ON THE COHOMOLOGY OF RAPOPORT-ZINK SPACES OF EL-TYPE

SUG WOO SHIN

Abstract. In this article, we are interested in determining the l-adic cohomology of Rapoport-Zink spaces associated to GL_n over an unramified extension of $\mathbb{Q}_p$ (called unramified "EL-type") in connection with the local Langlands correspondence for GL_n. In fact, we compute (the alternating sum of) certain representation-theoretic functors defined in terms of their cohomology on the level of Grothendieck groups. In case Rapoport-Zink spaces parametrize p-divisible groups of dimension one, the above alternating sum is determined by [HT01] (without the unramifiedness assumption). For p-divisible groups of dim > 1, Fargues ([Far04]) has obtained an answer for the supercuspidal part when the p-divisible groups are basic (i.e. their associated Newton polygons are straight lines). Our main result is an identity among the cohomology of Rapoport-Zink spaces that appear in the Newton stratification of the same unitary Shimura variety. Our result implies a theorem of Fargues, providing a second proof, and reveals new information about the cohomology of Rapoport-Zink spaces for non-basic p-divisible groups. We propose an inductive procedure to completely determine the above alternating sum, assuming a strengthening of our main result along with a conjecture of Harris.

1. Introduction

Rapoport-Zink spaces are moduli spaces of Barsotti-Tate groups (a.k.a. p-divisible groups) with additional structure ([RZ96]). They turn out to be closely related to the local geometry of Shimura varieties. This relationship may be thought of as a geometric counterpart of the interaction between the local and global Langlands correspondences. The problem of describing the l-adic cohomology of Rapoport-Zink spaces may be traced back to Lubin-Tate theory of formal groups and has had important consequences such as the proof of the local Langlands conjecture ([HT01]) and the local-global compatibility of the Langlands correspondence ([HT01], [Shic]). The idea for the latter application goes back to Deligne and Carayol.

In this paper, we are mainly concerned with the Rapoport-Zink spaces arising from an unramified Rapoport-Zink datum of EL-type where the relevant p-adic group is essentially GL_n. (For other cases see §8.4.) Let us set up some notation in order to be precise. We caution the reader that any notation in the introduction that is used without explanation is defined in §1.1. An unramified Rapoport-Zink datum of EL-type consists of (F, V, μ, b) where

- F is a finite unramified extension of $\mathbb{Q}_p$,
- V is an n-dimensional F-vector space, and set $G := R_{F/\mathbb{Q}_p} GL_F(V)$,
- $\mu : \mathbb{G}_m \to G$ is a $\overline{\mathbb{Q}}_p$-homomorphism up to $G(\overline{\mathbb{Q}}_p)$-conjugacy and
- b belongs to a finite set $B(G, -\mu)$.

Basically b and μ prescribe the Hodge and Newton polygons for the Barsotti-Tate groups in the moduli problem for Rapoport-Zink spaces. The set $B(G, -\mu)$ may be thought of as the set of Newton polygons which lie above the Hodge polygon given by μ and have the same end points. (The minus sign on μ reflects the fact that we work with the dual of the usual Dieudonné modules.) We can construct from (F, V, μ, b) a tower of Rapoport-Zink spaces $\mathcal{M}^{\mathrm{rig}}_{b,\mu} = \{\mathcal{M}^{\mathrm{rig}}_{b,\mu,U}\}$ over $\mathrm{Frac}W(\overline{\mathbb{F}}_p)$, indexed by open compact subgroups of $G(\mathbb{Q}_p)$. Each $\mathcal{M}^{\mathrm{rig}}_{b,\mu,U}$ may be viewed as a rigid analytic space or a Berkovich analytic space. Let E be the reflex field associated with (F, V, μ, b). The field E is finite over $\mathbb{Q}_p$ and independent of b. The l-adic cohomology of $\mathcal{M}^{\mathrm{rig}}_{b,\mu}$ is equipped with an action of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$, where J_b is an inner form of a Levi subgroup of G. Define $\mathrm{Mant}_{b,\mu}$: $\mathrm{Groth}(J_b(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_E)$ by the formula

$$\mathrm{Mant}_{b,\mu}(\rho) = \sum_{i,j\geq 0} (-1)^{i+j} \mathrm{Ext}^i_{J_b(\mathbb{Q}_p)}(H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu}), \rho))(-\dim \mathcal{M}^{\mathrm{rig}}_{b,\mu})$$

where $H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu})$ denotes the l-adic cohomology of $\mathcal{M}^{\mathrm{rig}}_{b,\mu}$ defined by Berkovich. (See Definition 4.7 for a careful treatment. In fact, the above definition is not entirely correct.) Our main problem is:

Problem 1.1. Describe $\mathrm{Mant}_{b,\mu}$ in terms of the local Langlands correspondence and other natural representation-theoretic operations.

This is close to but not the same as describing the l-adic cohomology of Rapoport-Zink spaces. (When b is basic, there is a precise conjecture ([Rap95, Conj 5.1]), which is stated also in the PEL case, about the "supercuspidal" part of the l-adic cohomology of Rapoport-Zink spaces. Corollary 1.3 is an instance of that conjecture; in the situation of the corollary, the Ext-functor in the definition of $\mathrm{Mant}_{b,\mu}$ vanishes in all positive degrees.) However, there are a few good reasons why one may want to study $\mathrm{Mant}_{b,\mu}$, or even prefer to study them. First, $\mathrm{Mant}_{b,\mu}$ naturally appear in the description of the l-adic cohomology of Shimura varieties (Proposition 1.4). Second, $\mathrm{Mant}_{b,\mu}$ satisfy (or are expected to satisfy) some nice identities as in Theorem 1.2 and Conjecture 8.6 when it is not clear that the same identities would hold for $H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu})$ themselves.

Before stating our main result towards Problem 1.1, we need to set up more notation. Let us indicate by $\mathrm{Irr}(\cdot)$ the set of irreducible admissible representations of a given p-adic group. Let M_b be a Levi subgroup of G which is also an inner form of J_b. Since M_b is a product of general linear groups, we can make sense of the Jacquet-Langlands map JL on the set of square-integrable representations. Define $\mathrm{Red}^{(b)} : \mathrm{Groth}(G(\mathbb{Q}_p)) \to \mathrm{Groth}(J_b(\mathbb{Q}_p))$ to be a sign factor $e(J_b) \in \{\pm 1\}$ times the composite of the Jacquet module functor from G to M_b and the Jacquet-Langlands map on the level of Grothendieck groups from M_b to J_b. Let r_μ denote the representation of ${}^LG$ over $\overline{\mathbb{Q}}_l$ determined by μ (§6.4). In particular $r_\mu|_{\widehat{G}}$ has highest weight μ. For $\pi \in \mathrm{Irr}(G(\mathbb{Q}_p))$, let $\mathrm{LL} : W_{\mathbb{Q}_p} \to {}^LG$ denote the semisimplified local Langlands image of π. Denote by $|\cdot|$ the modulus character on W_E such that $|\sigma|^{-1}$ equals the cardinality of the residue field of E for any lift $\sigma \in W_E$ of the geometric Frobenius element.

Our main result is the following theorem, where a triple (F, V, μ) as above is fixed and b varies in $B(G, -\mu)$. Roughly speaking, $\pi \in \mathrm{Irr}_l(G(\mathbb{Q}_p))$ is said to be *accessible* if π appears in the cohomology of some simple Shimura variety related to the given Rapoport-Zink datum. There is an abundance of accessible representations in the sense that the set of such representations is Zariski dense in the Bernstein variety for $G(\mathbb{Q}_p)$. (Refer to §2, §3 and Lemma 7.4.)

Theorem 1.2. *(Theorem 7.5) Suppose that $\pi \in \mathrm{Irr}(G(\mathbb{Q}_p))$ is accessible. Then*

$$\sum_{b\in B(G,-\mu)} \mathrm{Mant}_{b,\mu}(\mathrm{Red}^{(b)}(\pi)) = [\pi] \left[r_\mu \circ \mathrm{LL}(\pi)|_{W_E} \ |\cdot|^{-\sum_\tau p_\tau q_\tau/2} \right] \tag{1.1}$$

in $\mathrm{Groth}(G(\mathbb{Q}_p) \times W_E)$.

We say that b is basic if the corresponding Newton polygon has a single slope. There is a unique basic element in $B(G, -\mu)$, and b is basic if and only if J_b is an inner form of G (that is to say, $M_b = G$). If we take $\pi = JL(\rho)$ as a supercuspidal representation in (1.1), the summand for every non-basic b vanishes since π is killed by the Jacquet module. Thus we obtain the following corollary, which answers Problem 1.1 in the supercuspidal case. It was previously proved by Fargues, who used a somewhat different method.

Corollary 1.3. *([Far04, Thm 8.1.4]) Let (F, V, μ, b) be an unramified Rapoport-Zink EL datum. Suppose that $b \in B(G, -\mu)$ is basic (Definition 4.3). For a representation $\rho \in \mathrm{Irr}(J_b(\mathbb{Q}_p))$ such that $\mathrm{JL}(\rho)$ is supercuspidal, the following holds in $\mathrm{Groth}(G(\mathbb{Q}_p) \times W_E)$.*

$$\mathrm{Mant}_{b,\mu}(\rho) = e(J_b)[\mathrm{JL}(\rho)] \left[r_\mu \circ \mathrm{LL}(\mathrm{JL}(\rho))|_{W_E} \; |\cdot|^{-\sum_\tau p_\tau q_\tau/2} \right] \tag{1.2}$$

Certainly Theorem 1.2 does not provide a solution to Problem 1.1 in general, but it is not too far away at least conceptually, in that the problem would be solved if we could make progress towards two conjectures that seem to us natural and perhaps easier to attack. On the one hand, we conjecture that Theorem 1.2 holds without the accessibility assumption. This can be reduced to proving a certain algebraic property of $\mathrm{Mant}_{b,\mu}$ in the parameter $\rho \in \mathrm{Groth}(J_b(\mathbb{Q}_p))$. For the precise formulation in terms of trace functions, see Conjecture 8.3 due to Taylor. On the other hand, Harris conjectured that $\mathrm{Mant}_{b,\mu}$ for non-basic b should be computable by a simple induction formula from the basic case. The improvement of Theorem 1.2 and a proof of Harris's conjecture will enable us to completely determine $\mathrm{Mant}_{b,\mu}$ for any unramified datum of EL-type (F, V, μ, b) by inductive steps. See §8.2 for details.

Let us sketch the proof of Theorem 1.2. The three main ingredients of proof are Proposition 1.4, Proposition 1.5 and Theorem 1.6 below, which are global in nature. Let Sh denote a projective system of Shimura varieties associated to a unitary similitude group $\mathbf{G}$ with trivial endoscopy. We may arrange that Sh is proper over its field of definition $\mathbf{E}$, which is finite over $\mathbb{Q}$. The basic idea is to make use of the interplay among Shimura varieties, Igusa varieties and Rapoport-Zink spaces. The first two are global objects whereas the last one is purely local in nature. Mantovan proved the following formula by relating the Newton strata of the special fiber of Shimura varieties to Igusa varieties and Rapoport-Zink spaces.

Proposition 1.4. *([Man05, Thm 22]) The following holds in $\mathrm{Groth}(\mathbf{G}(\mathbb{A}^\infty) \times W_{\mathbf{E}_w})$.*

$$H(\mathrm{Sh}, \overline{\mathbb{Q}}_l) = \sum_{\mathbf{b} \in B(\mathbf{G}_{\mathbb{Q}_p}, -\boldsymbol{\mu})} \mathrm{Mant}_{\mathbf{b},\boldsymbol{\mu}}(H_c(\mathrm{Ig}_{\mathbf{b}}, \overline{\mathbb{Q}}_l))$$

In the formula, w is a place of $\mathbf{E}$ dividing p, $\mathbf{b}$ is the parameter for each Newton stratum, $\mathrm{Ig}_{\mathbf{b}}$ is the Igusa varieties for that stratum (depending on an additional choice), and $\boldsymbol{\mu} : \mathbb{G}_m \to \mathbf{G}$ comes from the datum for Shimura varieties. (Note that $H_c(\mathrm{Ig}_{\mathbf{b}}, \overline{\mathbb{Q}}_l) \in \mathrm{Groth}(\mathbf{G}(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$.)

On the other hand, for an irreducible representation Π^∞ which contributes to the cohomology of Sh, the associated virtual Galois representation $R_l(\Pi^\infty)$ in $H(\mathrm{Sh}, \overline{\mathbb{Q}}_l)$ can be computed easily from results of Kottwitz and Harris-Taylor, under a certain local condition on Π^∞. See [Far04, Thm A.7.2] (Proposition 6.11 of this article) for detailed explanation. In fact, the following identity holds up to some harmless constant which we omit in the introduction.

Proposition 1.5. *(Kottwitz, Harris-Taylor)*

$$[R_l(\Pi^\infty)|_{W_{\mathbf{E}_w}}] = [(r_{\iota_p^{-1}\boldsymbol{\mu}} \circ \iota_l^{-1}\mathrm{LL}(\Pi_p))|_{W_{\mathbf{E}_w}} \otimes |\cdot|^{-\dim \mathrm{Sh}}].$$

One of our main contributions is the following formula. It is a generalization of theorem V.5.4 of [HT01], which was one of their main results. The idea of proof is to compare the analytic trace formula for $\mathbf{G}$ and the counting-point formula for Igusa varieties ([Shi09, Thm 13.1]). In fact we compare the stabilized trace formulas in favor for conceptual clarity even though this could be avoided under our running assumptions on $\mathbf{G}$, which ensure that $\mathbf{G}$ has trivial endoscopy.

Theorem 1.6. *(Theorem 6.7) Under suitable assumptions on the PEL datum for Shimura varieties (which are made precise in Theorem 6.7), the following holds in $\mathrm{Groth}(\mathbf{G}(\mathbb{A}^{\infty,p}) \times J_{\mathbf{b}}(\mathbb{Q}_p))$.*

$$[\mathrm{Red}^{(\mathbf{b})}(H(\mathrm{Sh}, \overline{\mathbb{Q}}_l))] = (\text{constant}) \cdot [H_c(\mathrm{Ig}_{\mathbf{b}}, \overline{\mathbb{Q}}_l)]$$

(The notation $\mathrm{Red}^{(\mathbf{b})}$ *is defined similarly as* $\mathrm{Red}^{(b)}$*. The constant in the formula essentially matches the dimension of* $[R_l(\Pi^\infty)|_{W_{\mathbf{E}_w}}]$*.)*

Let us return to the setting of Theorem 1.2 and suppose that a triple (F, V, μ) and $\pi \in \mathrm{Irr}(G(\mathbb{Q}_p))$ are fixed. We can choose $\mathbf{G}$ and Sh so that $\mathbf{G}_{\mathbb{Q}_p} \simeq G$, $\boldsymbol{\mu} = \mu$ and $\mathbf{E}_w \simeq E$. (To be precise, the first two are true up to a GL_1 factor.) Let us also suppose that we can find Π^∞ as above such that the p-component of Π^∞ is isomorphic to π. This is roughly the condition in Theorem 1.2 that π is accessible. (In fact, it is enough to require that Π_p and π be isomorphic up to an unramified twist.) It is easy to deduce Theorem 1.2 at this point. Namely, we plug Proposition 1.4 and Theorem 1.6 into Proposition 1.5 and take the $\Pi^{\infty,p}$-isotypic part of the identity.

So far we have restricted ourselves to the unramified Rapoport-Zink spaces of EL-type. Some of our results would be readily generalized to the case of unramified unitary PEL-type in a weaker form, but there seem to be obstacles for extending our method to the case of unramified symplectic PEL-type or any ramified case. See §8.4 for further remarks.

Let us briefly explain the organization of the article. Section 2 revolves around the Bernstein varieties for inner forms of GL_n and related topics and is a preparation for §3 and §8.1. Section 3 is concerned with the density of the set of p-components of automorphic representations of $\mathbf{G}(\mathbb{A})$ in the whole space of representations of $\mathbf{G}(\mathbb{Q}_p)$, where $\mathbf{G}$ is a reductive group over $\mathbb{Q}$ satisfying certain conditions. The contents of §3 will be used only in Lemma 7.4 and §8.1. Thus it would be reasonable to skip sections 2 and 3 in the first reading. Sections 4 and 5 review basic definitions and facts concerning unramified Rapoport-Zink spaces of EL-type and those Shimura varieties and Igusa varieties which are relevant. The main argument is contained in sections 6 and 7. Section 6 introduces a crucial map $\mathrm{Red}^{(b)}$ and proves that the cohomology of Shimura varieties is related to that of Igusa varieties via $\mathrm{Red}^{(b)}$. The proof is based on the comparison of the two trace formulas originating from different sources: one from harmonic analysis and the other from a Grothendieck-Lefschetz fixed point formula for varieties in characteristic p. In section 7 we formulate the global setup to study Rapoport-Zink spaces and representations of p-adic groups by using Shimura varieties and automorphic representations, and then prove the main theorem based on the results in the previous sections. In the final section we propose further research directions by advertising conjectures and examine the relationship among them.

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1.1. **Notation.** Throughout this paper, p and l will always be distinct rational primes.

