_{\sigma,j}$ for $1 \le j \le m$ and $\gamma(\xi)_{\sigma,m+1} = \gamma^2_{\sigma,1}$.

If $\Pi_2 = \chi$ then for every $\sigma \in \Phi^+_\mathbb{C}$,

$$\gamma(\xi)_{\sigma,1} > \cdots > \gamma(\xi)_{\sigma,m} > \gamma(\xi)_{\sigma,m+1} = c_\sigma, \tag{7.2}$$

hence $\omega_*(\varphi_H) = 1$ and $\det(\omega_*(\varphi_H)) = 1$. Now suppose $\Pi_2 = \chi'$. For every $\sigma \in \Phi^+_\mathbb{C}\backslash\{\sigma_0\}$, (7.2) still holds if c_σ is replaced with c'_σ. On the other hand,

$$\gamma(\xi)_{\sigma_0,1} > \cdots > \gamma(\xi)_{\sigma_0,m-k} > \gamma(\xi)_{\sigma_0,m+1} = c'_{\sigma_0} > \gamma(\xi)_{\sigma_0,m-k+1} > \cdots > \gamma(\xi)_{\sigma_0,m}.$$

Thus $\omega_*(\varphi_H)$ is represented by an element of $(\mathcal{S}_{m+1})^{\Phi^+_{\mathbb{C}}}$ whose σ-component is trivial if $\sigma \neq \sigma_0$ and

$$(1,\dots,m+1)\mapsto(1,\dots,m-k,m+1,m-k+1,\dots,m)$$

if $\sigma=\sigma_0$. In particular, $\det(\omega_*(\varphi_H))=-1$ since k is odd and m is even. This completes the proof of the first assertion of the lemma.

As for the independence of the choice of τ, it is enough to show that the above computation of $\det(\omega_*(\varphi_H))$ does not depend on the choice of τ. The above argument depends only on $\tau|_E$ in that $\tau|_E$ determines the subset $\Phi^+_{\mathbb{C}}$ of $\Phi_{\mathbb{C}}$. So we are done if we get the same value of $\det(\omega_*(\varphi_H))$ for τ and τ^c. This follows from the evenness of m and the fact that every parameter flips sign if τ is changed to τ^c (and σ to σ^c) by conjugate self-duality. □

Proposition 7.4. *Let $m\geq 2$ be any integer. Keep the assumptions on (E,F,Π^0) as in the beginning of §7.1. For each prime l and an isomorphism $\iota_l:\overline{\mathbb{Q}}_l \overset{\sim}{\to}\mathbb{C}$, there exists a continuous semisimple representation $R_l(\Pi^0):\mathrm{Gal}(\overline{F}/F)\to GL_m(\overline{\mathbb{Q}}_l)$ such that*

(i) *At every place y of F such that $y\nmid l$ and $y|_{\mathbb{Q}}\notin \mathrm{Ram}_{E/\mathbb{Q}}$,*
$$[R_l(\Pi^0)|_{W_{F_y}}]=[\iota_l^{-1}\mathscr{L}_{m,F_y}(\Pi^0_y)] \tag{7.3}$$
in $\mathrm{Groth}(W_{F_y})$.

(ii) *Suppose $y\nmid l$. For any $\sigma\in W_{F_y}$, each eigenvalue α of $R_l(\Pi^0)(\sigma)$ satisfies $\alpha\in\overline{\mathbb{Q}}$ and $|\alpha|^2\in|k(y)|^{\mathbb{Z}}$ under any embedding $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$.*

(iii) *Let y be a prime of F not dividing l where Π^0_y is unramified. Then $R_l(\Pi^0)$ is unramified at y, and for all eigenvalues α of $R_l(\Pi^0)(\mathrm{Frob}_y)$ and for all embeddings $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$ we have $|\alpha|^2=|k(y)|^{m-1}$.*

(iv) *For every $y|l$, $R_l(\Pi^0)$ is potentially semistable at y.*

(v) *If l splits in E, then for every $y|l$ such that Π^0_y is unramified, $R_l(\Pi^0)$ is crystalline at y.*

(vi) *For each $\sigma:F\hookrightarrow\overline{\mathbb{Q}}_l$, (recall the definition of $j_{\iota_l\sigma}(\cdot)$ from (7.1))*
$$\dim \mathrm{gr}^j D_{\mathrm{DR},\sigma}(R_l(\Pi^0))=\begin{cases}1, & \text{if } j=j_{\iota_l\sigma}(k) \text{ for some } k\in[0,m-1]\\ 0, & \text{otherwise}\end{cases}$$

Proof. Fix l and $\iota_l:\overline{\mathbb{Q}}_l\overset{\sim}{\to}\mathbb{C}$ throughout the proof. Given (E,F,Π^0), define Π^1, Ξ^1 and n depending on the parity of m. In particular, choose $\Pi_{M,2}$ if m is even. Let ψ be a character satisfying (i)-(iv) of Lemma 7.2. Let $\xi_{\mathbb{C}}$ and Ξ be as in that lemma. Set $\xi:=\iota_l^{-1}\xi_{\mathbb{C}}$ and $\Pi:=\psi\otimes\Pi^1$. With these definitions and notations, we have put ourselves in (Case ST) (resp. (Case END)) of §6.1 when m is odd (resp. even). The assumptions of §6.1 in each case are easily verified.