Let G be a connected reductive group over a field F. Let G^* denote a quasi-split F-inner form of G. Write $Z(G)$ for the center of G and $Z_G(\gamma)$ for the centralizer of $\gamma \in G(F)$ in G. A semisimple $\gamma \in G(F)$ is called *F-elliptic* if $Z(Z_G(\gamma))^0/Z(G)^0$ is anisotropic over F. An *F-elliptic* torus T in G is one such that $T/Z(G)^0$ is anisotropic over F. Often a Levi subgroup of a parabolic subgroup of G will be called a Levi subgroup of G by abuse of terminology.

Let F be a finite extension of $\mathbb{Q}_p$. Then the Weil group W_F of F is defined ([Tat79]). To discuss the L-group ${}^LG$ of a connected reductive F-group G, we fix a $\mathrm{Gal}(\overline{F}/F)$-invariant splitting data $(\mathbb{B}, \mathbb{T}, \{X_\alpha\}_{\alpha\in\Delta})$ once and for all where Δ is the set of $\mathbb{B}$-positive roots for $\mathbb{T}$ in $\widehat{G}$. The L-group is defined as a semi-direct product ${}^LG := \widehat{G} \rtimes W_F$, where W_F acts on $\widehat{G}$ via $W_F \to \mathrm{Out}(G) \stackrel{\sim}{\to} \mathrm{Aut}(\widehat{G}, \mathbb{B}, \mathbb{T}, \{X_\alpha\}_{\alpha\in\Delta})$. We will view $\widehat{G}$ as a group over $\overline{\mathbb{Q}}_l$ rather than $\mathbb{C}$ when it is more convenient. (See [Bor79, §2] for details on L-groups.)

Keep assuming that F is finite over $\mathbb{Q}_p$. We denote by $\mathrm{Irr}(G(F))$ (resp. $\mathrm{Irr}_l(G(F))$) the set of all isomorphism classes of irreducible admissible representations of $G(F)$ on vector spaces over $\mathbb{C}$ (resp. $\overline{\mathbb{Q}}_l$). When π is an irreducible unitary representation of $G(F)$ (modulo split component in the center), π may also be viewed as an irreducible admissible representation by taking smooth vectors, so we may say $\pi \in \mathrm{Irr}(G(F))$. The subset $\mathrm{Irr}^2(G(F))$ of $\mathrm{Irr}(G(F))$ is the one consisting of (essentially) square-integrable representations. Let $C_c^\infty(G(F))$ denote the space of smooth and compactly supported $\mathbb{C}$-valued functions on $G(F)$. If $\phi \in C_c^\infty(G(F))$ and π is an admissible representation, $\pi(\phi) := \int_{G(F)} \phi(g)\pi dg$ is a finite rank operator (depending on the choice of the Haar measure on $G(F)$). The trace of $\pi(\phi)$ is written as $\mathrm{tr}\,\pi(\phi) = \mathrm{tr}\,(\phi|\pi)$. Let P be an F-rational parabolic subgroup of G with a Levi subgroup M. For each $\pi_M \in \mathrm{Irr}(M(F))$, we can define the normalized (resp. unnormalized) parabolic induction $\text{n-ind}_P^G(\pi_M)$ (resp. $\mathrm{Ind}_P^G(\pi_M)$) which is an admissible representation of $G(F)$. The induced representation $\text{n-ind}_P^G(\pi_M)$ will often be written as $\text{n-ind}_M^G(\pi_M)$ when working inside of $\mathrm{Groth}(G(F))$ or computing traces, since different choices of P give the same result. Define a character $\delta_P : M(F) \to \mathbb{R}_{>0}^\times$ by $\delta_P(m) = |\det(\mathrm{ad}(m))|_{\mathrm{Lie}\,(P)/\mathrm{Lie}\,(M)}|_F$. Let $e(G) \in \{\pm 1\}$ denote the Kottwitz sign defined in [Kot83].

The adèle ring over $\mathbb{Q}$ is written as $\mathbb{A}$. When S is a finite set of places of $\mathbb{Q}$, we denote by $\mathbb{A}^S$ the restricted product of $\mathbb{Q}_v$ for all $v \notin S$. Then $\mathrm{Irr}(G(\mathbb{A}^S))$ and $C_c^\infty(G(\mathbb{A}^S))$ have obvious meanings: the former is the set of isomorphism classes of irreducible admissible representations of $G(\mathbb{A}^S)$ and the latter is the space of $\mathbb{C}$-valued locally constant compactly supported functions on $G(\mathbb{A}^S)$. For $\phi \in C_c^\infty(G(\mathbb{A}^S))$ and an admissible representation π of $G(\mathbb{A}^S)$, the trace $\mathrm{tr}\,\pi(\phi) = \mathrm{tr}\,(\phi|\pi)$ is defined analogously as in the p-adic case.

Let F be any extension field of $\mathbb{Q}$. If G is an algebraic group over $\mathbb{Q}$, we write G_F for $G \times_{\mathbb{Q}} F$. Likewise, if B is a $\mathbb{Q}$-vector space or a $\mathbb{Q}$-algebra, write B_F for $B \otimes_{\mathbb{Q}} F$.

When $\gamma \in G(F)$ is semisimple and $\phi \in C_c^\infty(G(F))$, we write $O_\gamma^{G(F)}(\phi)$ and $SO_\gamma^{G(F)}(\phi)$ for orbital integrals and stable orbital integrals. (The definition can be found in [Kot88, §3], for instance.) When there is no danger of confusion about $G(F)$, we simply write $O_\gamma(\phi)$ and $SO_\gamma(\phi)$.

If F is a finite extension of a field K, then $R_{F/K}G$ denotes the Weil restriction of scalars (whose set of K-points is the same as $G(F)$).

We use the notation $\mathrm{Groth}(\cdot)$ for the Grothendieck group of admissible representations of topological groups. For instance, we will consider $\mathrm{Groth}(W_F)$, $\mathrm{Groth}(G(F))$, $\mathrm{Groth}(G(\mathbb{A}^S) \times \mathrm{Gal}(\overline{E}/E))$, etc, where F (resp. E) is a finite extension of $\mathbb{Q}_p$ (resp. $\mathbb{Q}$). For the precise definition of $\mathrm{Groth}(\cdot)$ in various situations, refer to [HT01, I.2].

2. Bernstein varieties for inner forms of GL_n

In this section we review the Bernstein varieties and a version of the trace Paley-Wiener theorem for inner forms of GL_n over a p-adic field, and then clarify the link between GL_n and its inner form. In §2.1 and §2.2, we use the following notations.

- F is a finite extension of $\mathbb{Q}_p$.
- $G := GL_n$ (as an F-group) and G' is an inner form of G over F. ($n \in \mathbb{Z}_{>0}$)
- $\mathscr{H}(G) = \mathscr{H}(G(F)) := C_c^\infty(G(F))$ and $\mathscr{H}(G') = \mathscr{H}(G'(F)) := C_c^\infty(G'(F))$.
- $\Psi^{\mathrm{ur}}(L)$ is the group of unramified characters $L(F) \to \mathbb{C}^\times$, where L is a Levi subgroup of either G or G'.

We fix Haar measures on $G(F)$ and $G'(F)$ so that they are compatible in the sense of [Kot88, p.631]. Throughout §2 we will use $\mathbb{C}$ as the coefficient field for convenience, but the contents of this section carry over without change if $\mathbb{C}$ is replaced by any algebraically closed field of characteristic 0 such as $\overline{\mathbb{Q}}_l$.

2.1. **The case of GL_n.** We will briefly recall the Bernstein variety $\mathfrak{z}_0^G = \mathfrak{z}_0^{G(F)}$ following [BD84] and its variant $\mathfrak{z}_2^G = \mathfrak{z}_2^{G(F)}$ as in [DKV84, A.4]. We will basically follow the treatment of Deligne, Kazhdan and Vigneras. Since we will use a version of the trace Paley-Wiener theorem (Proposition 2.6) in terms of $\mathfrak{z}_2^G$, our main interest lies in $\mathfrak{z}_2^G$. Nevertheless, $\mathfrak{z}_0^G$ arises naturally from the Bernstein center and was used in the proof of Proposition 2.6 by Deligne, Kazhdan and Vigneras ([DKV84, A.4.k]). Thus it is natural to review $\mathfrak{z}_2^G$ and $\mathfrak{z}_0^G$ together.

Define a set $\mathfrak{S}_2(G) = \mathfrak{S}_2(G(F))$ consisting of equivalence classes of pairs (L, D) where

- L is a Levi subgroup of G (we allow $L = G$),
- D is an $\Psi^{\mathrm{ur}}(L)$-orbit of square-integrable representations of $L(F)$ (namely, $D = \{\pi \otimes \chi | \chi \in \Psi^{\mathrm{ur}}(L)\}$ for some $\pi \in \mathrm{Irr}^2(L(F))$) and
- (L, D) and (L', D') are equivalent if there exists $g \in G(F)$ such that $L' = gLg^{-1}$ and $D' = gDg^{-1}$.

Let $W(L, D)$ be the stabilizer of D in $N_G(L)/L$. Then D and $V(L, D) := D/W(L, D)$ are naturally equipped with complex variety structures. Denote by $\mathcal{O}(L, D)$ the ring of regular functions on $V(L, D)$.

By replacing "square-integrable" by "supercuspidal" in the above definition, $\mathfrak{S}_0(G)$ is defined. For each $(L_0, D_0) \in \mathfrak{S}_0(G)$, the group $W(L_0, D_0)$ and the $\mathbb{C}$-variety $V(L_0, D_0) := D_0/W(L_0, D_0)$ are defined in the same manner as above. By definition,

$$\mathfrak{z}_2^G := \coprod_{(L,D)\in\mathfrak{S}_2(G)} V(L, D), \qquad \mathfrak{z}_0^G := \coprod_{(L_0,D_0)\in\mathfrak{S}_0(G)} V(L_0, D_0).$$

Let $(L, D) \in \mathfrak{S}_2(G)$. For each $\mathbb{C}$-point x of $V(L, D)$, choose any lift $y \in D(\mathbb{C})$ of x, which determines a square-integrable representation σ_y of $L(F)$. Then $\pi_x := \text{n-ind}_L^G(\pi_y)$ is well-defined as a member of $\mathrm{Groth}(G(F))$. Similarly, for $(L_0, D_0) \in \mathfrak{S}_0(G)$ and a $\mathbb{C}$-point x_0 of $V(L_0, D_0)$, we can associate $\pi_{x_0} \in \mathrm{Groth}(G(F))$, which is induced from a supercuspidal representation of $L_0(F)$. The following lemma is an immediate consequence of the definition.

Lemma 2.1. *The association $x \mapsto \pi_x$ (resp. $x_0 \mapsto \pi_{x_0}$) is a bijection from $\mathfrak{z}_2^G(\mathbb{C})$ (resp. $\mathfrak{z}_0^G(\mathbb{C})$) to the subset of* $\mathrm{Groth}(G(F))$ *consisting of parabolically induced representations from square-integrable (resp. supercuspidal) representations of Levi subgroups of G.*

Remark 2.2. In fact, there is a finite $\mathbb{C}$-morphism $\zeta : \mathfrak{z}_2^G \to \mathfrak{z}_0^G$ characterized as follows: for $x \in \mathfrak{z}_2^G(\mathbb{C})$, let (L_0, σ_0) be the supercuspidal support of π_x and D_0 be the $\Psi^{\mathrm{ur}}(L_0)$-orbit of σ_0. Then $\zeta(x)$ is the image of $\sigma_0 \in D_0$ in $V(L_0, D_0)$.

Example 2.3. Let $x \in \mathfrak{z}_2^G(\mathbb{C})$. There is a unique $(L, D) \in \mathfrak{S}_2(G)$ such that $x \in V(L, D)$. It is clear that π_x is an (irreducible) square-integrable representation if and only if $L = G$.

Suppose that $L = G$. Fix a uniformizer ϖ_F of $\mathcal{O}_F$. We can identify $\Psi^{\mathrm{ur}}(G)$ with $\mathbb{C}^\times$ by $\chi \mapsto \chi(\varpi_F)$. The unramified characters $\chi : F^\times \mapsto \mathbb{C}^\times$ such that $\pi \otimes (\chi \circ \det) \simeq \pi$ form a finite subgroup of $\Psi^{\mathrm{ur}}(G)$. Under the above identification we obtain a finite subgroup of $\mathbb{C}^\times$, which is denoted by S. Then $V(L, D)$ is isomorphic to the quotient variety $\mathbb{C}^\times/S$.

In view of the lemma, we will often write $\pi \in \mathfrak{z}_2^G(\mathbb{C})$ by abuse notation to mean that π is an induced representation as in Lemma 2.1.

Let $J_G = J_G(F)$ be the $\mathbb{C}$-subspace of $\mathscr{H}(G)$ generated by the functions of the form $g \mapsto \phi(g) - \phi(xgx^{-1})$ for every $\phi \in \mathscr{H}(G)$ and $x \in G(F)$. If $f \in J_G$, it is clear that $O_\gamma(f) = 0$ and $\mathrm{tr}\,\pi(f) = 0$ for every semisimple element $\gamma \in G(F)$ and every $\pi \in \mathrm{Irr}(G(F))$. Moreover,

Lemma 2.4. *(*[DKV84, A.4.h]*) The following are equivalent.*

(i) $f \in J_G$.
(ii) $O_\gamma(f) = 0$ *for every regular element* $\gamma \in G(F)$.
(iii) $O_\gamma(f) = 0$ *for every element* $\gamma \in G(F)$.
(iv) $\mathrm{tr}\,\pi(f) = 0$ *for every tempered* $\pi \in \mathrm{Irr}(G(F))$.
(v) $\mathrm{tr}\,\pi(f) = 0$ *for every* $\pi \in \mathrm{Irr}(G(F))$.

Remark 2.5. In fact, [DKV84] proves the above lemma for any inner form of GL_n. The generalization to arbitrary reductive G was proved by Kazhdan ([Kaz86, Thm 0]).

Now let $f \in \mathscr{H}(G)$. For each $(L, D) \in \mathfrak{S}_2(G)$, there is a complex-valued function $x \mapsto \mathrm{tr}\,\pi_x(f)$ defined on the $\mathbb{C}$-points of $V(L, D)$. Thus f defines a complex-valued function on $\mathfrak{z}_2^G(\mathbb{C})$. This function turns out to be a regular function and the following holds.

Proposition 2.6. *(*[DKV84, A.4.k]*) The above map from* $\mathscr{H}(G)$ *to the space of functions on* $\mathfrak{z}_2^G(\mathbb{C})$ *induces a* $\mathbb{C}$*-vector space isomorphism*

$$\mathscr{H}(G)/J_G \xrightarrow{\sim} \bigoplus_{(L,D)\in\mathfrak{S}_2(G)} \mathcal{O}(L, D). \tag{2.1}$$

Remark 2.7. Note that (2.1) depends on the choice of Haar measure on $G(F)$, as it is involved in the expression $\mathrm{tr}\,\pi_x(f)$ above.

The nice property of $\mathfrak{z}_2^G$ as in Proposition 2.6 does not hold in general if G is not a general linear group, since it relies on the Bernstein-Zelevinsky classification for GL_n ([BZ77], [Zel80]). Still, we will see analogous results in the next subsection when G is an inner form of GL_n.

2.2. **The case of an inner form of GL_n.** Basically the definitions in §2.1 carry over to an inner form G' of G. Namely, $\mathfrak{S}_2(G')$ is defined by replacing G with G' in the previous definition. For each $(L', D') \in \mathfrak{S}_2(G')$, we define $V(L', D')$ and $\mathcal{O}(L', D')$ and set

$$\mathfrak{z}_2^{G'} := \coprod_{(L',D')\in\mathfrak{S}_2(G')} V(L', D').$$

(Of course, $\mathfrak{z}_0^{G'}$ can be defined analogously.) The $\mathbb{C}$-points of $\mathfrak{z}_2^{G'}$ admit a similar description in terms of parabolic induction as in Lemma 2.1. As before, we sometimes write $\pi' \in \mathfrak{z}_2^{G'}(\mathbb{C})$ to mean that π' is a (full) parabolic induction from a square-integrable representation on a Levi subgroup of $G'(F)$. If $\pi' \in \mathfrak{z}_2^{G'}(\mathbb{C})$ is a reducible parabolic induction then π' is said to be a *reducible point*.

We are about to state Proposition 2.8, which extends Proposition 2.6. Let us explain the vertical maps in diagram (2.4). There is a canonical injection from the set of $G'(F)$-conjugacy classes of (F-rational) Levi subgroups L' of G' to the set of $G(F)$-conjugacy classes of Levi subgroups L of G. The map is induced by the transfer of parabolic subgroups from G' to G, which is an easy consequence of [Bor79, 3.1-3.2]. Suppose that L' maps to L (on the level of conjugacy classes). Then L' is an inner form of L and there is a natural isomorphism $\Psi^{\mathrm{ur}}(L') \xrightarrow{\sim} \Psi^{\mathrm{ur}}(L)$. The Jacquet-Langlands correspondence (e.g. [DKV84]), which is a bijection $\mathrm{Irr}^2(L'(F)) \xrightarrow{\sim} \mathrm{Irr}^2(L(F))$ compatible with twisting by $\Psi^{\mathrm{ur}}(L')$ and $\Psi^{\mathrm{ur}}(L)$ respectively, induces an injection $(L', D') \mapsto (L, D)$ from $\mathfrak{S}_2(G')$ to $\mathfrak{S}_2(G)$. A pair (L, D) belongs to the image precisely when L is the transfer of some Levi subgroup L' of G'. If (L', D') maps to (L, D), there are natural isomorphisms

$$V(L', D') \xrightarrow{\sim} V(L, D), \qquad \mathcal{O}(L', D') \xrightarrow{\sim} \mathcal{O}(L, D). \tag{2.2}$$

This induces the map $E_{G',G}$ of (2.4), whose image is exactly the direct sum over those $(L, D) \in \mathfrak{S}_2(G)$ which come from $\mathfrak{S}_2(G')$ (and zero outside those (L, D)).

The map $T_{G',G}$ in the left vertical arrow of (2.4) is the Langlands-Shelstad transfer of orbital integrals between inner forms. Namely, the association

$$T_{G',G} : f' \in \mathscr{H}(G')/J_{G'} \mapsto f \in \mathscr{H}(G)/J_G$$

is uniquely characterized by the identity of orbital integrals

$$O_{\gamma'}(f') = e(G') \cdot O_\gamma(f) \tag{2.3}$$

whenever $\gamma' \in G'(F)$ and $\gamma \in G(F)$ are regular elements with matching conjugacy classes. (The expression $O_\gamma(f)$ is well-defined as the orbital integral is independent of the lift of f to $\mathscr{H}(G)$. The same applies to $O_{\gamma'}(f')$. cf. Lemma 2.4.) Here compatible Haar measures are chosen on $Z_{G'}(\gamma')$ and $Z_G(\gamma)$, which are isomorphic. The sign $e(G') \in \{\pm 1\}$ in (2.3) may be viewed as the local transfer factor of our choice. It is easy to see that $T_{G',G}$ is injective from Lemma 2.4 and (2.3). (For every regular element γ', there exists an element γ with matching conjugacy class.)

Proposition 2.8. *(*[DKV84, Thm B.2.c]*) The following is a commutative diagram of $\mathbb{C}$-vector space morphisms, where the vertical arrows are explained above and the horizontal arrows are as in the last paragraph preceding Proposition 2.6.*

$$\begin{array}{ccc} \mathscr{H}(G')/J_{G'} & \xrightarrow{\sim} & \bigoplus_{(L',D') \in \mathfrak{S}_2(G')} \mathcal{O}(L',D') \\ \Big\downarrow{\scriptstyle e(G)\cdot T_{G',G}} & & \Big\downarrow{\scriptstyle E_{G',G}} \\ \mathscr{H}(G)/J_G & \xrightarrow{\sim} & \bigoplus_{(L,D) \in \mathfrak{S}_2(G)} \mathcal{O}(L,D) \end{array} \tag{2.4}$$

We want to define the notion of trace functions on $\mathrm{Groth}(G)$ and $\mathrm{Groth}(G')$. First, recall the following well-known fact.

Lemma 2.9. *(*[Zel80, Cor 7.5], [Tad90, p.56]*) The representations $\pi \in \mathfrak{z}_2^G$ form a $\mathbb{Z}$-basis of* $\mathrm{Groth}(G)$. *The analogue is true for G'.*

In particular, the obvious maps $\mathfrak{z}_2^G(\mathbb{C}) \to \mathrm{Groth}(G)$ and $\mathfrak{z}_2^{G'}(\mathbb{C}) \to \mathrm{Groth}(G')$ are injective.

Remark 2.10. The analogue of Lemma 2.9 for $\mathfrak{z}_0^{GL_2}$ fails already for $G = GL_2$. The Steinberg representation of $GL_2(F)$ is not in the $\mathbb{Z}$-span of $\mathfrak{z}_0^{GL_2}$ in $\mathrm{Groth}(GL_2(F))$.

Definition 2.11. A group homomorphism $\alpha : \mathrm{Groth}(G) \to \mathbb{C}$ is said to be a *trace function* on $\mathrm{Groth}(G)$ if the restriction of α to $\mathfrak{z}_2^G$ belongs to $\oplus_{(L,D)\in\mathfrak{S}_2(G)}\mathcal{O}(L,D)$. (In other words, α is a trace function if $\alpha|_{\mathfrak{z}_2^G}$ is a regular function and supported on finitely many components of $\mathfrak{S}_2(G)$.) A trace function on $\mathrm{Groth}(G')$ is defined in the exactly same way.

The terminology is justified by the following proposition.

Proposition 2.12. *Let α be as in Definition 2.11. There exists a function $f \in \mathscr{H}(G)$ such that $\alpha(\pi) = \mathrm{tr}\,\pi(f)$ for every $\pi \in \mathrm{Groth}(G)$ if and only if α is a trace function. The exact analogue is true for G'.*

Proof. Clear from Proposition 2.8. □

Remark 2.13. If G is an arbitrary reductive group over F, it seems more customary to define a trace function in terms of $\mathfrak{z}_0^G$. (The definition of $\mathfrak{z}_0^G$ easily extends to this generality.) Namely, $\alpha : \mathrm{Groth}(G) \to \mathbb{C}$ is defined to be a trace function if $\alpha|_{\mathfrak{z}_0^G}$ is a regular function and supported on finitely many components of $\mathfrak{S}_0(G)$. In this case, the analogue of Proposition 2.12 is established by

[BDK86]. In particular, the two definitions of a trace function coincide when G is an inner form of GL_n.

Badulescu([Bad07, Prop 3.3]) defined a group homomorphism $\mathrm{LJ} = \mathrm{LJ}_{G,G'} : \mathrm{Groth}(G) \to \mathrm{Groth}(G')$ uniquely characterized by the character identity

$$\operatorname{tr} \pi(f) = \operatorname{tr} \mathrm{LJ}(\pi)(f') \tag{2.5}$$

for any $f \in \mathscr{H}(G)$ and $f' \in \mathscr{H}(G')$ such that $f' \mapsto f$ via $e(G) \cdot T_{G',G}$ in the quotient rings. It is the inverse map of the usual Jacquet-Langlands bijection (e.g. [DKV84]), characterized by the same identity, on the set of isomorphism classes of square-integrable representations. In fact, Badulescu's characterization uses an identity involving the values of character functions on regular semisimple sets. This is easily shown to be equivalent to (2.5) by Weyl's integration formula ([DKV84, A.3.f]).

Using the injection $\mathfrak{S}_2(G') \hookrightarrow \mathfrak{S}_2(G)$, we can define

$$\widetilde{E}_{G',G} : \prod_{(L',D')\in\mathfrak{S}_2(G')} \mathcal{O}(L',D') \to \prod_{(L,D)\in\mathfrak{S}_2(G)} \mathcal{O}(L,D)$$

by extending the natural isomorphisms (2.2) by zero outside $\mathfrak{S}_2(G')$. Let $\widetilde{E}_{G,G'}$ be the projection map in the opposite direction which simply forgets the components outside $\mathfrak{S}_2(G')$. Clearly

$$\widetilde{E}_{G,G'} \circ \widetilde{E}_{G',G} = \mathrm{id}. \tag{2.6}$$

Moreover, there are induced maps

$$\widetilde{E}^*_{G',G} : \mathfrak{z}_2^G \to \mathfrak{z}_2^{G'}, \qquad \widetilde{E}^*_{G,G'} : \mathfrak{z}_2^{G'} \to \mathfrak{z}_2^G.$$

In light of Lemma 2.9, $\widetilde{E}^*_{G',G}$ and $\widetilde{E}^*_{G,G'}$ linearly extend to maps between Grothendieck groups. The latter maps are still to be written as $\widetilde{E}^*_{G',G}$ and $\widetilde{E}^*_{G,G'}$ by abuse of notation. It is easy to see from the construction that $\widetilde{E}^*_{G',G}$ is surjective (as a map from $\mathfrak{z}_2^G$ to $\mathfrak{z}_2^{G'}$, thus also on the level of Grothendieck groups).

Lemma 2.14. *The map* $\widetilde{E}^*_{G',G} : \mathrm{Groth}(G) \to \mathrm{Groth}(G')$ *is surjective and the same as* $\mathrm{LJ}_{G,G'}$.

Proof. The surjectivity is already explained above. To check that $\widetilde{E}^*_{G',G} = \mathrm{LJ}_{G,G'}$, we compare the two maps on each $\pi \in \mathfrak{z}_2^G(\mathbb{C})$. For any choice of $f' \in \mathscr{H}(G')$, set $f := e(G') \cdot T_{G',G}(f')$. By Proposition 2.8 and (2.5),

$$\operatorname{tr} \mathrm{LJ}_{G,G'}(\pi)(f') = \operatorname{tr} \pi(f) = \operatorname{tr} (\widetilde{E}^*_{G',G}(\pi))(f').$$

Therefore $\mathrm{LJ}_{G,G'}(\pi) = \widetilde{E}^*_{G',G}(\pi)$ in $\mathrm{Groth}(G')$. □

We use more intuitive notation $\mathrm{JL}_{G',G}$ for $\widetilde{E}^*_{G,G'} : \mathrm{Groth}(G') \to \mathrm{Groth}(G)$. By (2.6), $\mathrm{LJ}_{G,G'} \circ \mathrm{JL}_{G',G} = \mathrm{id}$. In particular, $\mathrm{JL}_{G',G}$ is the usual Jacquet-Langlands bijection on the set of square-integrable representations.

Lemma 2.15. *Let* $f \in \mathscr{H}(G)$. *Then there exists* $f' \in \mathscr{H}(G')$ *such that*

$$\operatorname{tr} \mathrm{JL}_{G',G}(\rho)(f) = \operatorname{tr} \rho(f').$$

Proof. Choose f' such that the image of f in $\oplus_{(L,D)\in\mathfrak{S}_2(G)}\mathcal{O}(L,D)$ is mapped to the image of f' in $\oplus_{(L',D')\in\mathfrak{S}_2(G')}\mathcal{O}(L',D')$ under $\widetilde{E}_{G,G'}$. The desired identity is easily deduced from Proposition 2.8. □

Remark 2.16. In fact, it can be checked from the definition that $\mathrm{JL}_{G',G}$ coincides with the map JL_r of [Bad07, §3.1].

2.3. **The case of a product of inner forms of general linear groups.** Let $r \in \mathbb{Z}_{>0}$. Let F_i be a finite extension of $\mathbb{Q}_p$ ($1 \le i \le r$). Consider

$$G = \prod_{i=1}^{r} R_{F_i/\mathbb{Q}_p} G_i \tag{2.7}$$

where each G_i is an inner form of GL_{m_i} over F_i ($m_i \in \mathbb{Z}_{>0}$). Define $\mathfrak{S}_2(G(\mathbb{Q}_p)) := \prod_{i=1}^r \mathfrak{S}_2(G_i(F_i))$ and $\mathfrak{z}_2^{G(\mathbb{Q}_p)} := \prod_{i=1}^r \mathfrak{z}_2^{G_i(F_i)}$. For $(L,D) = \{(L_i, D_i)\}_{i=1}^r \in \mathfrak{S}_2(G(\mathbb{Q}_p))$, set $V(L,D) := \prod_{i=1}^r V(L_i, D_i)$. A point $(x_i)_{i=1}^r$ of $V(L,D)$ is considered reducible if x_i is a reducible point (§2.2) of $V(L_i,D_i)$ for at least one $1 \le i \le r$. As before, $\mathcal{O}(L,D) := \otimes_{i=1}^r \mathcal{O}(L_i, D_i)$ denotes the ring of regular functions on $V(L,D)$. Note that $J_{G(\mathbb{Q}_p)}$ is defined exactly as in §2.1. We can naturally identify $\mathfrak{z}_2^G = \prod_{(L,D)} V(L,D)$ and

$$\mathscr{H}(G(\mathbb{Q}_p))/J_G(\mathbb{Q}_p) = \bigotimes_{i=1}^{r} \mathscr{H}(G_i(F_i))/J_{G_i}(F_i) \simeq \bigoplus_{(L,D)\in\mathfrak{S}_2(G(\mathbb{Q}_p))} \mathcal{O}(L,D) \tag{2.8}$$

where the last isomorphism comes from Proposition 2.6.

3. Density of local components of automorphic representations

In this section we prove under suitable conditions that the p-components of automorphic representations of a global group $\mathbf{G}$ form a Zariski dense subset in the Bernstein variety of $\mathbf{G}(\mathbb{Q}_p)$. This result must be well-known to experts, at least under the assumption (iii) below, and we claim no originality. Indeed, our argument is modeled after the ones in [DKV84, A.2.c-d] and [Kaz86, Appendix]. We will restrict ourselves to the case which will suffice for later applications.

Let $\mathbf{G}$ be a connected reductive group over $\mathbb{Q}$ such that

- (i) $\mathbf{G}/Z(\mathbf{G})$ is anisotropic over $\mathbb{Q}$,
- (ii) $\mathbf{G}_{\mathbb{R}}$ admits an elliptic torus and
- (iii) there exists an infinite set T of finite primes of $\mathbb{Q}$ such that for each $p \in T$, $\mathbf{G}_{\mathbb{Q}_p}$ is isomorphic to a product of general linear groups as in (2.7).
- (iv) If an automorphic representation Π of $\mathbf{G}(\mathbb{A})$ has a supercuspidal component Π_y for some $y \in T$, then Π_z is generic for every $z \in T$.

(See a comment on these conditions in Remark 3.2.) Conditions (i)-(iv) are satisfied when $\mathbf{G}$ is a unitary (similitude) group. See the proof of Lemma 7.4.

Fix a finite subset S of T. Set $\mathbb{A}_S := \otimes_{p\in S}\mathbb{Q}_p$ and $\mathscr{H}(\mathbf{G}(\mathbb{A}_S)) := \otimes_{p\in S}\mathscr{H}(\mathbf{G}(\mathbb{Q}_p))$. Let $J_{\mathbf{G}(\mathbb{A}_S)}$ be the subspace of $\mathscr{H}(\mathbf{G}(\mathbb{A}_S))$ generated by the functions $g \mapsto \phi(g) - \phi(xgx^{-1})$ for $\phi \in \mathscr{H}(\mathbf{G}(\mathbb{A}_S))$ and $x \in \mathbf{G}(\mathbb{A}_S)$. Note that (2.8) tells us that there is an isomorphism of $\mathbb{C}$-vector spaces

$$\mathscr{H}(\mathbf{G}(\mathbb{A}_S))/J_{\mathbf{G}(\mathbb{A}_S)} \simeq \bigoplus_{\{(L_p,D_p)\}_{p\in S}} \left(\bigotimes_{p\in S} \mathcal{O}(L_p, D_p) \right)$$

induced by the map sending $\phi \in \mathscr{H}(\mathbf{G}(\mathbb{A}_S))$ to the regular function $x \mapsto \operatorname{tr} \pi_x(\phi)$.

Fix an irreducible algebraic representation ξ of $\mathbf{G}$ over $\mathbb{C}$. Let $A_{\mathbf{G},\infty} := A_{\mathbf{G}}(\mathbb{R})^0$ where $A_{\mathbf{G}}$ is the maximal $\mathbb{Q}$-split torus in $Z(\mathbf{G})$. By restricting ξ to $A_{\mathbf{G},\infty}$, we obtain a character $\chi_\xi : A_{\mathbf{G},\infty} \to \mathbb{C}^\times$. Consider the automorphic spectrum

$$\mathscr{A}_\xi(\mathbf{G}) := L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}), \chi_\xi^{-1})$$

which consists of square-integrable (modulo $A_{\mathbf{G},\infty}$) functions f on $\mathbf{G}(\mathbb{A})$ such that $f(hg) = f(g)$ for all $h \in \mathbf{G}(\mathbb{Q})$ and $f(zg) = \chi_\xi(z)f(g)$ for all $z \in A_{\mathbf{G},\infty}$. The space $\mathscr{A}_\xi(\mathbf{G})$ is discrete by assumption

(i) and equipped with a left $\mathbf{G}(\mathbb{A})$-module structure via right translation action. Set $f_\xi := \phi_\xi$ where ϕ_ξ is defined in §6.1 using $\mathbf{G}_\mathbb{R}$ in place of G there. Define a subset

$$\mathscr{A}_\xi(\mathbf{G}, S) \subset \mathrm{Irr}(\mathbf{G}(\mathbb{A}_S))$$

to be the set of (isomorphism classes of) representations $\Pi_S = \otimes_{p\in S}\Pi_p$ such that $\Pi_p \in \mathfrak{z}_2^{\mathbf{G}(\mathbb{Q}_p)}$ for every $p \in S$ (i.e. Π_p is generic), arising as the S-component of some $\Pi \subset \mathscr{A}_\xi(\mathbf{G})$ which satisfies $\mathrm{tr}\,\Pi_\infty(f_\xi) \neq 0$.

Proposition 3.1. *Assume (i)-(iv) in the beginning of §3. Let $\{(L_p, D_p)\}_{p\in S}$ be any collection of pairs such that $(L_p, D_p) \in \mathfrak{S}_2^{\mathbf{G}(\mathbb{Q}_p)}$ for each $p \in S$. Then the set*

$$Y := \left(\prod_{p\in S} V(L_p, D_p)\right) \cap \mathscr{A}_\xi(\mathbf{G}, S) \tag{3.1}$$

is Zariski dense in $\prod_{p\in S} V(L_p, D_p)$.

Proof. Without loss of generality, we can assume that $L_p = \mathbf{G}_{\mathbb{Q}_p}$ and D_p is a supercuspidal orbit for some $p \in S$. Indeed, if there is no such p, the conclusion immediately follows from the case where S is enlarged to include an auxiliary prime $q \in T\backslash S$ and choose $(L_q, D_q) \in \mathfrak{S}_2^{\mathbf{G}(\mathbb{Q}_p)}$ such that $L_q = \mathbf{G}_{\mathbb{Q}_q}$ and D_q is a supercuspidal orbit.