Now we can run the argument of §6 to obtain $R'_l(\Pi)$ as in Corollary 6.8 if m is odd and $R''_l(\Pi)$ as in Corollary 6.10 if m is even. If m is even, we may freely change the choice of $\Pi_{M,2}$ using Lemma 7.3, if necessary, to ensure that the case $e_1=e_2$ occurs. With the definition

$$R_l(\Pi^0):=R'_l(\Pi)\ (m:\text{odd}),\quad\text{and}\quad R_l(\Pi^0):=R''_l(\Pi)\ (m:\text{even}), \tag{7.4}$$

the condition (7.3) is already verified at every $y\nmid l$ such that $y|_{\mathbb{Q}}$ splits in E. The properties (ii)-(v) of $R_l(\Pi^0)$ follow from Proposition 5.3 and (vi) from Corollaries 6.8 and 6.10. We remark that the proof of (v), in which we suppose that l splits in E, requires choices be made such that

- ψ satisfies all (i)-(v) of Lemma 7.2,
- ϖ satisfies an exact analogue of (v) of Lemma 7.2, and
- if m is even, Π_2 is unramified at places of F dividing u (where u is as in Lemma 7.2).

(Obviously there exists ϖ which satisfies the above condition as well as the conditions in Lemma 7.1.)

It remains to prove (7.3) for y such that $y|_{\mathbb{Q}}$ is inert in E and $y\nmid l$. Set $p:=y|_{\mathbb{Q}}$. We can find infinitely many real quadratic fields A not contained in F such that p is inert in A and $\mathrm{Ram}_{A/\mathbb{Q}}\subset\mathrm{Spl}_{E/\mathbb{Q}}$. (So $\mathrm{Ram}_{A/\mathbb{Q}}\subset\mathrm{Spl}_{F/F^+,\mathbb{Q}}$.) Choose one such A. Let E' be the quadratic subfield of AE different from A and E. Then E' is an imaginary quadratic field where p splits. Let $F':=AF$ and $(F')^+:=AF^+$. We claim that A can be chosen so that $BC_{F'/F}(\Pi^0)$ is cuspidal. To prove the claim, assume to the contrary that $BC_{F'/F}(\Pi^0)$ is not cuspidal for some $F'=AF$. Then by [AC89, Thm 4.2, p.202], it must be the case that m is even and that Π^0 is an automorphic induction from a cuspidal automorphic representation of $GL_{m/2}(\mathbb{A}_{F'})$. This can happen for only finitely many quadratic extensions F' of F. Hence there exists a choice of A (satisfying the previous conditions on A) such that $BC_{AF/F}(\Pi^0)$ is cuspidal.

By strong multiplicity one, we deduce that $BC_{F'/F}(\Pi^0)^\vee \simeq BC_{F'/F}(\Pi^0) \circ c$. It is easy to verify that $(E', F', BC_{F'/F}(\Pi^0))$ satisfies the assumptions in the beginning of §7.1. So there exists $R_l(BC_{F'/F}(\Pi^0))$, defined as previously in the current proof, with the property that for any place z of F' such that $z|_{\mathbb{Q}}$ splits in E' and $z \nmid l$,

$$[R_l(BC_{F'/F}(\Pi^0))|_{W_z}] = [\iota_l^{-1}\mathscr{L}_{m,F'_z}(\Pi^0_z)]. \tag{7.5}$$

The Cebotarev density theorem implies that $R_l(BC_{F'/F}(\Pi^0))$ is isomorphic to the restriction of $R_l(\Pi^0)$ to $\mathrm{Gal}(\overline{F}/F')$. We know that y splits in F' since p splits in E'. Let y' be a place of F' above y. Applying (7.5) to $z = y'$, we deduce that

$$[R_l(\Pi^0)|_{W_y}] = [\iota_l^{-1}\mathscr{L}_{m,F_y}(\Pi^0_y)].$$

□

7.2. **Removing assumptions from §7.1.** We are going to improve Proposition 7.4 by removing the first three assumptions in the beginning of §7.1.

Theorem 7.5. *Let $m \in \mathbb{Z}_{\geq 2}$ be an integer and F be any CM field. Let Π^0 be a cuspidal automorphic representation of $GL_m(\mathbb{A}_F)$ satisfying*

- $(\Pi^0)^\vee \simeq \Pi^0 \circ c$,
- *Π^0_∞ is cohomological for some irreducible algebraic representation Ξ^0 and*
- *in addition, Ξ^0 is slightly regular (§7.1) if m is even.*

For each prime l and an isomorphism $\iota_l : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, there exists a continuous semisimple representation $R_l(\Pi^0) : \mathrm{Gal}(\overline{F}/F) \to GL_m(\overline{\mathbb{Q}}_l)$ such that for any place y of F not dividing l,

$$[R_l(\Pi^0)|_{W_{F_y}}] = [\iota_l^{-1}\mathscr{L}_{m,F_y}(\Pi^0_y)] \tag{7.6}$$

holds in $\mathrm{Groth}(W_{F_y})$. Moreover (ii)-(vi) of Proposition 7.4 are verified, with (v) replaced by

(v)′ *For every $y|l$ such that Π^0_y is unramified, $R_l(\Pi^0)$ is crystalline at y.*

Remark 7.6. Let $m \in \mathbb{Z}_{\geq 2}$. Let F be any *totally real* field and Π^0 a cuspidal automorphic representation of $GL_m(\mathbb{A}_F)$ such that Π^0_∞ is cohomological and $\Pi^0 \simeq \Pi^0 \otimes (\psi \circ \det)$ for some character $\psi : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$. Suppose that Π^0_∞ is cohomological for a slightly regular representation if m is even. (Slight regularity is defined analogously as in the case when F is a CM field.) Then a precise analogue of Theorem 7.5 (along with Theorem 7.11 and Corollary 7.13) for F and Π^0 can be proved in the same way Theorem 3.6 of [Tay04] (which considers the case $F = \mathbb{Q}$ for simplicity) was deduced from [HT01, Thm VII.1.9]. See [Tay04] for more detail.

Remark 7.7. One may compare the theorem with [Clo91, Thm 5.7], [HT01, Thm VII.1.9] and [Mor10, Cor 8.4.9]. See also [CHLa]. Refer to §1 for more details.

Remark 7.8. The method of proof is to construct Galois representations of $\mathrm{Gal}(\overline{F}/F')$ for many quadratic extensions F' of F (for which technical assumptions are satisfied) by using Proposition 7.4, and then to "patch" them to produce a representation of $\mathrm{Gal}(\overline{F}/F)$. This type of argument was used in [BR89] and [HT01], and generalized to soluble extensions ([Sor]).

Proof. We may fix l and $\iota_l : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ throughout the proof. Let Π^0 be as in the theorem. In what we call Step (I), we prove the theorem under the following assumptions on (E, F, Π^0), with an exception that (7.6) is established only at $y|_{\mathbb{Q}} \notin \mathrm{Ram}_{E/\mathbb{Q}}$. (We get rid of the conditions on (E, F, Π^0) and y in Step (II).)

- E is an imaginary quadratic field.
- $F = EF^+$.
- l splits in E.
- $\mathrm{Ram}_{\mathbb{Q}}(\Pi^0) \subset \mathrm{Spl}_{E/\mathbb{Q}}$.
- $\mathrm{Ram}_{F/\mathbb{Q}} \subset \mathrm{Ram}_{E/\mathbb{Q}} \coprod \mathrm{Spl}_{E/\mathbb{Q}}$.
- Any finite place y of F^+ is unramified in F if $y|_{\mathbb{Q}}$ is ramified in E.

Let $\mathscr{F}(F)$ be the set of all imaginary quadratic extensions F' over F^+ such that

- Any finite place y of F^+ splits in F' if $y|_{\mathbb{Q}}$ is ramified in E.
- If $y \in \mathrm{Ram}_{F'/F^+}$ then any place y' of F^+ such that $y'|_{\mathbb{Q}} = y|_{\mathbb{Q}}$ splits in F.
- $BC_{FF'/F}(\Pi^0)$ is cuspidal.

Note that the last condition excludes finitely many F'. (See the proof of Proposition 7.4 where the cuspidality of a quadratic base change is discussed.) For each $F' \in \mathscr{F}(F)$, it is verified that $(E, FF', BC_{FF'/F}(\Pi^0))$ satisfies the assumptions in the beginning of §7.1. So there exists $R_l(BC_{FF'/F}(\Pi^0))$ as in Proposition 7.4. Moreover, for any finite extension M over F, clearly it is possible to find $F' \in \mathscr{F}(F)$ such that F' is linearly disjoint from M over F. In this situation we may use the argument of [HT01, p.230-231] to construct a representation $R_l(\Pi^0) : \mathrm{Gal}(\overline{F}/F) \to GL_n(\overline{\mathbb{Q}}_l)$. Moreover, there is a certain finite extension M_0 over F (which depends on a choice made in the course of constructing $R_l(\Pi^0)$) such that for any $F' \in \mathscr{F}(F)$ which is linearly disjoint from M_0 over F, we have

$$[R_l(\Pi^0)|_{\mathrm{Gal}(\overline{F}/FF')}] = [R_l(BC_{FF'/F}(\Pi^0))]. \tag{7.7}$$

(Note that our F, FF' and M_0 play the roles of L, F_A and MA_1 in the notation of [HT01, p.230-231], respectively.) The properties (ii), (iii), (iv) and (vi) of $R_l(\Pi^0)$ are inherited from those of $R_l(BC_{FF'/F}(\Pi^0))$. To verify (7.6) for $R_l(\Pi^0)$, let us fix a finite place $y \nmid l$ of F such that $y|_{\mathbb{Q}}$ does not ramify in E. Choose $F' \in \mathscr{F}(F)$ such that

- F' is linearly disjoint from M_0 over F.
- $y|_{F^+}$ splits in F'.

Then y splits as $y'y''$ in FF'. We deduce (7.6) from

$$[R_l(BC_{FF'/F}(\Pi^0))|_{W_{FF'_{y'}}}] = [\iota_l^{-1}\mathscr{L}_{n,FF'_{y'}}(BC_{FF'/F}(\Pi^0))]$$

and the restriction of (7.7) to $W_{FF'_{y'}} = W_{F_y}$. To show (v)$'$ for $R_l(\Pi^0)$, let y be a place of F above l. Choose $F' \in \mathscr{F}(F)$ which satisfies the two conditions in the above bullet list so that $y = y'y''$ in FF'. By (7.7) we have that $[R_l(\Pi^0)|_{W_y}]$ is the same as $[R_l(BC_{FF'/F}(\Pi^0))|_{W_{F_{y'}}}]$ where the latter is crystalline by (v) of Proposition 7.4. This finishes Step (I).