Suppose that Y is not Zariski dense in $\prod_{p\in S} V(L_p, D_p)$. Then we can choose a nonzero regular function f_S on $\prod_{p\in S} V(L_p, D_p)$ which vanishes on Y. The last vanishing means that

$$\mathrm{tr}\,\pi_S(f_S) = 0, \quad \forall \pi_S \in Y \tag{3.2}$$

whereas f_S being a nonzero function means that f_S is not contained in $J_{\mathbf{G}(\mathbb{A}_S)}$ as an element of $\mathscr{H}(\mathbf{G}(\mathbb{A}_S))$.

Let f_S be any function as above satisfying (3.2). Let f^S be the characteristic function on a compact open subset U^S of $\mathbf{G}(\mathbb{A}^{S\cup\{\infty\}})$. Let f_ξ be as earlier in §3. The trace formula for compact quotients applies to $\mathbf{G}$ by initial assumption (i). We claim that the spectral side vanishes for the test function $f^S f_S f_\xi$. If so,

$$\sum_{\gamma\in\mathbf{G}(\mathbb{Q})/\sim} O_\gamma(f^S f_S f_\xi) = 0 \tag{3.3}$$

where the sum runs over the set of representatives for $\mathbf{G}(\mathbb{Q})$-conjugacy classes in $\mathbf{G}(\mathbb{Q})$. (It is automatic that every γ is semisimple and elliptic over $\mathbb{Q}$ by assumption (i).) Let us prove the claim. If the claim is false, there exists an automorphic representation π of $\mathbf{G}(\mathbb{A})$ such that $\mathrm{tr}\,\pi(f^S f_S f_\xi) \neq 0$. In view of condition (iv) and the assumption in the beginning of the proof, π_S is generic. Therefore $\pi_S \in Y$, but this contradicts (3.2). The claim is proved.

In view of the property that $f_S \notin J_{\mathbf{G}(\mathbb{A}_S)}$, there exists a regular semisimple element $\delta_S \in \mathbf{G}(\mathbb{A}_S)$ such that $O_{\delta_S}(f_S) \neq 0$ (Lemma 2.4). Since the orbital integral is locally constant on the set of regular elements, there exists an open neighborhood U_S of δ_S consisting of regular elements such that $O_{\delta'_S}(f_S) \neq 0$ for every $\delta'_S \in U_S$. In view of Lemma 6.1, we can easily find a regular elliptic element $\delta_\infty \in \mathbf{G}(\mathbb{R})$ such that $O_{\delta_\infty}(f_\xi) \neq 0$. Choose an open neighborhood U_∞ of δ_∞ consisting of regular elements such that $O_{\delta'_\infty}(f_\infty) \neq 0$ for every $\delta'_\infty \in U_\infty$. By weak approximation (cf. [Kaz86, Lem 5.(a), Appendix]), there exists an element $\gamma_0 \in \mathbf{G}(\mathbb{Q})$ such that $\gamma_0 \in U_S U_\infty$. (So γ_0 is regular, and also elliptic in $\mathbf{G}(\mathbb{R})$ by Lemma 6.1.) In particular,

$$O_{\gamma_0}(f_S) \neq 0 \quad \text{and} \quad O_{\gamma_0}(f_\xi) \neq 0. \tag{3.4}$$

Fix one such γ_0 and choose an open compact subgroup U^S of $\mathbf{G}(\mathbb{A}^{S\cup\{\infty\}})$ containing γ_0. In particular,

$$O_{\gamma_0}(f^S) \neq 0. \tag{3.5}$$

By shrinking U^S if necessary, we can arrange that any $\gamma \in \mathbf{G}(\mathbb{Q})$ such that $O_\gamma(f^S f_S f_\xi) \neq 0$ must be $\mathbf{G}(\mathbb{A}^{S\cup\{\infty\}})$-conjugate to γ_0. (There are finitely many $G(\mathbb{Q})$-conjugacy orbits intersecting nontrivially with $U_S U^S U_\infty$, and they are regular by the choice of U^S. They give rise to finitely many $G(\mathbb{A}^{S\cup\infty})$-conjugacy orbits, and one of them is the conjugacy class of γ_0. Therefore if U^S is small enough then U^S intersects nontrivially with the conjugacy class of γ_0 only. For this, consider the union of the closures of the other $G(\mathbb{A}^{S\cup\infty})$-conjugacy orbits, and take its complement in U^S. The closures do not meet the conjugacy class of γ_0 since all conjugacy classes under consideration are regular.)

Such a γ is conjugate to γ_0 in $\mathbf{G}(\mathbb{C})$ and $\mathbf{G}(\overline{\mathbb{Q}}_p)$ for any finite prime p. By assumption (iii), γ is conjugate to γ_0 in $\mathbf{G}(\mathbb{A}_S)$. On the other hand, $O_\gamma(f_\xi) = O_{\gamma_0}(f_\xi)$ by the formula (6.2). Indeed, since γ and γ_0 are elliptic regular, $|d(I_\gamma)| = |d(I_{\gamma_0})| = 1$ and $q(I_\gamma) = q(I_{\gamma_0}) = 0$ in the notation of §6.1. Therefore (3.3) implies that

$$O_{\gamma_0}(f^S f_S f_\xi) = 0.$$

The last equality clearly contradicts (3.4) and (3.5). Hence the proof is complete. □

Remark 3.2. It is worth examining the assumptions (i)-(iv) in the beginning of this section. The condition (i) is not essential. When (i) is dropped, one can appeal to the simple trace formula by imposing certain test functions outside S. The last condition (iv) is not essential but imposed for convenience. It allows us not to worry about non-generic representations which have the same supercuspidal supports as square-integrable representations. Although (ii) is built into the definition of the set $\mathscr{A}_\xi(\mathbf{G}, S)$, Proposition 3.1 can be proved by the same argument if the condition for Π_∞ is removed in the definition of $\mathscr{A}_\xi(\mathbf{G}, S)$. Perhaps (iii) is the most serious condition, which saves us from dealing with the issue of local endoscopy at $p \in S$. To formulate an analogue of Proposition 3.1 without (iii), it seems natural to work with stable orbital integrals and L-packets.

Remark 3.3. A result stronger than Proposition 3.1 was obtained by Clozel ([Clo86, §4]) in a very general setting (without conditions (i)-(iv)), provided that Π_p is square-integrable for every $p \in S$.

Remark 3.4. The author also proved a stronger result ([Shia]) which implies Proposition 3.1 by a different method after finishing this paper. We decided to keep section 3 (including the two remarks above) in its original form, believing that the argument has its own interesting point.

4. Rapoport-Zink spaces of EL-type

Fix a prime l different from p until the end of the paper. In this section, we briefly recall the definition of Rapoport-Zink spaces based on [RZ96] and [Far04]. (Also see [Man04], [Man05].) Rapoport-Zink spaces are important since they are closely related to the local geometry of Shimura varieties, and their l-adic cohomology is expected to realize the local Langlands correspondence in a suitable sense.

4.1. **Rapoport-Zink datum of EL-type.** We will use the following notations for a connected reductive group G_0 over $\mathbb{Q}_p$.

- $L := \mathrm{Frac} W(\overline{\mathbb{F}}_p)$.
- σ is the automorphism of L inducing $x \mapsto x^p$ on the residue field.
- $B(G_0)$ is the set of σ-conjugacy classes in $G_0(L)$.
- $B(G_0, \mu_0)$ is the finite subset of $B(G_0)$ defined in [Kot97, 6.2], where $\mu_0 : \mathbb{G}_m \to G_0$ is a cocharacter over $\overline{\mathbb{Q}}_p$. (In the case of interest, a combinatorial description of $B(G_0, \mu_0)$ will be given below.)

Definition 4.1. ([Far04, 2.2.1], cf. [RZ96, 3.82]) An *unramified Rapoport-Zink datum of EL-type* is a quadruple (F, V, μ, b) where

(i) F is a finite unramified extension of $\mathbb{Q}_p$.

(ii) V is a finite dimensional F-vector space. Let $G := R_{F/\mathbb{Q}_p} GL_F(V)$.
(iii) $\mu : \mathbb{G}_m \to G$ is a homomorphism over $\overline{\mathbb{Q}}_p$ (up to $G(\overline{\mathbb{Q}}_p)$-conjugacy) which induces a weight decomposition $V \otimes_{\mathbb{Q}} \mathbb{Q}_p^{\mathrm{ur}} = V_0 \oplus V_1$ where $\mu(z)$ acts on V_i by z^i for $i = 0, 1$.
(iv) b is an element of $B(G, -\mu)$. (We adopt the sign convention of [Shi09], which seems to us easy to work with as we use the covariant Dieudonné modules, which are the duals of the usual contravariant Dieudonné modules. But keep in mind that many authors use a different convention.)

By a *partial unramified Rapoport-Zink datum* of EL-type, we mean (F, V, μ) satisfying (i)-(iii) above.

Remark 4.2. Refer to [RZ96, 1.37] for the definition of a Rapoport-Zink datum of EL-type and PEL-type in general. Note that the "Drinfeld case" ([RZ96, 1.44, 3.54]), a well-known case of EL-type, is left out from our discussion.

Let $n := \dim_F V$. Whenever it is convenient we will identify $G = R_{F/\mathbb{Q}_p} GL_n$ by choosing an F-basis of V. The identification is conjugated by an element of $G(\mathbb{Q}_p)$ if a different basis is chosen, but this conjugation is harmless for many purposes. For instance, an (admissible) representation of $G(\mathbb{Q}_p)$ yields a representation of $GL_n(F)$ whose isomorphism class is independent of the basis.

Giving μ is equivalent to giving a pair of nonnegative integers (p_τ, q_τ) for each $\tau \in \mathrm{Hom}_{\mathbb{Q}_p}(F, \overline{\mathbb{Q}}_p)$ such that $p_\tau + q_\tau = n$. Given such data, the corresponding μ is represented by the $\overline{\mathbb{Q}}_p$-homomorphism $\mathbb{G}_m \to \prod_{\tau \in \mathrm{Hom}_{\mathbb{Q}_p}(F, \overline{\mathbb{Q}}_p)} GL_n$ such that

$$z \mapsto \prod_\tau \mathrm{diag}(\underbrace{z, \dots, z}_{p_\tau}, \underbrace{1, \dots, 1}_{q_\tau}). \tag{4.1}$$

For later use in section 8, we define

$$\dim(\mu) := \sum_\tau p_\tau. \tag{4.2}$$

The reflex field E is the subfield of $\overline{\mathbb{Q}}_p$ fixed under the stabilizer in $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of the pairs $\{(p_\tau, q_\tau)\}_{\tau \in \mathrm{Hom}_{\mathbb{Q}_p}(F, \overline{\mathbb{Q}}_p)}$. (The element b is irrelevant in the definition of E.) Since F is unramified over $\mathbb{Q}_p$, so is E.

Fix $\widetilde{b} \in G(L)$ in the σ-conjugacy class defined by $b \in B(G, -\mu)$ so that $\widetilde{b}$ is decent in the sense of [RZ96, Def 1.8]. Define a connected reductive group J_b over $\mathbb{Q}_p$ by the rule

$$J_b(R) = \{g \in G(L \otimes_{\mathbb{Q}_p} R) \mid g = \widetilde{b}\sigma(g)\widetilde{b}^{-1}\} \tag{4.3}$$

for any $\mathbb{Q}_p$-algebra R. (A different choice of $\widetilde{b}$ does not change the $\mathbb{Q}_p$-isomorphism class of J_b.)

Given (F, V, μ), it is easy to write down a complete list of all possible $b \in B(G, -\mu)$. We copy here Example 4.3 of [Shi09] for the ease of reference. Let $\bar{\nu}_G : B(G) \to N(G)$ denote the Newton map ([RR96, 1.9]). To describe the set $N(G)$, consider a maximal $\mathbb{Q}_p$-split torus $A = (GL_1)^n$ inside G. The Weyl group for (G, A) is denoted by Ω, which is isomorphic to the symmetric group S_n. By definition $N(G) = X_*(A)^\Omega \otimes_{\mathbb{Z}} \mathbb{Q}$, thus there is a natural bijection $N(G) \simeq (\mathbb{Q}^n)^{S_n}$ coming from the natural bijection $X_*(GL_1) \simeq \mathbb{Z}$ by sending the cocharacter $z \mapsto z$ to 1. In turn, $N(G)$ is identified with the set of the following data:

$$(r, \{\lambda_i\}_{1 \le i \le r}, \{m_i\}_{1 \le i \le r}) \quad \text{such that} \quad \lambda_i \in \mathbb{Q},\ r, m_i \in \mathbb{Z}_{>0},\ \lambda_1 < \cdots < \lambda_r,\ \sum_{i=1}^r m_i = n. \tag{4.4}$$

It is convenient to describe the finite subset $B(G, -\mu)$ of $B(G)$ in terms of its image under $\bar{\nu}_G$, since $\bar{\nu}_G$ is injective in our case.

Given μ as in (4.1), set $n' := n[F:\mathbb{Q}_p]$ and

$$(y_1,\dots,y_{n'}) := (\underbrace{1,\dots,1}_{\sum_\tau p_\tau},\underbrace{0,\dots,0}_{\sum_\tau q_\tau}).$$

For an element $b \in B(G)$ such that $\bar{\nu}_G(b) = (r,\{\lambda_i\},\{m_i\})$, set

$$(x_1,\dots,x_{n'}) := (\underbrace{-\lambda_1,\dots,-\lambda_1}_{[F:\mathbb{Q}_p]m_1},\dots,\underbrace{-\lambda_r,\dots,-\lambda_r}_{[F:\mathbb{Q}_p]m_r}).$$

Then $b \in B(G,-\mu)$ if and only if

$$\sum_{i=1}^{j} x_i \le \sum_{i=1}^{j} y_i \quad \text{for} \quad 1 \le j < n' \qquad \text{and} \qquad \sum_{i=1}^{n'} x_i = \sum_{i=1}^{n'} y_i. \tag{4.5}$$

(Unfortunately there is a sign mistake in the line right above formula (9) on page 522 of [Shi09], where $b \in B(G,\mu)$ should read $b \in B(G,-\mu)$.) In particular the condition (4.5) implies that $1 \ge -\lambda_1 > \cdots > -\lambda_r \ge 0$.

The $\mathbb{Q}_p$-group J_b has the following concrete description

$$J_b \simeq R_{F/\mathbb{Q}_p} \prod_{i=1}^{r} GL_{m_i/h_i}(D_{-\lambda_i}) \tag{4.6}$$

where $D_{-\lambda_i}$ denotes the division algebra with center F and Hasse invariant $-\lambda_i \in \mathbb{Q}/\mathbb{Z}$, and $h_i := [D_{-\lambda_i} : F]^{1/2}$.

Let M_b be the $\mathbb{Q}_p$-group defined in §3.2 of [Shib]. Rather than recalling the definition of M_b, we give an explicit description, which the reader may take as an alternative definition. We may identify

$$M_b = R_{F/\mathbb{Q}_p} \prod_{i=1}^{r} GL_{m_i}, \tag{4.7}$$

where the latter is viewed as the subgroup of $G = R_{F/\mathbb{Q}_p} GL_n$ via the obvious block diagonal embedding. Define P_b to be the parabolic subgroup which consists of block upper triangular matrices and has M_b as a Levi subgroup. (Actually P_b is the same as the parabolic subgroup $P(\nu_G(\widetilde{b}))$ for the triple $(G,\nu_G(\widetilde{b}),M)$ as in [Shib, §3.3] in the notation thereof. The fact that P_b is upper triangular corresponds to the ordering $-\lambda_1 > \cdots > -\lambda_r$.)

For later use, we define the notion of basic elements.

Definition 4.3. An element $b \in B(G,-\mu)$ is called *basic* if J_b is an inner form of G (over $\mathbb{Q}_p$). Equivalently, b is basic if $M_b = G$. In the above description, b is basic exactly when $r = 1$, as can be seen from (4.7). (For other equivalent conditions, see [RR96, 1.12], cf. [Kot85, §5].)

Remark 4.4. For a given μ, there is a unique basic element in $B(G,-\mu)$. (cf. [Kot97, 6.4])

Remark 4.5. The values $-\lambda_i$ for those λ_i in (4.4) are called the *slopes* associated to b. Slopes are always in the closed interval $[0,1]$. Thus a basic element b has a unique slope.

4.2. **Rapoport-Zink spaces without level structure.** It will be convenient to define certain categories of Barsotti-Tate groups with "G-structure". Refer to [Mes72] for generalities about Barsotti-Tate groups.

Definition 4.6. Let S be a scheme in which p is Zariski-locally nilpotent. Denote by BT_S^G the category where

- An object is a pair (Σ, i_Σ) consisting of a Barsotti-Tate group Σ over S and a $\mathbb{Z}_p$-morphism $i_\Sigma : \mathcal{O}_F \hookrightarrow \mathrm{End}_S(\Sigma)$ and

- A morphism from (Σ_1, i_{Σ_1}) to (Σ_2, i_{Σ_2}) is a morphism $f \in \mathrm{Hom}_S(\Sigma_1, \Sigma_2)$ such that $i_{\Sigma_2} \circ f = f \circ i_{\Sigma_1}$.