Step (II) is to prove the theorem in general. Let F and Π^0 be as in the theorem. Let $\mathscr{E}(F)$ be the set of all imaginary quadratic fields E not contained in F such that

- l splits in E.
- $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi^0) \subset \mathrm{Spl}_{E/\mathbb{Q}}$.
- If a finite place y of F^+ is such that $y|_{\mathbb{Q}} \in \mathrm{Ram}_{E/\mathbb{Q}}$ then $y \in \mathrm{Unr}_{F/F^+}$.
- $BC_{EF/F}(\Pi^0)$ is cuspidal.

As before, the last condition excludes only finitely many E. For each $E \in \mathscr{E}(F)$, it is verified that $(E, EF, BC_{EF/F}(\Pi^0))$ satisfies the assumptions in Step (I) of the current proof. So there exists $R_l(BC_{EF/F}(\Pi^0))$ satisfying (i)-(vi) of Proposition 7.4. For any finite extension M over F, we can find $E \in \mathscr{E}(F)$ such that EF is linearly disjoint from M over F. As before we use the argument of [HT01, p.230-231] to construct $R_l(\Pi^0)$. A similar argument as in Step (I) shows that (7.6) and the assertions (ii)-(vi) of Proposition 7.4 (with (v) replaced by (v)$'$) hold for $R_l(\Pi^0)$, by reducing to the case considered in Step (I). (To verify (7.6) for $R_l(\Pi^0)|_{W_{F_z}}$ at an *arbitrary* place z of F, we choose $E \in \mathscr{E}(F)$ such that $z|_{\mathbb{Q}}$ splits in E and imitate the argument in Step (I), with EF in place of FF'.) □

The Ramanujan-Petersson conjecture for GL_n states that every non-archimedean local component of a cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$ for a number field F is (essentially) tempered.

Corollary 7.9. *Let m, F, Π^0 be as in Theorem 7.5. Then Π^0_w is tempered at every finite place w of F.*

Remark 7.10. Compare the corollary with [Clo91, Cor 5.8], [HT01, Cor VII.1.11] and [Mor10, Cor 8.4.10]. (cf. Remark 7.7.)

Proof. This follows from (ii) of Theorem 7.5 and [HT01, Cor VII.2.18]. □

7.3. **Strengthening of the local-global compatibility.** The aim of this last subsection is to improve the identity (7.6) of Theorem 7.5 as in the following theorem. It is worth pointing out that we make use of Corollary 7.9 in the proof, among others. Fix a prime l and an isomorphism $\iota_l : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ throughout §7.3.

Theorem 7.11. *In the setting of Theorem 7.5, we have the following isomorphism of Weil-Deligne representations at every $y \nmid l$.*

$$\mathrm{WD}(R_l(\Pi^0)|_{\mathrm{Gal}(\overline{F}_y/F_y)})^{\mathrm{F-ss}} \simeq \iota_l^{-1}\mathscr{L}_{n,F_y}(\Pi^0_y) \tag{7.8}$$

Taylor and Yoshida proved the above result ([TY07, Thm 1.2]) in the setting of [HT01]. (Boyer ([Boy09]) proved the weight monodromy conjecture for the vanishing cycle complexes arising from Shimura varieties in the same setting, providing an alternative approach to work of Taylor and Yoshida.) To prove Theorem 7.11, it suffices to prove an analogue of [TY07, Thm 1.5] in our setting, namely that $\mathrm{WD}(R_l(\Pi^0)|_{\mathrm{Gal}(\overline{F}_y/F_y)})$ is pure for every $y \nmid l$, by the remark above the cited theorem. For this, we basically repeat the argument of sections 3 and 4 of Taylor-Yoshida's paper with only minor changes. Note that we only need to consider the case "$l \neq p$" in that paper (except a temporary digression to the case $l = p$ in Lemma 7.12 and Corollary 7.13). We devote this subsection to sketch the proof of Theorem 7.11, which amounts to explaining how their argument should be modified. Obviously we claim no originality.

First of all, we briefly recall our Shimura varieties that were used earlier. This replaces the beginning of section 2 of [TY07]. (We do not need the later part of that section.) We put ourselves in the situation of §7.1. So we begin with a triple (E, F, Π^0) satisfying the assumptions there and choose a PEL datum and other data. Recall that we consider (Case ST) with $n = m$ if m is odd and (Case END) with $n = m + 1$ if m is even. In the latter case, choose $\Pi_{M,2}$ so that $R''_l(\Pi)$ has dimension m (rather than 1) in Corollary 6.10. Such a choice is possible by Lemma 7.3.

For each sufficiently small open compact subgroup K of $G(\mathbb{A}^\infty)$, let X_K denote the Shimura variety Sh_K constructed from the above PEL datum (§5.2). We list the modifications to be made in section 3 of [TY07] so that things make sense in our setting. The notations B, $\mathcal{O}_B$, B^{op} and $\mathcal{O}_B^{\mathrm{op}}$ there should be replaced by F, $\mathcal{O}_F$, $M_n(F)$ and $M_n(\mathcal{O}_F)$, respectively. Fix a prime $p \in \mathrm{Spl}_{E/\mathbb{Q}}$ and a place w of F above p. Choose $\iota_p : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ such that $\iota_p^{-1}\tau$ induces the place w. We also fix $\iota_l : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. The groups $U_p^w(\mathfrak{m})$, $\mathrm{Ma}(\mathfrak{m})$ and $\mathrm{Iw}(\mathfrak{m})$ can be defined as obvious analogues, as well as $U_0 := U^p \times \mathrm{Ma}(\mathfrak{m})$ and $U := U^p \times \mathrm{Iw}(\mathfrak{m})$. Set $\mathscr{G}_A := A[w^\infty]$ for abelian schemes A in the moduli problem of our Shimura variety (without multiplying the idempotent ϵ as in Taylor-Yoshida). Let $\mathscr{G}$ denote the Barsotti-Tate $\mathcal{O}_{F_w}$-module associated to the universal abelian scheme for X_{U_0} . We explained in §5.2 that X_{U_0} has a smooth projective integral model over $\mathcal{O}_{F_w}$. Recall that the special fiber $\overline{X}_{U_0}$ over $\mathrm{Spec}\, k(w)$ admits a stratification into $\overline{X}_{U_0}^{(h)}$ for $0 \leq h \leq n-1$. Note that $\overline{X}_{U_0}^{(0)}$ is nonempty of dimension 0 as we can exhibit an $\overline{\mathbb{F}}_p$-point in the corresponding Igusa variety which is a covering of $\overline{X}_{U_0}^{(0)}$, as it was done in [HT01, Lem III.4.3, Cor V.4.5]. By analogues of [HT01, Lem III.4.1.2] and [TY07, Lem 3.1] in our setting (which are proved in the same way), $\overline{X}_{U_0}^{(h)}$ are of pure dimension h for $0 \leq h \leq n-1$. The integral model for X_U over $\mathcal{O}_{F_w}$ and the schemes $Y_{U,i}$, $Y_{U,\mathscr{S}}$ and $Y^0_{U,\mathscr{S}}$ over $\mathrm{Spec}\, k(w)$ are defined as in [TY07]. Notice that m and S in their paper are denoted by $\mathfrak{m}$ and $\mathscr{S}$, respectively, in order to avoid conflict with our notation. Apart from the changes already mentioned, the material in section 3 of Taylor-Yoshida's paper goes through without further modification.

This is a good place to record a useful fact, which will not be needed in the proof of Theorem 7.11. Only in this paragraph, assume that $w|l$ and $l = p$. We know that X_U is a proper scheme over $\mathcal{O}_{F_w}$ with semistable reduction ([TY07, Prop 3.4]), so the universal abelian scheme $\mathscr{A}_U$ over X_U also has semistable reduction over $\mathcal{O}_{F_w}$. Since $H^k(X_U \times_{\mathcal{O}_{F_w}} \overline{F}_w, \mathscr{L}_\xi)$ is a direct summand of $H^{k+m_\xi}(\mathscr{A}_U \times_{\mathcal{O}_{F_w}} \overline{F}_w, \overline{\mathbb{Q}}_l)$ up to a Tate twist for an integer m_ξ ([TY07, p.477]), we deduce that $H^k(X_U \times_{\mathcal{O}_{F_w}} \overline{F}_w, \mathscr{L}_\xi)$ is a semistable representation of $\mathrm{Gal}(\overline{F}_w/F_w)$ ([Tsu99]). Write each $\pi_l \in \mathrm{Irr}_l(G(\mathbb{Q}_l))$ as $\pi_l = \pi_{l,0} \otimes \pi_w \otimes (\otimes_{i>1}\pi_{w_i})$, following our previous convention.

Lemma 7.12. *Let $\pi^\infty \in \mathrm{Irr}_l(G(\mathbb{A}^\infty))$ and assume that $\pi_{l,0}^{\mathbb{Z}_l^\times} \neq 0$. If $\pi_w^{\mathrm{Iw}_{n,w}} \neq 0$ and $R^k_{\xi,l}(\pi^\infty) \neq 0$ for some k, then $R^k_{\xi,l}(\pi^\infty)$ is a semistable representation of* $\mathrm{Gal}(\overline{F}_w/F_w)$.

Proof. Recall $U = U^p \times (\mathrm{Iw}_{n,w} \times U_p^w(\mathfrak{m}) \times \mathbb{Z}_p^\times)$. We can arrange that $(\pi^\infty)^U \neq 0$ by choosing sufficiently small U^p and $U_p^w(\mathfrak{m})$. Then $R^k_{\xi,l}(\pi^\infty)$ is semistable since it appears with nonzero multiplicity as a subrepresentation of $H^k(X_U \times_{\mathcal{O}_{F_w}} \overline{F}_w, \mathscr{L}_\xi)$.

□

Corollary 7.13. *In the setting of Theorem 7.5, if Π^0_y has a nonzero Iwahori fixed vector at $y|l$ then $R_l(\Pi^0)$ is semistable at y.*

Proof. The proof is the same as in the crystalline case. Namely, the corollary is derived from Lemma 7.12 in the same way as the assertion (v)' of Theorem 7.5 was deduced from Proposition 5.3.(v). □

We return to the case $l \neq p$. Now we adapt section 4 of Taylor-Yoshida to our situation. We work under the setting of §6 of our paper, in either (Case ST) or (Case END), depending on the parity of m. Choose a finite set S under the assumptions in the beginning of §6. (In addition, we already assumed that the conditions (i)-(v) above Lemma 5.1 are satisfied.) All additional assumptions will be removed at the end. In fact, let us consider only (Case ST) for now. In particular $\Pi = \psi \otimes \Pi^0$ is cuspidal. (The argument is essentially the same in (Case END), which will be briefly discussed in Remark 7.16.) Let $\pi_p \in \mathrm{Irr}_l(G(\mathbb{Q}_p))$ be such that $BC(\iota_l \pi_p) \simeq \Pi_p$ as before. Write $\pi_p = \pi_{p,0} \otimes \pi_w \otimes (\otimes_{i>1} \pi_{w_i})$ so that $\iota_l \pi_{p,0} = \psi_u$ for $u := w|_E$ and $\iota_l \pi_{w_i} \simeq \Pi^1_{w_i}$ for all i.