Define $\mathrm{BT}^{0,G}_S$ to be the category whose objects are the same as BT^G_S but morphisms are those $f \in \mathrm{Hom}_S(\Sigma_1, \Sigma_2) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $i_{\Sigma_2}(x) \circ f = f \circ i_{\Sigma_1}(x)$ for all $x \in \mathcal{O}_F$.

Fix a Barsotti-Tate-group $(\Sigma, i_\Sigma) \in \mathrm{BT}^G_{\mathrm{Spec}\,\overline{\mathbb{F}}_p}$ such that its associated isocrystal is of type b. To explain what this means, let $(\mathbb{D}(\Sigma), \Phi)$ denote the contravariant Dieudonné module associated to Σ, where Φ stands for the σ-semilinear endomorphism of $\mathbb{D}(\Sigma)$. Then $(\mathbb{D}(\Sigma)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}, \Phi^* \otimes 1)$ is an isocrystal equipped with an F-action induced by i_Σ, where Φ^* is the endomorphism of $\mathbb{D}(\Sigma)^\vee$ induced by Φ. (We adopted the convention of [Shi09, p.525] to take dual of the contravariant Dieudonné module.) The latter isocrystal is said to be of type b if it is isomorphic to $(V \otimes_{\mathbb{Q}_p} L, b(1 \otimes \sigma))$ as isocrystals with F-action. (See [RR96] or [Kot97], for instance, about standard notations and generalities on isocrystals.) The existence of (Σ, i_Σ) as above is guaranteed by [KR03, Thm 5.1]. Moreover, the isomorphism class of (Σ, i_Σ) is unique in $\mathrm{BT}^{0,G}_{\mathrm{Spec}\,\overline{\mathbb{F}}_p}$.

We consider the following moduli functor

$$\mathcal{M}_{b,\mu} : \begin{pmatrix} W(\overline{\mathbb{F}}_p)\text{-schemes} \\ \text{where } p \text{ is locally nilpotent} \end{pmatrix} \longrightarrow (\mathrm{Sets})$$
$$S \mapsto \{(H, i, \beta)\}/\sim$$

where

- (H, i) is an object of BT^G_S.
- $\beta : \Sigma \times_{\overline{\mathbb{F}}_p} \overline{S} \to H \times_S \overline{S}$ is a quasi-isogeny which induces an isomorphism $(\Sigma \times_{\overline{\mathbb{F}}_p} \overline{S}, i_\Sigma) \overset{\sim}{\to} (H \times_S \overline{S}, i)$ in $\mathrm{BT}^{0,G}_{\overline{S}}$, where $\overline{S}$ is the closed subscheme of S defined by the ideal sheaf $p\mathcal{O}_S$.
- (Determinant condition) The equality of polynomials $\det_{\mathcal{O}_S}(a \,|\, \mathrm{Lie}\, H) = \det_{\mathbb{Q}_p^{\mathrm{ur}}}(a \,|\, V_1)$ holds for all $a \in \mathcal{O}_B$, in the sense of [Kot92b, §5] or [RZ96, 3.23.(a)].
- (H_1, i_1, β_1) is equivalent to (H_2, i_2, β_2) if there exists an isomorphism $f : H_1 \to H_2$ such that f sends i_1 to i_2 and $f \times_S \overline{S}$ sends β_1 to β_2.

The above functor is representable by a formal scheme of locally finite type over $\mathrm{Spf}\, W(\overline{\mathbb{F}}_p)$ ([RZ96, Thm 3.25]). The representing formal scheme is called a *Rapoport-Zink space of EL-type* (without level structure) and denoted $\mathcal{M}_{b,\mu}$. The functors $\mathcal{M}_{b,\mu}$ for any two choices of (Σ_1, i_{Σ_1}) and (Σ_2, i_{Σ_2}) of type b (with b and μ fixed) are naturally isomorphic in that an isomorphism $(\Sigma_1, i_{\Sigma_1}) \overset{\sim}{\to} (\Sigma_2, i_{\Sigma_2})$ in $\mathrm{BT}^{0,G}_{\mathrm{Spec}\,\overline{\mathbb{F}}_p}$ induces an isomorphism of the corresponding functors.

4.3. **l-adic cohomology of Rapoport-Zink spaces.** There is a standard construction to obtain a rigid analytic space $\mathcal{M}^{\mathrm{rig}}_{b,\mu}$ over $L = \mathrm{Frac} W(\overline{\mathbb{F}}_p)$ from the formal scheme $\mathcal{M}_{b,\mu}$. One also constructs a tower of coverings $\mathcal{M}^{\mathrm{rig}}_{b,\mu,U}$ over $\mathcal{M}^{\mathrm{rig}}_{b,\mu}$ for open compact subgroups U of $G(\mathbb{Q}_p)$ ([RZ96, Ch 5] cf. [Far04, 2.3.9]). Recall that l is a fixed prime such that $l \neq p$. We consider the étale cohomology of Rapoport-Zink spaces in the sense of Berkovich ([Ber93]), for which we use abbreviated notation:

$$H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu,U}) := H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu,U} \times_{\widehat{E^{\mathrm{ur}}}} \widehat{\overline{E^{\mathrm{ur}}}}, \overline{\mathbb{Q}}_l).$$

This $\overline{\mathbb{Q}}_l$-vector space has the structure of a smooth representation of $J_b(\mathbb{Q}_p) \times W_E$. The action of $\delta \in J_b(\mathbb{Q}_p)$ on the moduli data of $\mathcal{M}_{b,\mu}$ is described as

$$(H, \lambda, i, \beta) \mapsto (H, \lambda, i, \beta \circ \delta).$$

This action extends to every covering $\mathcal{M}^{\mathrm{rig}}_{b,\mu,U}$, thus induces an action on the cohomology. The action of $I_E \simeq \mathrm{Gal}(\widehat{\overline{E^{\mathrm{ur}}}}/\widehat{E^{\mathrm{ur}}})$ is obviously defined on $H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu,U})$ and extended to an action of W_E via Weil descent data. Moreover $G(\mathbb{Q}_p)$ acts on the tower of $\mathcal{M}^{\mathrm{rig}}_{b,\mu,U}$ via Hecke correspondences. Details about these actions can be found in [RR96, Ch 5], [Far04, Ch 4] and [Man04].

Definition 4.7. The map $\mathrm{Mant}_{b,\mu} : \mathrm{Groth}(J_b(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_E)$ is defined by

$$\mathrm{Mant}_{b,\mu}(\rho) := \sum_{i,j\geq 0} (-1)^{i+j} \varinjlim_{U\subset G(\mathbb{Q}_p)} \mathrm{Ext}^i_{J_b(\mathbb{Q}_p)}(H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu,U}), \rho))(-\dim \mathcal{M}^{\mathrm{rig}}_{b,\mu})$$

where U runs over the set of open compact subgroups of $G(\mathbb{Q}_p)$. The notation $(-\dim \mathcal{M}^{\mathrm{rig}}_{b,\mu})$ indicates the corresponding Tate twist. The Ext-groups are taken in the category of smooth representations of $J_b(\mathbb{Q}_p)$ (as left $J_b(\mathbb{Q}_p)$-modules).

Remark 4.8. The map $\mathrm{Mant}_{b,\mu}$ has been considered by several authors. See [Har01], [Far04] and [Man05] for example.

Note that $\mathrm{Mant}_{b,\mu}$ is well-defined. First of all, the Ext-groups in Definition 4.7 vanish beyond a certain degree and yield finite length representations for each U. (See [Far04, §4.4].) Next, whenever we have an exact sequence of true representations $0 \to \rho_1 \to \rho_2 \to \rho_3 \to 0$, there is a resulting Ext long exact sequence, and this shows that $\mathrm{Mant}_{b,\mu}(\rho_2) = \mathrm{Mant}_{b,\mu}(\rho_1) + \mathrm{Mant}_{b,\mu}(\rho_3)$ in $\mathrm{Groth}(G(\mathbb{Q}_p) \times W_E)$.

We briefly remark about left and right actions. The group $J_b(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \times W_E$ acts on the left of $H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu,U})$. In Definition 4.7, Ext is taken with respect to left $J_b(\mathbb{Q}_p)$-modules. We view $\mathrm{Mant}_{b,\mu}(\rho)$ as a left module of $G(\mathbb{Q}_p) \times W_E$ by composing with the group inverse, as in the usual construction of contragredient representations.

Our first task is to understand the effect of an unramified character twist on $\mathrm{Mant}_{b,\mu}$ functor. The determinant character $\det : G(\mathbb{Q}_p) = GL_F(V) \to F^\times$ restricted to $M_b(\mathbb{Q}_p)$ may be transferred to $J_b(\mathbb{Q}_p)$ by using the fact that the maximal abelian quotients of $M_b(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$ are canonically isomorphic. The transferred character coincides with the reduced norm map $N_{J_b} : J_b(\mathbb{Q}_p) \to F^\times$.

Lemma 4.9. *Let (F, V, μ, b) be an unramified Rapoport-Zink EL datum. Suppose that $\omega : F^\times \to \overline{\mathbb{Q}}_l^\times$ is an unramified character. The following holds in* $\mathrm{Groth}(G(\mathbb{Q}_p) \times W_E)$.

$$\mathrm{Mant}_{b,\mu}(\rho \otimes (\omega \circ N_{J_b})) = \mathrm{Mant}_{b,\mu}(\rho) \otimes (\omega \circ \det) \otimes (\omega \circ \mathrm{Art}_F^{-1})^{-\sum_\tau p_\tau}$$

Proof. Define a character χ of $J_b(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \times W_E$ by

$$\chi := (\omega \circ N_{J_b}) \otimes (\omega \circ \det) \otimes (\omega \circ \mathrm{Art}_F^{-1})^{-\sum_\tau p_\tau}.$$

Then in $\mathrm{Groth}(J_b(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \times W_E)$,

$$\begin{aligned} & \varinjlim_U \mathrm{Ext}^i_{J_b(\mathbb{Q}_p)}(H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu,U}), \rho \otimes (\omega \circ N_{J_b})) \\ \simeq\ & \varinjlim_U \mathrm{Ext}^i_{J_b(\mathbb{Q}_p)}(H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu,U}) \otimes \chi, \rho) \otimes (\omega \circ \det) \otimes (\omega \circ \mathrm{Art}_F^{-1})^{-\sum_\tau p_\tau}. \end{aligned}$$

So it suffices to prove that there is an isomorphism of $\overline{\mathbb{Q}}_l$-vector spaces

$$H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu,U}) \simeq H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu,U}) \otimes \chi. \tag{4.8}$$

compatible with the action of $J_b(\mathbb{Q}_p) \times (U\backslash G(\mathbb{Q}_p)/U) \times W_E$.

From here on, we will freely adopt the notation and results from [Far04, pp.73-74]. There is a $J_b(\mathbb{Q}_p)$-equivariant map $\mathcal{M}^{\mathrm{rig}}_{b,\mu,U} \to \Delta$ where $\Delta = \mathrm{Hom}_{\mathbb{Z}}(X^*(G), \mathbb{Z})$. Moreover, there is a natural way to define an action of $J_b(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \times W_E$ on Δ such that the map $\mathcal{M}^{\mathrm{rig}}_{b,\mu,U} \to \Delta$ is equivariant with respect to the action of $J_b(\mathbb{Q}_p) \times (U\backslash G(\mathbb{Q}_p)/U) \times W_E$. (Note that U acts trivially on Δ.) The subgroup $(J_b(\mathbb{Q}_p)\times G(\mathbb{Q}_p)\times W_E)^1$ acts trivially on Δ. The last fact can be interpreted as χ being trivial on $(J_b(\mathbb{Q}_p)\times G(\mathbb{Q}_p)\times W_E)^1$ by examining how the action of $J_b(\mathbb{Q}_p)\times G(\mathbb{Q}_p)\times W_E$ is defined on Δ. We know

$$\varinjlim_{U'} H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu,U'}) \simeq \text{c-ind}^{J_b(\mathbb{Q}_p)\times G(\mathbb{Q}_p)\times W_E}_{(J_b(\mathbb{Q}_p)\times G(\mathbb{Q}_p)\times W_E)^1}\left(\varinjlim_{U'} H^j_c(\mathcal{M}^{\mathrm{rig},(0)}_{b,\mu,U'})\right).$$

Hence $\varinjlim_{U'} H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu,U'}) \simeq \varinjlim_{U'} H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,\mu,U'})\otimes\chi$. By taking U-invariant parts, we deduce (4.8).

□

5. Cohomology of Shimura varieties and Igusa varieties. I

The cohomology of PEL-type Shimura varieties and that of Igusa varieties are closely related via the cohomology of Rapoport-Zink spaces when the Shimura varieties are proper over the reflex field. After a short review of basic definitions, we will quote a result due to Mantovan ([Man05, Thm 22]), which generalizes the "first basic identity" of Harris and Taylor ([HT01, Thm IV.2.8]).

As in previous sections, fix primes p and l such that $p\neq l$. Also fix $\mathbb{Q}$-algebra maps $\iota:\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$, $\iota_l:\mathbb{Q}_l\overset{\sim}{\to}\mathbb{C}$ and $\iota_p:\mathbb{Q}_p\overset{\sim}{\to}\mathbb{C}$.

Definition 5.1. ([Kot92b, §5], cf. [Shi09, Def 5.1, 5.2])

A *PEL datum* is a quintuple $(\mathbf{B},*,\mathbf{V},\langle\cdot,\cdot\rangle,h)$ where

- $\mathbf{B}$ is a finite-dimensional simple $\mathbb{Q}$-algebra.
- $*$ is a positive involution on $\mathbf{B}$.
- $\mathbf{V}$ is a finite semisimple $\mathbf{B}$-module.
- $\langle\cdot,\cdot\rangle:\mathbf{V}\times\mathbf{V}\to\mathbb{Q}$ is a $*$-Hermitian pairing with respect to the $\mathbf{B}$-action.
- $h:\mathbb{C}\to\mathrm{End}_{\mathbb{R}}(\mathbf{V}_{\mathbb{R}})$ is an $\mathbb{R}$-algebra homomorphism satisfying the equality $\langle h(z)v,w\rangle = \langle v,h(z^c)w\rangle$ $(\forall v,w\in\mathbf{V}_{\mathbb{R}},\ z\in\mathbb{C})$ and such that the bilinear pairing $(v,w)\mapsto\langle v,h(\sqrt{-1})w\rangle$ is symmetric and positive definite.

An *unramified integral PEL datum* (which depends on the choice of p) is a septuple $(\mathbf{B},\mathcal{O}_{\mathbf{B}},*,\mathbf{V},\Lambda_0,\langle\cdot,\cdot\rangle,h)$ where $(\mathbf{B},*,\mathbf{V},\langle\cdot,\cdot\rangle,h)$ is a PEL datum such that $\mathbf{B}_{\mathbb{Q}_p}$ is a product of matrix algebras over finite unramified extensions of $\mathbb{Q}_p$ and

- $\mathcal{O}_{\mathbf{B}}$ is a $\mathbb{Z}_{(p)}$-maximal order in $\mathbf{B}$ that is preserved by $*$ such that $\mathcal{O}_{\mathbf{B}}\otimes_{\mathbb{Z}}\mathbb{Z}_p$ is a maximal order in $\mathbf{B}_{\mathbb{Q}_p}$.
- Λ_0 is a $\mathbb{Z}_p$-lattice in $\mathbf{V}_{\mathbb{Q}_p}$ that is preserved by $\mathcal{O}_{\mathbf{B}}$ and self-dual for $\langle\cdot,\cdot\rangle$.

We fix an unramified integral PEL datum. Define a reductive group $\mathbf{G}$ over $\mathbb{Q}$ by the relation

$$\mathbf{G}(R)=\{(\delta,g)\in R^\times\times\mathrm{End}_{\mathbf{B}\otimes_{\mathbb{Q}}R}(\mathbf{V}\otimes_{\mathbb{Q}}R)\mid\langle gv_1,gv_2\rangle=\delta\langle v_1,v_2\rangle,\ \forall v_1,v_2\in\mathbf{V}\otimes_{\mathbb{Q}}R\}\tag{5.1}$$

for any $\mathbb{Q}$-algebra R. Assume that it is of type (A). (Refer to [Kot92b, §5] for the classification of PEL data.) Then $\mathbf{G}$ is a unitary similitude group. We use the following notation.

- $\mathbf{F}:=Z(\mathbf{B})$ and $\mathbf{F}^+:=\mathbf{F}^{*=1}$.
- $n:=[\mathbf{B}:\mathbf{F}]^{1/2}$.
- $i:\mathbf{F}\hookrightarrow\overline{\mathbb{Q}}$ is a $\mathbb{Q}$-embedding, fixed once and for all.
- $\mathbf{E}$ is the reflex field, which is a subfield of $\mathbb{C}$ and finite over $\mathbb{Q}$.
- $\boldsymbol{\mu}=\boldsymbol{\mu}_h:\mathbb{G}_m\to\mathbf{G}$ is a group homomorphism over $\mathbb{C}$ such that the induced map on $\mathbb{C}$-points is the composite map

$$\mathbb{C}^\times\hookrightarrow\mathbb{C}^\times\times\mathbb{C}^\times\simeq(\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C})^\times\overset{(h,\mathrm{id})}{\to}(\mathrm{End}_{\mathbf{B}}(\mathbf{V})\otimes_{\mathbb{Q}}\mathbb{C})^\times$$

  where the first map is $z\mapsto(z,1)$ and the inverse of the second map is induced by the algebra map given by $z_1\otimes z_2\mapsto(z_1z_2,z_1\bar{z}_2)$.