Let $I^{(h)}_{U^p,\mathfrak{m}}$ be the Igusa variety of the first kind defined in [HT01, p.121]. (Substitute our Shimura varieties in the definition.) The Iwahori-Igusa variety $I^{(h)}_U$ over $\overline{X}^{(h)}_{U_0}$ is defined as on page 487 of [TY07]. The results of page 487 carry over without change. If $0 \le h \le n-1$ corresponds to b as in (5.3), we will write $\mathrm{Ig}^{(h)}$ for Ig_b and $J^{(h)}(\mathbb{Q}_p)$ for $J_b(\mathbb{Q}_p)$.

At this point we need to mention that we will follow the sign convention of [TY07] in order to minimize confusion. This means that the signs of $H_c(I^{(h)}, \mathscr{L}_\xi)$ (and its variants) and $H(X, \mathscr{L}_\xi)$ differ from the usual convention by $(-1)^h$ and $(-1)^{n-1}$, respectively. Accordingly, we change the definition of $H_c(\mathrm{Ig}^{(h)}, \mathscr{L}_\xi)$ by multiplying $(-1)^h$.

One major change occurs in the middle of page 488 where theorem V.5.4 of [HT01] is cited. Let us elaborate on this point. Put $D := D_{F_w,1/(n-h)}$. Write $\mathcal{O}_D$ for the maximal order in D. It follows from the definition of $H_c(I^{(h)}, \mathscr{L}_\xi)$ that

$$H_c(I^{(h)}, \mathscr{L}_\xi) = H_c(\mathrm{Ig}^{(h)}, \mathscr{L}_\xi)^{\mathbb{Z}_p^\times \times \mathcal{O}_D^\times}$$

where $\mathbb{Z}_p^\times \times \mathcal{O}_D^\times$ is viewed as the subgroup $\mathbb{Z}_p^\times \times (\mathcal{O}_D^\times \times (1)) \times \prod_{i>1}(1)$ of $J^{(h)}(\mathbb{Q}_p)$ via the expression (5.4). Applying Theorem 6.1, we have that (cf. (5.16))

$$BC^p(H_c(I^{(h)}, \mathscr{L}_\xi)[\Pi^S]) = (-1)^{n-1} C_G [\iota_l^{-1} \Pi^{\infty,p}][\pi_{p,0}^{\mathbb{Z}_p^\times} \otimes \mathrm{Red}^{n-h,h}(\pi_w)^{\mathcal{O}_D^\times} \otimes (\otimes_{i>1} \pi_{w_i})]$$

in $\mathrm{Groth}(\mathbb{G}(\mathbb{A}^{\infty,p}) \times J^{(h)}(\mathbb{Q}_p))$, where BC^p denotes the local base change at the places away from p and ∞ (§4.2). We remark that $e_p(J^{(h)})$ does not show up in the formula as we are following the sign convention of [TY07]. According to page 488, Frob_w acts on $H_c(I^{(h)}, \mathscr{L}_\xi)$ as

$$(1, p^{-[k(w):\mathbb{F}_p]}, \varpi_D^{-1}, 1, 1) \in G(\mathbb{A}^{\infty,p}) \times (\mathbb{Q}_p^\times/\mathbb{Z}_p^\times) \times (D^\times/\mathcal{O}_D^\times) \times GL_h(F_w) \times \left(\prod_{i>1} GL_n(F_{w_i})\right)$$

where ϖ_D is any uniformizer of D. It is easy to check that

$$\begin{aligned} BC^p(H_c(I^{(h)}_{\mathrm{Iw}(\mathfrak{m})}, \mathscr{L}_\xi)[\Pi^S]) &= BC^p(H_c(I^{(h)}, \mathscr{L}_\xi)[\Pi^S])^{U_p^w(\mathfrak{m}) \times \mathrm{Iw}_{h,w}} \\ &= (-1)^{n-1} C_G [\iota_l^{-1} \Pi^{\infty,p}][\mathrm{Red}^{(h)}(\pi_w \otimes \pi_{p,0})] \cdot \dim[(\otimes_{i>1} \pi_{w_i})^{U_p^w(\mathfrak{m})}] \end{aligned}$$

in $\mathrm{Groth}(\mathbb{G}(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}_w^{\mathbb{Z}})$ where $\mathrm{Red}^{(h)}$ is defined on page 488.

It is easy to deduce the following analogue of [TY07, Lem 4.3].

$$\begin{aligned} BC^p(H(Y_{\mathrm{Iw}(\mathfrak{m}),\mathscr{S}}, \mathscr{L}_\xi)[\Pi^S]) &= (-1)^{n-1} C_G [\iota_l^{-1} \Pi^{\infty,p}] \dim[(\otimes_{i>1} \pi_{w_i})^{U_p^w(\mathfrak{m})}] \times \\ & \left(\sum_{h=0}^{n-\#\mathscr{S}} (-1)^{n-\#\mathscr{S}-h} \binom{n-\#\mathscr{S}}{h} \iota_l^{-1} [\mathrm{Red}^{(h)}(\Pi^1_w \otimes \psi_u)] \right) \end{aligned} \tag{7.9}$$

We proceed to prove the following analogue of [TY07, prop 4.4] by imitating the original argument.

Proposition 7.14. *Keep the previous notation. Suppose that* $\pi_p^{\mathrm{Iw}(\mathfrak{m})} \neq 0$. *Then*

$$BC^p(H^j(Y_{\mathrm{Iw}(\mathfrak{m}),\mathscr{S}},\mathscr{L}_\xi)[\Pi^S]) = 0$$

for $j \neq n - \#\mathscr{S}$.

Proof. Let $D(\Pi) := (-1)^{n-1}C_G \cdot \dim(\otimes_{i>1}\Pi_{w,i})^{U_p^w(\mathfrak{m})}$ for each $\pi^\infty \in \mathscr{R}_l(\Pi)$. (Note that our $D(\Pi)$ differs from D of [TY07] by the dimension of the Iwahori invariants at w.) The assumption implies that $D(\Pi) \neq 0$. By (7.9),

$$BC^p(H(Y_{\mathrm{Iw}(\mathfrak{m}),\mathscr{S}},\mathscr{L}_\xi))\{\Pi^{\infty,p}\} = D(\Pi)\sum_{h=0}^{n-\#\mathscr{S}}(-1)^{n-\#\mathscr{S}-h}\binom{n-\#\mathscr{S}}{h}\iota_l^{-1}[\mathrm{Red}^{(h)}(\Pi_w^1\otimes\psi_u)]$$

in $\mathrm{Groth}(\mathrm{Frob}_w^{\mathbb{Z}})$. We will be done if the above expression is shown to be zero.

The initial assumption says that Π_w^1 has a nonzero Iwahori fixed vector. Moreover Π_w^1 is tempered by Corollary 7.9. So Π_w^1 has the form

$$\text{n-ind}_{P(F_w)}^{GL_n(F_w)}(\mathrm{Sp}_{s_1}(\pi_1)\otimes\cdots\otimes\mathrm{Sp}_{s_t}(\pi_t))$$

for unramified characters $\pi_i : F_w^\times \to \mathbb{C}^\times$ and $\sum_i s_i = n$. Then $\mathrm{Red}^{(h)}(\Pi_w^1\otimes\psi_u)$ can be computed as in [TY07]. We obtain

$$BC^p(H(Y_{\mathrm{Iw}(\mathfrak{m}),\mathscr{S}},\mathscr{L}_\xi))\{\Pi^{\infty,p}\} = D(\Pi)\sum_{s_i=\#\mathscr{S}}\frac{(n-\#\mathscr{S})!}{\prod_{j\neq i}s_j!}[V_i] \tag{7.10}$$

where V_i is defined on page 490 of [TY07]. Since V_i are strictly pure of weight $m_\xi - 2t_\xi + (n-\#\mathscr{S})$ (m_ξ and t_ξ are defined in [HT01, p.98]), the Weil conjecture implies that

$$BC^p(H(Y_{\mathrm{Iw}(\mathfrak{m}),\mathscr{S}},\mathscr{L}_\xi))\{\Pi^{\infty,p}\} = 0$$

for $j \neq n - \#\mathscr{S}$.

□

So far we have considered BC^p on the level of Grothendieck groups. Now we work with genuine admissible representations. For each $k \geq 0$, define (cf. (5.5))

$$BC^p(H^k(X_{\mathrm{Iw}(\mathfrak{m})},\mathscr{L}_\xi)[\Pi^S]) := \bigoplus_{\pi^\infty}\dim(\pi_p^{\mathrm{Iw}(\mathfrak{m})})\cdot BC(\pi^{\infty,p})\otimes R^k_{\xi,l}(\pi^\infty)$$

where the sum runs over $\pi^\infty \in \mathrm{Irr}_l(G(\mathbb{A}^\infty))$ such that π^S is unramified and $BC(\iota_l\pi^S)\simeq\Pi^S$. Theorem 6.4 and its proof show that

$$BC^p(H^{n-1}(X_{\mathrm{Iw}(\mathfrak{m})},\mathscr{L}_\xi)[\Pi^S]) \simeq (\dim\pi_p^{\mathrm{Iw}(\mathfrak{m})})\cdot\iota_l^{-1}\Pi^{\infty,p}\otimes\widetilde{R}_l^{n-1}(\Pi) \tag{7.11}$$

as admissible representations of $\mathbb{G}(\mathbb{A}^{\infty,p})\times\mathrm{Gal}(\overline{F}/F)$.

Corollary 7.15. *In the setting of Proposition 7.14,* $\mathrm{WD}(\widetilde{R}_l^{n-1}(\Pi)|_{\mathrm{Gal}(\overline{F}_w/F_w)})$ *is pure of weight* $m_\xi - 2t_\xi + n - 1$.

Proof. In view of [TY07, Lem 1.4.(1), Lem 1.7], it suffices to show that $\mathrm{WD}(\widetilde{R}_l^{n-1}(\Pi)^{\mathrm{ss}}|_{\mathrm{Gal}(\overline{F}_w/F_w)})^{\mathrm{F-ss}}$ is pure of the designated weight. Here the superscript "ss" means the semisimplification of the $\mathrm{Gal}(\overline{F}/F)$-action.