- U_p^{hs} is the hyperspecial subgroup of $\mathbf{G}(\mathbb{Q}_p)$ stabilizing Λ_0.
- $\mathbf{G}_1$ is the kernel of the map $\mathbf{G} \to \mathbb{G}_m$ given by $(\delta, g) \mapsto \delta$. If Φ is a subset of $\mathrm{Hom}_{\mathbb{Q}}(\mathbf{F}, \mathbb{C})$ such that $\mathrm{Hom}_{\mathbb{Q}}(\mathbf{F}, \mathbb{C})$ is the disjoint union of Φ and $c \circ \Phi$ (in which case the set Φ is called a CM-type for $\mathbf{F}$) then there is a natural isomorphism
$$\mathbf{G}_1(\mathbb{R}) \simeq \prod_{\tau \in \Phi} U(p_\tau, q_\tau)$$
for pairs of integers (p_τ, q_τ) satisfying $p_\tau + q_\tau = n$.

The definition of an unramified integral PEL datum implies that $\mathbf{F}$ is a finite extension of $\mathbb{Q}$ unramified at p. Then $\mathbf{E}$ is also unramified over $\mathbb{Q}$ as it is contained in the Galois closure of $\mathbf{F}$ (in $\mathbb{C}$).

We are about to discuss Shimura varieties and Igusa varieties associated to an unramified integral PEL datum. We will omit details and refer to [Kot92b, §5] for Shimura varieties and [Man05, §4] for Igusa varieties. (The reader might find [Shi09, §5] and [Shib, §4.1] also helpful.) Associated to a PEL datum $(\mathbf{B}, *, \mathbf{V}, \langle\cdot,\cdot\rangle, h)$ is a system $\mathrm{Sh} = \{\mathrm{Sh}_U\}$ of smooth quasi-projective varieties over $\mathbf{E}$ where U runs over the set of sufficiently small open compact subgroups of $\mathbf{G}(\mathbb{A}^\infty)$. On the other hand, an unramified integral PEL datum gives rise to a system of integral models $\mathrm{Sh}_p = \{\mathrm{Sh}_{U^p}\}$ over $\mathcal{O}_{\mathbf{E},(p)}$ such that the generic fiber of each Sh_{U^p} is naturally identified with $\mathrm{Sh}_{U^p U_p^{\text{hs}}}$, where U^p runs over the set of sufficiently small open compact subgroups of $\mathbf{G}(\mathbb{A}^{\infty,p})$.

From now on, only a PEL datum of type (A) satisfying (A1) and (A2) below will be considered.

(A1) $\mathbf{B}$ is a central division algebra over $\mathbf{F}$,
(A2) $\mathbf{V}$ is a simple $\mathbf{B}$-module.

Consequences of (A1) and (A2) are as follows.

(C1) $\mathbf{G}/Z(\mathbf{G})$ is anisotropic over $\mathbb{Q}$,
(C2) Sh is (not only quasi-projective but) projective over $\mathcal{O}_{\mathbf{E},(p)}$ and
(C3) $\mathfrak{K}(I_\gamma/\mathbb{Q})$, as defined in [Kot86, 4.6], is trivial for every semisimple element $\gamma \in \mathbf{G}(\mathbb{Q})$ where $I_\gamma := Z_{\mathbf{G}}(\gamma)$. (Since $\mathbf{G}^{\text{der}}$ is simply connected, I_γ is connected.)

Indeed, (C1) is easy to see as $\mathbf{G}$ naturally embeds into the $\mathbb{Q}$-group $GL_{\mathbf{B}}(\mathbf{V}) \simeq GL_1(\mathbf{B}^{op})$. For (C2) and (C3), see [Kot92b, p.392] and [Kot92a, Lem 2].

Remark 5.2. Loosely speaking, (C3) may be rephrased as "$\mathbf{G}$ has no endoscopy" from the viewpoint of the trace formula. More precisely, (C3) forces the vanishing of the terms for $H \not\simeq \mathbf{G}^*$ in the stable trace formula (on the geometric side). See the proofs of Propositions 6.3 and 6.6.

The subfield $\mathbf{E}$ of $\mathbb{C}$ may be embedded into $\overline{\mathbb{Q}}_p$ via ι_p^{-1}. This induces a place w of $\mathbf{E}$ dividing p as well as an embedding $\mathcal{O}_{\mathbf{E},(p)} \hookrightarrow \mathbb{Z}_p^{\text{ur}}$ where $\mathbb{Z}_p^{\text{ur}}$ is the integral closure of $\mathbb{Z}_p$ in $\mathbb{Q}_p^{\text{ur}}$. By taking reduction at w, we obtain a map $\mathcal{O}_{\mathbf{E},(p)} \to \overline{\mathbb{F}}_p$ factoring through the residue field $k(w)$ at w. Using this map, we take the special fiber $\mathrm{Sh}_p \times_{\mathcal{O}_{\mathbf{E},(p)}} \overline{\mathbb{F}}_p$ of $\mathrm{Sh}_p = \{\mathrm{Sh}_{U^p}\}$.

On the other hand, for each $\mathbf{b} \in B(\mathbf{G}_{\mathbb{Q}_p}, -\boldsymbol{\mu})$, choose a Barsotti-Tate group $\Sigma_{\mathbf{b}}$ over $\overline{\mathbb{F}}_p$ of isogeny type $\mathbf{b}$ as in [Shi09, §5]. Let $J_{\mathbf{b}}$ be the reductive group over $\mathbb{Q}_p$ defined from $\mathbf{G}$ and $\mathbf{b}$, exactly as in (4.3). In this context, we can define the Igusa variety, which is closely related to $\mathrm{Sh}_p \times_{\mathcal{O}_{\mathbf{E},(p)}} \overline{\mathbb{F}}_p$. (See [Man05, §4] for details. Section 5 of [Shi09] might be helpful as well.) The Igusa variety is a projective system of smooth varieties $\mathrm{Ig}_{\Sigma_{\mathbf{b}}} = \{\mathrm{Ig}_{\Sigma_{\mathbf{b}}, U^p, m}\}$ over $\overline{\mathbb{F}}_p$ where U^p are sufficiently small open compact subgroups of $\mathbf{G}(\mathbb{A}^{\infty,p})$ as before and $m \in \mathbb{Z}_{>0}$. (The varieties $\mathrm{Ig}_{\Sigma_{\mathbf{b}}, U^p, m}$ are usually not proper over $\overline{\mathbb{F}}_p$.) Let ξ be an irreducible algebraic representation of $\mathbf{G}$ over $\overline{\mathbb{Q}}_l$. Then ξ gives rise to l-adic sheaves on Sh and $\mathrm{Ig}_{\Sigma_{\mathbf{b}}}$, which will be denoted by $\mathscr{L}_\xi$ (by abuse of notation). Define

$$H(\mathrm{Sh}, \mathscr{L}_\xi) := \sum_k (-1)^k \varinjlim_{U} H^k(\mathrm{Sh}_U, \mathscr{L}_\xi), \qquad H_c(\mathrm{Ig}_{\Sigma_{\mathbf{b}}}, \mathscr{L}_\xi) := \sum_k (-1)^k \varinjlim_{U^p, m} H_c^k(\mathrm{Ig}_{\Sigma_{\mathbf{b}}, U^p, m}, \mathscr{L}_\xi).$$

The virtual representations $H(\mathrm{Sh}, \mathscr{L}_\xi)$ and $H_c(\mathrm{Ig}_{\Sigma_{\mathbf{b}}}, \mathscr{L}_\xi)$ may be viewed as objects of $\mathrm{Groth}(\mathbf{G}(\mathbb{A}^\infty) \times \mathrm{Gal}(\overline{\mathbf{E}}/\mathbf{E}))$ and $\mathrm{Groth}(\mathbf{G}(\mathbb{A}^{\infty,p}) \times J_{\mathbf{b}}(\mathbb{Q}_p))$, respectively. In particular, $H(\mathrm{Sh}, \mathscr{L}_\xi)$ may be written as

$$H(\mathrm{Sh}, \mathscr{L}_\xi) = \sum_{\Pi^\infty} [\Pi^\infty][R_{l,\xi,\boldsymbol{\mu}}(\Pi^\infty)] \tag{5.2}$$

for some $R_{l,\xi,\boldsymbol{\mu}}(\Pi^\infty) \in \mathrm{Groth}(\mathrm{Gal}(\overline{\mathbf{E}}/\mathbf{E}))$, where Π^∞ runs over $\mathrm{Irr}_l(\mathbf{G}(\mathbb{A}^\infty))$. Define

$$\mathscr{A}^\infty_{\iota_l \xi}(\mathbf{G}) := \left\{ \Pi^\infty \in \mathrm{Irr}(\mathbf{G}(\mathbb{A}^\infty)) \;\middle|\; \begin{array}{c} \Pi^\infty \otimes \Pi_\infty \text{ is automorphic for some} \\ \iota_l\xi\text{-cohomological representation } \Pi_\infty \text{ of } \mathbf{G}(\mathbb{R}) \end{array} \right\} \tag{5.3}$$

where the notion of $\iota_l\xi$-cohomological representation is defined on page 198 of [HT01], for instance, in terms of the non-vanishing of the relative Lie algebra cohomology of $\Pi_\infty \otimes \iota_l\xi$.

Remark 5.3. Let $\phi_{\iota_l\xi}$ be as in §6.1. Then Π_∞ is $\iota_l\xi$-cohomological if $\mathrm{tr}\,\Pi_\infty(\phi_{\iota_l\xi}) \neq 0$. Indeed, $\mathrm{tr}\,\Pi_\infty(\phi_{\iota_l\xi})$ computes the Euler-Poincaré characteristic of the relative Lie algebra cohomology of $\Pi_\infty \otimes \iota_l\xi$. (cf. [Kot92a, Lem 3.2].)

By Matsushima's formula, it is not hard to see that $R_{l,\xi,\boldsymbol{\mu}}(\Pi^\infty) \neq 0$ only if $\iota_l\Pi^\infty \in \mathscr{A}^\infty_{\iota_l\xi}(\mathbf{G})$ (cf. [HT01, Cor VI.2.4]). We recall a formula of Mantovan that will be needed in §7.

Proposition 5.4. *(*[Man05, Thm 22]*) When* Sh *is proper over* $\mathbf{E}$*, the following equality holds in* $\mathrm{Groth}(\mathbf{G}(\mathbb{A}^\infty) \times W_{\mathbf{E}_w})$.

$$H(\mathrm{Sh}, \mathscr{L}_\xi) = \sum_{\mathbf{b} \in B(\mathbf{G}_{\mathbb{Q}_p}, -\boldsymbol{\mu})} \mathrm{Mant}_{\mathbf{b},\boldsymbol{\mu}}(H_c(\mathrm{Ig}_{\Sigma_{\mathbf{b}}}, \mathscr{L}_\xi))$$

(We regard $\mathrm{Mant}_{\mathbf{b},\boldsymbol{\mu}}$ *as the identity on* $\mathrm{Groth}(\mathbf{G}(\mathbb{A}^{\infty,p}))$.*)*

Remark 5.5. Fargues has obtained an analogous formula for the basic stratum ([Far04, Cor 4.6.3]).

6. Cohomology of Shimura varieties and Igusa varieties. II

After recalling a certain property of the test function at infinity (§6.1) and introducing an important representation-theoretic operation $\mathrm{Red}^{(b)}$ (§6.2), we will compare the trace formulas for Shimura varieties and Igusa varieties via stabilization in §6.3. Although the two trace formulas look similar, they come from sources of different nature. Namely, one is a consequence of the analytic trace formula while the other is derived from a version of the Grothendieck-Lefschetz fixed-point formula for l-adic cohomology. The upshot is an equality relating the cohomology of Igusa varieties to that of Shimura varieties (Theorem 6.7) under simplifying assumptions. Most notably, we only deal with the case of trivial endoscopy, which still suffices for the application (§7) we have in mind.

6.1. **The function ϕ_ξ at infinity.** Let G denote a connected reductive group over $\mathbb{R}$ (rather than over $\mathbb{Q}_p$) only in §6.1. Suppose that G has an elliptic torus. We use the following notations.

- A_G is the maximal split torus in $Z(G)$ and $A_{G,\infty} := A_G(\mathbb{R})^0$.
- K_∞ is a maximal compact subgroup of $G(\mathbb{R})$.
- $\epsilon_{\mathbf{G}} := [K_\infty : K_\infty^0]$.
- ξ is an irreducible algebraic representation of G over $\mathbb{C}$. (Outside §6.1, ξ is a representation over $\overline{\mathbb{Q}}_l$.)
- $\chi_\xi : A_{G,\infty} \to \mathbb{C}^\times$ is the character given by restricting ξ to $A_{G,\infty}$.
- $\Pi_{\mathrm{temp}}(\chi_\xi^{-1})$ is the set of isomorphism classes of irreducible admissible representations of $G(\mathbb{R})$ which are tempered modulo $A_{G,\infty}$ and whose central character is χ_ξ^{-1} on $A_{G,\infty}$.
- $\Pi_{\mathrm{disc}}(\xi^\vee)$ is the subset of $\Pi_{\mathrm{temp}}(\chi_\xi^{-1})$ consisting of those representations which are square-integrable modulo $A_{G,\infty}$ and have the same infinitesimal and central characters as $\xi^\vee$.

- Let T be an elliptic torus of G. Define
$$q(G) := \frac{1}{2}\dim(G(\mathbb{R})/K_\infty A_{G,\infty}), \qquad d(G) := |\ker(H^1(\mathbb{R},T)\to H^1(\mathbb{R},G))|.$$
(Note that $d(G)$ is independent of the choice of T.)
- When $\gamma\in G(\mathbb{R})$ is elliptic semisimple, set $I_\gamma := Z_G(\gamma)$ (which is connected since G^{der} is simply connected) and define $q(I_\gamma)$ and $d(I_\gamma)$ similarly as above. Denote by $\overline{I}_\gamma$ a compact-mod-center inner form of I_γ.
- Let $\chi : A_{G,\infty}\to\mathbb{C}^\times$ be any continuous character. Define $C_c^\infty(G(\mathbb{R}),\chi)$ to be the space of $\mathbb{C}$-valued functions f on $G(\mathbb{R})$ such that $f(zg)=\chi(z)f(g)$ for every $z\in A_{G,\infty}$ and $g\in G(\mathbb{R})$.

Let $\phi_\xi\in C_c^\infty(G(\mathbb{R}),\chi_\xi)$ be a function such that for any $\pi\in\Pi_{\mathrm{temp}}(\chi_\xi^{-1})$,
$$\mathrm{tr}\,\pi(\phi_\xi)=\begin{cases}(-1)^{q(G)}, & \text{if } \pi\in\Pi_{\mathrm{disc}}(\xi^\vee)\\ 0, & \text{otherwise}\end{cases} \tag{6.1}$$
The existence of ϕ_ξ is guaranteed by [CD90, Prop 4, Cor] (cf. [Art89, Lem 3.1]). In fact, ϕ_ξ may be taken as $(-1)^{q(G)}$ times the sum of pseudo-coefficients for all $\pi\in\Pi_{\mathrm{disc}}(\xi^\vee)$.

Lemma 6.1. *If $\gamma\in G(\mathbb{R})$ is elliptic semisimple then*
$$O_\gamma^{G(\mathbb{R})}(\phi_\xi) = \mathrm{vol}(\overline{I}_\gamma(\mathbb{R})/A_{\overline{I}_\gamma,\infty})^{-1}(-1)^{q(I_\gamma)}d(I_\gamma)\cdot\mathrm{tr}\,\xi(\gamma) \tag{6.2}$$
$$SO_\gamma^{G(\mathbb{R})}(\phi_\xi) = \mathrm{vol}(\overline{I}_\gamma(\mathbb{R})/A_{\overline{I}_\gamma,\infty})^{-1}\cdot e(I_\infty)\cdot d(G)\cdot\mathrm{tr}\,\xi(\gamma) \tag{6.3}$$
and otherwise $SO_\gamma^{G(\mathbb{R})}(\phi_\xi)=O_\gamma^{G(\mathbb{R})}(\phi_\xi)=0$.

Proof. The first formula and the last vanishing are implied by [Art89, Thm 5.1]. The second formula follows from the proof of [Kot92a, Lem 3.1], which is applicable not only to the groups considered in that article but also to our case. Alternatively, (6.3) can be deduced from (6.2) using the argument in the proof of [CL99, Thm A.1.1]. □

6.2. **Definition of** $\mathrm{Red}^{(\mathbf{b})}$**.** We return to the setting of section 5 and keep assumptions (A1) and (A2) from there. Let us make the following additional assumptions.

(B1) $\mathbf{F}=\mathbf{F}^+\mathbf{K}$ for the totally real field $\mathbf{F}^+=\mathbf{F}^{*=1}$ and an imaginary quadratic field $\mathbf{K}$.
(B2) The prime p is inert in $\mathbf{F}^+$ and splits in $\mathbf{K}$.

The embedding $\mathbf{F}\overset{i}{\hookrightarrow}\overline{\mathbb{Q}}\overset{\iota_p^{-1}\iota}{\hookrightarrow}\overline{\mathbb{Q}}_p$ induces a place v of $\mathbf{F}$. Then there is a natural isomorphism $\mathbf{F}_{\mathbb{Q}_p}\simeq\mathbf{F}_v\times\mathbf{F}_{v^c}$. Correspondingly we have a decomposition $\mathbf{V}_{\mathbb{Q}_p}=\mathbf{V}_v\oplus\mathbf{V}_{v^c}$ such that $\mathbf{V}_v$ (resp. $\mathbf{V}_{v^c}$) is an $\mathbf{F}_v$-vector space of dimension n^2, as well as a decomposition $\Lambda_0=\Lambda_v\oplus\Lambda_{v^c}$. We may choose an isomorphism $\mathbf{B}_{\mathbb{Q}_p}\simeq M_n(\mathbf{F}_v)\times M_n(\mathbf{F}_{v^c})$ such that Λ_v and Λ_{v^c} are invariant under $M_n(\mathcal{O}_{\mathbf{F}_v})$ and $M_n(\mathcal{O}_{\mathbf{F}_{v^c}})$, respectively, and also that $*$ is transported to the involution of $M_n(\mathbf{F}_v)\times M_n(\mathbf{F}_{v^c})$ sending (a_v,a_{v^c}) to $(a_{v^c}^c,a_v^c)$. Choose an idempotent $\epsilon_v\in M_n(\mathcal{O}_{\mathbf{F}_v})$ such that $\epsilon_v\Lambda_v$ is a free $\mathcal{O}_{\mathbf{F}_v}$-module of rank n. Set $\epsilon_{v^c}:=\epsilon_v^*$ and $\epsilon:=(\epsilon_v,\epsilon_{v^c})$.

There is a natural $\mathbb{Q}_p$-embedding
$$\mathbf{G}_{\mathbb{Q}_p}\hookrightarrow GL_1\times R_{\mathbf{F}_v/\mathbb{Q}_p}GL_{\mathbf{F}_v}(\epsilon_v\mathbf{V}_v)\times R_{\mathbf{F}_{v^c}/\mathbb{Q}_p}GL_{\mathbf{F}_{v^c}}(\epsilon_{v^c}\mathbf{V}_{v^c}).$$
The projection onto the first two components induces an isomorphism
$$\mathbf{G}_{\mathbb{Q}_p}\overset{\sim}{\to}GL_1\times R_{\mathbf{F}_v/\mathbb{Q}_p}GL_{\mathbf{F}_v}(\epsilon_v\mathbf{V}_v) \tag{6.4}$$
There is a corresponding decomposition
$$B(\mathbf{G}_{\mathbb{Q}_p})\simeq B(GL_1)\times B(R_{\mathbf{F}_v/\mathbb{Q}_p}GL_{\mathbf{F}_v}(\epsilon_v\mathbf{V}_v)). \tag{6.5}$$

Using (6.4) and (6.5) we write

$$\mathbf{b} = (\mathbf{b}_0, \mathbf{b}_v), \quad \text{and} \quad \boldsymbol{\mu} = (\boldsymbol{\mu}_0, \boldsymbol{\mu}_v).$$

Correspondingly, there are $\mathbb{Q}_p$-isomorphisms

$$J_{\mathbf{b}} \xrightarrow{\sim} GL_1 \times J_{\mathbf{b}_v} \quad \text{and} \quad M_{\mathbf{b}} \xrightarrow{\sim} GL_1 \times M_{\mathbf{b}_v} \tag{6.6}$$

where $M_{\mathbf{b}}$ is defined in [Shib, §3.2]. The $\mathbf{F}_v$-groups $J_{\mathbf{b}_v}$ and $M_{\mathbf{b}_v}$ are defined from $\mathbf{b}_v$ as in §4.1.

The unramified Rapoport-Zink datum of PEL type $(\mathbf{F}_{\mathbb{Q}_p}, *, \epsilon\mathbf{V}_{\mathbb{Q}_p}, \mathbf{b}, \iota_p^{-1}\boldsymbol{\mu}_h)$, where ϵ is defined above, gives rise to a tower of Rapoport-Zink spaces $\{\mathcal{M}^{\mathrm{rig}}_{\mathbf{b},\boldsymbol{\mu},U}\}$ for open compact subgroups U of $\mathbf{G}(\mathbb{Q}_p)$ (see [RZ96, Ch 5], [Far04, Ch 2.3]) and $\mathrm{Mant}_{\mathbf{b},\boldsymbol{\mu}} : \mathrm{Groth}(J_{\mathbf{b}}(\mathbb{Q}_p)) \to \mathrm{Groth}(\mathbf{G}(\mathbb{Q}_p) \times W_{\mathbf{E}_w})$, where the latter is defined exactly as in Definition 4.7. Set

$$(F, V, \mu, b) := (\mathbf{F}_v, \epsilon_v\mathbf{V}_v, \boldsymbol{\mu}_v, \mathbf{b}_v).$$

Then

$$G = R_{\mathbf{F}_v/\mathbb{Q}_p} GL_{\mathbf{F}_v}(\epsilon_v\mathbf{V}_v), \quad J_b = J_{\mathbf{b}_v}, \qquad M_b = M_{\mathbf{b}_v}, \qquad E = \mathbf{E}_w.$$

These are obvious except maybe the last one, whose proof is given at the end of this subsection. Note that $e(J_b) = e(J_{\mathbf{b}_v}) = e(J_{\mathbf{b}})$. We have an isomorphism

$$\mathrm{Mant}_{\mathbf{b},\iota_p^{-1}\boldsymbol{\mu}} = \mathrm{Mant}_{\mathbf{b}_0,\boldsymbol{\mu}_0} \otimes \mathrm{Mant}_{b,\mu} \tag{6.7}$$

compatibly with (6.4) and (6.6) since there is a corresponding isomorphism on the level of Rapoport-Zink spaces (cf. [Far04, 2.3.7.1]).

Let N_b^{op} denote the unipotent radical of P_b^{op} (the opposite parabolic of P_b). There is an unnormalized Jacquet module functor

$$\mathrm{Jac}^G_{P_b^{\mathrm{op}}} : \mathrm{Groth}(G(\mathbb{Q}_p)) \to \mathrm{Groth}(M_b(\mathbb{Q}_p))$$

sending $\pi \in \mathrm{Irr}(G(\mathbb{Q}_p))$ to the $N_b^{\mathrm{op}}(\mathbb{Q}_p)$-coinvariant of $\pi|_{P_b^{\mathrm{op}}(\mathbb{Q}_p)}$. In light of the descriptions (4.6) and (4.7), define

$$\mathrm{LJ}_{M_b,J_b} : \mathrm{Groth}(M_b(\mathbb{Q}_p)) \to \mathrm{Groth}(J_b(\mathbb{Q}_p))$$

using the map LJ of §2.2. Finally define

$$\mathrm{Red}^{(b)} := e(J_b) \cdot \mathrm{LJ}_{M_b,J_b} \circ \mathrm{Jac}^G_{P_b^{\mathrm{op}}} \quad \text{and} \quad \mathrm{Red}^{(\mathbf{b})} := \mathrm{id} \otimes \mathrm{Red}^{(b)}$$

where $e(J_b) \in \{\pm 1\}$ is the Kottwitz sign for J_b.

Lemma 6.2. *In the above setting, we have $E = \mathbf{E}_w$ as subfields of $\overline{\mathbb{Q}}_p$ (where $\mathbf{E}_w$ embeds into $\overline{\mathbb{Q}}_p$ by continuously extending the embedding $\mathbf{E} \hookrightarrow \overline{\mathbb{Q}}_p$ given in the paragraph below Remark 5.2).*

Proof. The above embedding $\mathbf{E} \hookrightarrow \overline{\mathbb{Q}}_p$ factors through $\overline{\mathbb{Q}}$ as follows.

$$\mathbf{E} \hookrightarrow \overline{\mathbb{Q}} \overset{\iota}{\hookrightarrow} \mathbb{C} \overset{\iota_p^{-1}}{\simeq} \overline{\mathbb{Q}}_p$$

Let α be the p-adic valuation on $\overline{\mathbb{Q}}$ induced by $\iota_p^{-1}\iota$. Recall that $F \simeq \mathbf{F}_v$ and $v|_{\mathbf{K}}$ is inert in $\mathbf{F}$. Let $u : \mathbf{K} \hookrightarrow \overline{\mathbb{Q}}$ be the map such that $\iota_p^{-1}\iota u$ induces $v|_{\mathbf{K}}$ on $\mathbf{K}$. There is a bijection from $\mathrm{Hom}_{\mathbf{K},v|_{\mathbf{K}}}(\mathbf{F}, \overline{\mathbb{Q}})$ to $\mathrm{Hom}_{\mathbb{Q}_p}(F, \overline{\mathbb{Q}}_p)$ sending τ to the unique continuous map $F \hookrightarrow \overline{\mathbb{Q}}_p$ extending $\iota_p^{-1}\iota\tau : \mathbf{F} \hookrightarrow \overline{\mathbb{Q}}_p$ (where F is identified with $\mathbf{F}_v$). We have

$$\begin{aligned}
\mathrm{Gal}(\overline{\mathbb{Q}}_p/E) &= \{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \,|\, p_{\sigma\tau} = p_\sigma, \ \forall \tau \in \mathrm{Hom}_{\mathbb{Q}_p}(F, \overline{\mathbb{Q}}_p)\} \\
&= \{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \,|\, \alpha \circ \sigma = \alpha, \ p_{\sigma\tau} = p_\sigma, \ \forall \tau \in \mathrm{Hom}_{\mathbf{K},v|_{\mathbf{K}}}(\mathbf{F}, \overline{\mathbb{Q}})\} \\
&= \{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbf{E}) \,|\, \alpha \circ \sigma = \alpha, \} \\
&= \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbf{E}_w).
\end{aligned}$$

□

6.3. **Comparison of trace formulas.** Continuing the discussion of §6.2, we compare the trace formulas for Shimura varieties and Igusa varieties. The upshot is Theorem 6.7, a simple relation between the cohomology spaces of Shimura varieties and Igusa varieties. Although (C3) of §5 allows us to avoid the full stabilization of the trace formulas (as was avoided in [Kot92a]), we decided to use the stable trace formulas, believing that such an approach is more conceptual and easier to extend to a more general case. The stabilization relies on the proof of the fundamental lemma and the Langlands-Shelstad transfer conjecture due to Ngô, Waldspurger and others. (See [Ngo] for instance.)

Let (H, s, η) be an elliptic endoscopic triple for $\mathbf{G}$ over $\mathbb{Q}$ ([Kot84, 7.4]). As usual, $\tau(H)$ stands for the Tamagawa number for H (see [Kot88] for instance). For a test function f^H in $C_c^\infty(H(\mathbb{A}^\infty)) \otimes C_c^\infty(H(\mathbb{R}), \chi_H)$, define

$$ST_e^H(f^H) := \sum_{\gamma_H} \tau(H) \cdot SO_{\gamma_H}^{H(\mathbb{A})}(f^H) \tag{6.8}$$

where γ_H runs over a set of representatives for stable elliptic conjugacy classes in $H(\mathbb{Q})$. (The subscript in ST_e^H stands for "elliptic".) Note that every H as above has a simply connected derived subgroup.

Fix a quasi-split inner form $\mathbf{G}^*$ of $\mathbf{G}$ over $\mathbb{Q}$. If $\mathbf{G}$ is quasi-split over $\mathbb{Q}_v$ (in particular if $v = p$), there is an isomorphism $\mathbf{G}_{\mathbb{Q}_v} \simeq \mathbf{G}^*_{\mathbb{Q}_v}$ canonical up to $\mathbf{G}(\overline{\mathbb{Q}}_v)$-conjugacy. We fix such an isomorphism. There is a unique (up to isomorphism) elliptic endoscopic triple (H, s, η) such that $H \simeq \mathbf{G}^*$. For each finite place v of $\mathbb{Q}$, let $\phi_v \in C_c^\infty(\mathbf{G}(\mathbb{Q}_v))$. The transfer $\phi_v^* \in C_c^\infty(\mathbf{G}^*(\mathbb{Q}_v))$ of ϕ_v is defined as follows. If $\mathbf{G}$ is quasi-split over $\mathbb{Q}_v$ then put $\phi_v^* := \phi_v$. This is the case at all but finitely many places. Otherwise, ϕ_v^* is given by the relation

$$SO_{\gamma_v^*}(\phi_v^*) = e(\mathbf{G}_{\mathbb{Q}_v}) \cdot SO_{\gamma_v}(\phi_v) \tag{6.9}$$

for every semisimple $\gamma_v^* \in \mathbf{G}^*(\mathbb{Q}_v)$, where $\gamma_v \in \mathbf{G}(\mathbb{Q}_v)$ is the transfer of γ_v^* if it exists (then it is unique up to stable conjugacy), and the right hand side is viewed as zero if such an element γ_v does not exist. As usual, compatible measures are used in (6.9) and $e(\mathbf{G}_{\mathbb{Q}_v})$ is the Kottwitz sign. Note that $e(\mathbf{G}_{\mathbb{Q}_v})$ may be viewed as the transfer factor. (Since the product of $e(\mathbf{G}_{\mathbb{Q}_v})$ over all places v of $\mathbb{Q}$ is 1 by [Kot83, p.297], the product formula of [LS87, 6.4] is satisfied for our transfer factors. One could work with a different constant as the transfer factor at each place as long as the product formula holds.) The transfer $\phi_\xi^* \in C_c^\infty(\mathbf{G}^*(\mathbb{R}), \chi_\xi)$ of ϕ_ξ is defined by the same formula as (6.9). (We treated ϕ_ξ^* separately only for the notational problem, as ϕ_ξ^* is compactly supported modulo $A_{\mathbf{G}^*_{\mathbb{R}},\infty} = A_{\mathbf{G}_{\mathbb{R}},\infty}$.) The definition of ϕ_v^* extends to the adelic setting in the evident manner.

Proposition 6.3. *For any function $\phi^\infty \in C_c^\infty(\mathbf{G}(\mathbb{A}^\infty))$,*

$$\mathrm{tr}\,(\phi^\infty | \iota_l H(\mathrm{Sh}, \mathscr{L}_\xi)) = \epsilon_{\mathbf{G}} \cdot |\ker^1(\mathbb{Q}, \mathbf{G})| \cdot ST_e^{\mathbf{G}^*}((\phi^*)^\infty \phi_\xi^*).$$

Remark 6.4. The above proposition works for any $\mathbf{G}$ arising from a PEL datum as long as $\mathbf{G}$ is anisotropic modulo center over $\mathbb{Q}$. When $\mathbf{G}$ is not anisotropic modulo center over $\mathbb{Q}$, an analogous stable trace formula is still available but has to include more than the elliptic part. It is worked out in an unpublished manuscript of Kottwitz ([Kot]).

Proof. By theorem 6.1 of [Art89] along with remark 3 to the theorem (the latter explains the appearance of $\epsilon_{\mathbf{G}}$),

$$\mathrm{tr}\,(\phi^\infty | \iota_l H(\mathrm{Sh}, \mathscr{L}_\xi)) = \epsilon_{\mathbf{G}} \cdot |\ker^1(\mathbb{Q}, \mathbf{G})| \sum_{\gamma \in \mathbf{G}(\mathbb{Q})/\sim} O_\gamma^{\mathbf{G}(\mathbb{A}^\infty)}(\phi) \cdot O_\gamma^{\mathbf{G}(\mathbb{R})}(\phi_\xi) \tag{6.10}$$

where γ runs over semisimple conjugacy classes in $\mathbf{G}(\mathbb{Q})$ which are elliptic over $\mathbb{R}$. The factor $|\ker^1(\mathbb{Q}, \mathbf{G})|$ shows up since our moduli Shimura variety is the disjoint union of $|\ker^1(\mathbb{Q}, \mathbf{G})|$-copies of the canonical model of Shimura ([Del71]). See [Kot92b, §8] for explanation.

By the proof of [Kot86, Thm 9.6], (6.10) is stabilized as

$$\mathrm{tr}\,(\phi^\infty|\iota_l H(\mathrm{Sh},\mathscr{L}_\xi)) = \epsilon_{\mathbf{G}}\cdot|\ker^1(\mathbb{Q},\mathbf{G})|\sum_{(H,s,\eta)}\iota(\mathbf{G},H)ST^H_e(f^H)$$

where the sum runs over the isomorphism classes of elliptic endoscopic triples for $\mathbf{G}$. Note that the above stabilization is unconditional thanks to the proof of the fundamental lemma. We claim that the summand vanishes unless $H\simeq\mathbf{G}^*$. Indeed, in situation of lemma 9.7 of [Kot86], every κ is trivial by (C3) of §5. It is clear from the proof of that lemma that the trivial κ corresponds to $H\simeq\mathbf{G}^*$ under the bijection of lemma 9.7 of Kottwitz. In light of the characterization of f^H in paragraph 5.4 of [Kot86], the stable orbital integral of f^H must vanish on every $(\mathbf{G},H)$-regular semisimple element of $H(\mathbb{A})$. Thus the claim is proved.

In case $H=\mathbf{G}^*$, the function $f^{\mathbf{G}^*}$ is by definition a transfer of $\phi^\infty\phi_\xi$ to $\mathbf{G}^*$, so we may take $f^{\mathbf{G}^*}=(\phi^*)^\infty\phi^*_\xi$. The proposition follows.