We use a slightly different form of the spectral sequence of [TY07, Prop 3.5], which can be derived from its proof. With the notation of that proposition, consider the spectral sequence

$$BC^p(E_1^{i,j}(\mathrm{Iw}(\mathfrak{m}),\xi)[\Pi^S]) \Rightarrow BC^p(WD(H^{i+j}(X_{\mathrm{Iw}(\mathfrak{m})},\mathscr{L}_\xi)^{\mathrm{ss}}|_{\mathrm{Gal}(\overline{F}_w/F_w)})^{\mathrm{F-ss}}). \tag{7.12}$$

Here each side is viewed as a semisimple representation of $\mathbb{G}(\mathbb{A}^{\infty,p})\times\mathrm{Frob}_w$ (after semisimplifying the action of $\mathbb{G}(\mathbb{A}^{\infty,p})\times\mathrm{Frob}_w$ on the left hand side) with a nilpotent operator N. The above spectral sequence can be obtained in the following way. First, we semisimplify the action of $G(\mathbb{A}^{\infty,p})\times\mathrm{Frob}_w$ in the Rapoport-Zink weight spectral sequence, which is the second last formula of [TY07, p.485]. Next, separate the $[\Pi^S]$-part and apply BC^p to the spectral sequence.

Proposition 7.14 tells us that $BC^p(E_1^{i,j}(\mathrm{Iw}(\mathfrak{m}),\xi)[\Pi^S])$ vanishes unless $i+j=n-1$. So the semisimplified spectral sequence (7.12) degenerates at E_1 and

$$\mathrm{WD}(BC^p(H^{n-1}(X_{\mathrm{Iw}(\mathfrak{m})},\mathscr{L}_\xi)[\Pi^S])^{\mathrm{ss}}|_{\mathrm{Gal}(\overline{F}_w/F_w)})^{\mathrm{F-ss}} \tag{7.13}$$

is pure of the desired weight. This concludes the proof in view of (7.11).

□

Remark 7.16. So far we have been dealing with (Case ST) and odd m. In (Case END) with even m, Proposition 7.14 and Corollary 7.15 are still valid. In (7.9) and the proof of Proposition 7.14, we apply (ii) of Theorem 6.1 to compute $BC^p(H(Y_{\mathrm{Iw}(\mathfrak{m}),\mathscr{S}},\mathscr{L}_\xi)[\Pi^S])$. The proof of Proposition 7.14 mostly goes through except that one of the V_i's will be missing on the right side of (7.10). Corollary 7.15 in (Case END) is proved similarly as in (Case ST).

We are ready to complete the proof of Theorem 7.11. Allow m to be either odd or even. Let us forget the additional assumptions of §5-§6 and put ourselves in the situation of §7.2, but let L denote the CM field to begin with, instead of F. So Π^0 is a cuspidal automorphic representation of $GL_m(\mathbb{A}_L)$. (Of course, unlike [TY07], we do not assume that Π^0 is square integrable at a finite place.) Let $R_l(\Pi^0):\mathrm{Gal}(\overline{L}/L)\to GL_m(\overline{\mathbb{Q}}_l)$ be given by Theorem 7.5. Our plan is to imitate page 492 of [TY07] to find a certain finite soluble extension F over L so that the proof for L and Π^0 can be reduced to the proof for F and $BC_{F/L}(\Pi^0)$.

Fix a place v of L above p where $p\neq l$. Recall the remark below the statement of Theorem 7.11 that it suffices to prove $\mathrm{WD}(R_l(\Pi^0)|_{\mathrm{Gal}(\overline{F}_v/F_v)})$ is pure. Find a CM field F such that (as usual $F^+:=F^{c=1}$)

- $[F^+:\mathbb{Q}]$ is even,
- $F=EF^+$ for an imaginary quadratic field E in which p splits,
- F is soluble and Galois over L,
- $\mathrm{Ram}_{F/\mathbb{Q}}\cup\mathrm{Ram}_{\mathbb{Q}}(\Pi^0)\subset\mathrm{Spl}_{F/F^+,\mathbb{Q}}$,
- $\Pi^0_F:=BC_{F/L}(\Pi^0)$ is a cuspidal automorphic representation of $GL_m(\mathbb{A}_F)$, and
- there is a place w of F above v such that $\Pi^0_{F,w}$ has an Iwahori fixed vector.

We justify that it is possible to choose F as above. As a first step, we find a CM field F_0 which is soluble and Galois over L and a place w_0 of F_0 above v such that the last two conditions in the list are satisfied for F_0 and w_0 in place of F and w. Next we find F from F_0 by taking quadratic extensions of F_0 twice as in the proof of Theorem 7.5. We elaborate on this point. Choose $E\in\mathscr{E}(F_0)$ such that p splits in E and $E\subsetneq F_0$. Since EF_0 verifies the assumptions of Step (I) in that proof, we may choose $F'\in\mathscr{F}(EF_0)$ different from EF_0 and take $F:=F'EF_0$. Let w be any place of F above w_0. It is easy to see that F satisfies every condition in the above list.

With (E,F,Π^0_F) in hand, consider the setting of §7.1. Let Π_F denote the representation Π of §7.1 obtained by substituting Π^0_F for Π^0 in that section. The proofs of Corollaries 6.8 and 6.10 tell us that

$$C_G\cdot R_l(\Pi^0)|_{\mathrm{Gal}(\overline{F}/F)}\simeq C_G\cdot R_l(\Pi^0_F)\simeq \widetilde{R}_l^{n-1}(\Pi_F)^{\mathrm{ss}}\otimes R_l(\psi)^{-1}.$$

It follows from Corollary 7.15 and [TY07, Lem 1.7] that $\mathrm{WD}(R_l(\Pi^0)|_{\mathrm{Gal}(\overline{L}_v/L_v)})$ is pure. The proof of Theorem 7.11 is concluded.

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