□

Our next task is to obtain an analogue of Proposition 6.3 for the cohomology of Igusa varieties, starting from the stabilization of the counting-point formula for Igusa varieties ([Shib, Thm 7.2]). Choose any acceptable function $\phi^{\infty,p}\times\phi'_p$ in $C^\infty_c(\mathbf{G}(\mathbb{A}^{\infty,p})\times J_{\mathbf{b}}(\mathbb{Q}_p))$ in the sense of [Shi09, Def 6.2]. We already defined $(\phi^*)^{\infty,p}$, which is a transfer of $\phi^{\infty,p}$. Set

$$\widetilde{\phi}^*_p := h^{\mathbf{G}^*}_p\in C^\infty_c(\mathbf{G}(\mathbb{Q}_p))\quad\text{and}\quad \widetilde{\phi}^*_\infty := h^{\mathbf{G}^*}_\infty\in C^\infty_c(\mathbf{G}^*(\mathbb{R}),\chi_\xi)$$

where $h^{\mathbf{G}^*}_p$ and $h^{\mathbf{G}^*}_\infty$ are the functions in §6.3 and §5.2 of [Shib]. (The explicit construction of $h^{\mathbf{G}^*}_\infty$ is due to Kottwitz.) Note that $h^{\mathbf{G}^*}_p$ and $h^{\mathbf{G}^*}_\infty$ depend on ϕ'_p and ξ, respectively. Recall from [Kot90, (7.4)] that for any elliptic element $\gamma^*\in\mathbf{G}^*(\mathbb{R})$,

$$SO^{\mathbf{G}^*(\mathbb{R})}_{\gamma^*}(\widetilde{\phi}^*_\infty)=\begin{cases}\mathrm{vol}(\overline{I}_{\gamma^*}(\mathbb{R})/A_{\overline{I}_{\gamma^*},\infty})^{-1}\cdot e(\overline{I}_{\gamma^*})\cdot e(\mathbf{G}_{\mathbb{R}})\cdot\mathrm{tr}\,\xi(\gamma^*) & \text{if}\quad \gamma^*\text{ is elliptic},\\ 0 & \text{otherwise.}\end{cases}\tag{6.11}$$

Note that $e(\mathbf{G}_{\mathbb{R}})$ plays the role of transfer factor at ∞. The following property of $\widetilde{\phi}^*_p$ will be of importance.

Lemma 6.5. *For every $\pi\in\mathrm{Irr}(\mathbf{G}(\mathbb{Q}_p))$,*

$$\mathrm{tr}\,\left(\mathrm{Red}^{(\mathbf{b})}(\pi)\right)(\phi'_p)=\mathrm{tr}\,\pi(\widetilde{\phi}^*_p).\tag{6.12}$$

Proof. This is proved precisely as the case $\vec{n}=(n)$ of [Shic, Lem 5.10], which occurs when the endoscopic group H equals $\mathbf{G}^*$. Note that our $\widetilde{\phi}^*_p$ is the same as the function $\phi^{\vec{n}}_{\mathrm{Ig},p}$ of that paper if $\vec{n}=(n)$. □

Proposition 6.6. *Let $\phi^{\infty,p}\phi'_p\in C^\infty_c(\mathbf{G}(\mathbb{A}^{\infty,p})\times J_{\mathbf{b}}(\mathbb{Q}_p))$ be any acceptable function. Then*

$$\mathrm{tr}\,(\phi^{\infty,p}\phi'_p|\iota_l H_c(\mathrm{Ig}_{\Sigma_b},\mathscr{L}_\xi))=|\ker^1(\mathbb{Q},\mathbf{G})|\cdot d(\mathbf{G}_{\mathbb{R}})^{-1}\cdot ST^{\mathbf{G}^*}_e((\phi^*)^{\infty,p}\widetilde{\phi}^*_p\phi^*_\xi).$$

Proof. Theorem 7.2 of [Shib] says that

$$\mathrm{tr}\,(\phi^{\infty,p}\phi'_p|\iota_l H_c(\mathrm{Ig}_{\Sigma_b},\mathscr{L}_\xi))=|\ker^1(\mathbb{Q},\mathbf{G})|\sum_{(H,s,\eta)}\iota(\mathbf{G},H)ST^H_e(h^H).$$

The summand vanishes unless $H\simeq\mathbf{G}^*$ by lemma 7.1 of [Shib]. Indeed, in the notation of that lemma, (C3) of §5 implies that $(H,s,\eta,\gamma_H)\notin\mathcal{EQ}^{\mathrm{ell}}(\mathbf{G})$ unless $H\simeq\mathbf{G}^*$. (As in the proof of Proposition 6.3, this may be interpreted as the fact that the quadruple (H,s,η,γ_H) satisfying $H\simeq\mathbf{G}^*$ does not arise as the transfer of any (elliptic) conjugacy class of $\mathbf{G}(\mathbb{Q})$ if κ is trivial.) Thus we have

$$\mathrm{tr}\,(\phi^{\infty,p}\phi'_p|\iota_l H_c(\mathrm{Ig}_{\Sigma_b},\mathscr{L}_\xi))=|\ker^1(\mathbb{Q},\mathbf{G})|\cdot ST^{\mathbf{G}^*}_e((\phi^*)^{\infty,p}\widetilde{\phi}^*_p\widetilde{\phi}^*_\infty).$$

The proof is finished by comparing (6.11) and Lemma 6.1 in light of the transfer identity (6.9). □

Theorem 6.7. *Under assumptions (A1)-(A2) and (B1)-(B2) on the PEL datum, the following holds in* $\mathrm{Groth}(\mathbf{G}(\mathbb{A}^{\infty,p})\times J_{\mathbf{b}}(\mathbb{Q}_p))$.

$$[\mathrm{Red}^{(\mathbf{b})}(H(\mathrm{Sh},\mathscr{L}_\xi))]=\epsilon_{\mathbf{G}}\cdot d(\mathbf{G}_{\mathbb{R}})\cdot[H_c(\mathrm{Ig}_{\Sigma_{\mathbf{b}}},\mathscr{L}_\xi)]$$

Proof. Proposition 6.3, Proposition 6.6 and Lemma 6.5 imply that for any acceptable function $\phi^{\infty,p}\phi'_p$,

$$\begin{aligned}\epsilon_{\mathbf{G}}\cdot d(\mathbf{G}_{\mathbb{R}})\cdot\mathrm{tr}\,(\phi^{\infty,p}\phi'_p|\iota_l H_c(\mathrm{Ig}_{\Sigma_{\mathbf{b}}},\mathscr{L}_\xi)) &= \mathrm{tr}\,(\phi^{\infty,p}\widetilde{\phi}^*_p|\iota_l H(\mathrm{Sh},\mathscr{L}_\xi))\\ &= \mathrm{tr}\,(\phi^{\infty,p}\phi'_p|\iota_l\mathrm{Red}^{(\mathbf{b})}(H(\mathrm{Sh},\mathscr{L}_\xi))).\end{aligned}$$

Now we use lemma 6.4 of [Shi09] to conclude. □

Remark 6.8. We can be explicit about the constants $\epsilon_{\mathbf{G}}$ and $d(\mathbf{G}_{\mathbb{R}})$. Recall the definition of (p_τ,q_τ) from §5. By an easy computation $\epsilon_{\mathbf{G}}=1$ unless $p_\tau=q_\tau$ for every $\tau\in\Phi$ in which case $\epsilon_{\mathbf{G}}=2$. (Recall $\epsilon_{\mathbf{G}}=[K_\infty:K^0_\infty]$ where K_∞ is a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$. The group K^0_∞ is the preimage of $\mathbb{R}^\times_{>0}$ in K_∞ under the map $\mathbf{G}(\mathbb{R})\to\mathbb{G}_m(\mathbb{R})=\mathbb{R}^\times$ sending (δ,g) to g. If $p_\tau\neq q_\tau$ for some $\tau\in\Phi$ then $K_\infty=K^0_\infty$. Otherwise the image of K_∞ in $\mathbb{R}^\times$ is $\mathbb{R}^\times$ and $[K_\infty:K^0_\infty]=2$.) In both cases

$$\epsilon_{\mathbf{G}}\cdot d(\mathbf{G}_{\mathbb{R}})=\prod_{\tau\in\Phi}\binom{p_\tau+q_\tau}{p_\tau}.$$

Of course, the values of $\epsilon_{\mathbf{G}}$ and $d(\mathbf{G}_{\mathbb{R}})$ do not depend on the choice of the CM-type Φ for $\mathbf{F}$.

Remark 6.9. Theorem 6.7 may be viewed as an extension of theorem V.5.4 of [HT01], which deals with only those pairs (p_τ,q_τ) such that (p_τ,q_τ) equals $(1,n-1)$ for one τ and $(0,n)$ for all other $\tau\in\Phi$. However our result relies on the assumption that the PEL datum is unramified. Such an assumption was unnecessary in [HT01].

We end with a variant of Lemma 6.5 which will be used in §8.1.

Lemma 6.10. *Let* $f'\in C^\infty_c(J_b(\mathbb{Q}_p))$. *Then there exists* $f\in C^\infty_c(G(\mathbb{Q}_p))$ *such that for every* $\pi\in\mathrm{Groth}(G(\mathbb{Q}_p))$,

$$\mathrm{tr}\left(\mathrm{Red}^{(b)}(\pi)\right)(f')=\mathrm{tr}\,\pi(f).$$

Proof. Recall from §6.2 that $\mathbf{G}(\mathbb{Q}_p)\simeq\mathbb{Q}_p^\times\times G(\mathbb{Q}_p)$ and $J_{\mathbf{b}}(\mathbb{Q}_p)\simeq\mathbb{Q}_p^\times\times J_b(\mathbb{Q}_p)$. The lemma is proved by the same method used to prove Lemma 6.5. (One only has to disregard the $\mathbb{Q}_p^\times$-factor.) □

6.4. **Description via the local Langlands correspondence.** The dual group of $\mathbf{G}_{\mathbb{Q}_p}$ may be described as

$$\widehat{\mathbf{G}_{\mathbb{Q}_p}}\simeq\widehat{GL_1}\times\widehat{G}\simeq GL_1(\mathbb{C})\times\prod_{\tau\in\mathrm{Hom}_{\mathbb{Q}_p}(F,\overline{\mathbb{Q}}_p)}GL_n(\mathbb{C}).\tag{6.13}$$

The L-group ${}^L\mathbf{G}_{\mathbb{Q}_p}:=\widehat{\mathbf{G}_{\mathbb{Q}_p}}\rtimes W_{\mathbb{Q}_p}$ is given by the rule

$$w(g_0,(g_\tau))w^{-1}=(g_0,(g'_\tau)),\qquad\text{where } g'_\tau=g_{w^{-1}\tau}$$

for every $w\in W_{\mathbb{Q}_p}$ and $(g_0,(g_\tau))\in\widehat{\mathbf{G}_{\mathbb{Q}_p}}$. Likewise ${}^LG$ is defined, by ignoring the $GL_1(\mathbb{C})$-factor.

Recall from §6.2 the identification that $\boldsymbol{\mu}=(\boldsymbol{\mu}_0,\boldsymbol{\mu}_v)=(\boldsymbol{\mu}_0,\mu)$. We can associate to $\boldsymbol{\mu}$ a finite dimensional representation $r_{\boldsymbol{\mu}}$ of ${}^L\mathbf{G}_E=\widehat{\mathbf{G}}\rtimes W_E$ as in [Lan79, p.238]. In particular, the restriction of $r_{\boldsymbol{\mu}}$ to $\widehat{G}$ has highest weight $\boldsymbol{\mu}$. A representation r_μ of ${}^LG_E$ (resp. $r_{\boldsymbol{\mu}_0}$ of ${}^L(GL_1)_{\mathbb{Q}_p}$) is defined analogously.

We can be explicit about $r_{\boldsymbol{\mu}_0}$, r_μ and $r_{\boldsymbol{\mu}}$. The one-dimensional representation $r_{\boldsymbol{\mu}_0}$ is given by $r_{\boldsymbol{\mu}_0}(g,w) = g^{-1}$. If $(\mathrm{Std})_a$ denotes the standard representation of GL_a ($a \in \mathbb{Z}_{>0}$) then

$$r_\mu|_{\widehat{G}} = \otimes_\tau \left(\bigwedge^{p_\tau} (\mathrm{Std})_n^\vee \right), \qquad r_{\boldsymbol{\mu}}|_{\widehat{\mathbf{G}_{\mathbb{Q}_p}}} = (\mathrm{Std})_1^{-1} \otimes r_\mu|_{\widehat{G}}$$

and for $w \in W_E$, $r_{\boldsymbol{\mu}}(1 \rtimes w)$ sends $(v_0, (v_\tau))$ to $(v_0, (v'_\tau))$ with $v'_\tau = v_{w^{-1}\tau}$, where $v_\tau \in \wedge^{p_\tau}(\mathrm{Std})_n^{p_\tau}$. This is well-defined since $p_{w^{-1}\tau} = p_\tau$ for every $w \in W_E$. To define $r_\mu(1 \rtimes w)$ for $w \in W_E$, we simply ignore v_0 from the description of $r_{\boldsymbol{\mu}}(1 \rtimes w)$.

Suppose that a prime q splits in $\mathbf{K}$ as $q = xx^c$. In analogy with (6.4), there is an isomorphism $\mathbf{G}(\mathbb{Q}_q) \simeq \mathbb{Q}_q^\times \times \prod_{y|x} GL_n(\mathbf{F}_y)$. For $\Pi^\infty \in \mathrm{Irr}(\mathbf{G}(\mathbb{A}^\infty))$, its q-component will be written as

$$\Pi_q = \Pi_{q,0} \otimes \left(\bigotimes_{y|x} \Pi_y \right).$$

In particular, when $q = p$, we write $\Pi_p = \Pi_{p,0} \otimes \Pi_v$.

We need to define

$$\mathrm{LL}(\Pi_{p,0}) : W_{\mathbb{Q}_p} \to {}^L GL_1, \quad \mathrm{LL}(\Pi_v) : W_{\mathbb{Q}_p} \to {}^L G, \quad \mathrm{LL}(\Pi_p) : W_{\mathbb{Q}_p} \to {}^L \mathbf{G}_{\mathbb{Q}_p} \tag{6.14}$$

(up to isomorphism) which may be thought of as the images of $\Pi_{p,0}$, Π_v and Π_p under the semisimple Langlands map. The objects of (6.14) may be identified with the continuous cohomology classes in $H^1(W_{\mathbb{Q}_p}, \widehat{GL_1})$, $H^1(W_{\mathbb{Q}_p}, \widehat{G})$ and $H^1(W_{\mathbb{Q}_p}, \widehat{\mathbf{G}_{\mathbb{Q}_p}})$, respectively. Define $\mathrm{LL}(\Pi_{p,0})$ as the character $W_{\mathbb{Q}_p} \to \mathbb{C}^\times$ given as $\Pi_{p,0} \circ \mathrm{Art}_{\mathbb{Q}_p}^{-1}$ where $\mathrm{Art}_{\mathbb{Q}_p} : \mathbb{Q}_p^\times \xrightarrow{\sim} W_{\mathbb{Q}_p}^{\mathrm{ab}}$ is the local Artin map matching $p \in \mathbb{Q}_p^\times$ with a lift of the geometric Frobenius. Noting that

$$H^1(W_{\mathbb{Q}_p}, \widehat{G}) = H^1(W_{\mathbb{Q}_p}, \mathrm{Ind}_{W_F}^{W_{\mathbb{Q}_p}} GL_n(\mathbb{C})) = H^1(W_F, GL_n(\mathbb{C})) = \mathrm{Hom}(W_F, GL_n(\mathbb{C})),$$

we define $LL(\Pi_v)$ so that its image in $\mathrm{Hom}(W_F, GL_n(\mathbb{C}))$ corresponds to $\Pi_v \in \mathrm{Irr}(G(\mathbb{Q}_p)) = \mathrm{Irr}(GL_n(F))$ via the semisimplification of the local Langlands map ([HT01, p.2]). Finally $\mathrm{LL}(\Pi_p)$ is the image of $(\mathrm{LL}(\Pi_{p,0}), LL(\Pi_v))$ via

$$H^1(W_{\mathbb{Q}_p}, \widehat{\mathbf{G}_{\mathbb{Q}_p}}) \simeq H^1(W_{\mathbb{Q}_p}, \widehat{GL_1}) \times H^1(W_{\mathbb{Q}_p}, \widehat{G})$$

induced from (6.13).

Let us mention the l-adic analogue of the previous discussion. The dual groups and L-groups may be defined by taking $\overline{\mathbb{Q}}_l$-points (rather than $\mathbb{C}$-points). Denote them by $\widehat{G}(\overline{\mathbb{Q}}_l)$, ${}^L G(\overline{\mathbb{Q}}_l)$, and so on. (In particular ${}^L G$ and ${}^L G(\overline{\mathbb{Q}}_l)$ may be identified via ι_l if topology is disregarded.) We can define $r_{\boldsymbol{\mu}_0}$, r_μ and $r_{\boldsymbol{\mu}}$ as l-adic representations (rather than complex representations) of ${}^L GL_1(\overline{\mathbb{Q}}_l)$, ${}^L G(\overline{\mathbb{Q}}_l)$ and ${}^L \mathbf{G}_{\mathbb{Q}_p}(\overline{\mathbb{Q}}_l)$, respectively, in the same way as before. By abuse of notation, the l-adic analogues will still be denoted by $r_{\boldsymbol{\mu}_0}$, r_μ and $r_{\boldsymbol{\mu}}$.

Proposition 6.11. *(Kottwitz, Harris-Taylor) Let $\Pi^\infty \in \mathscr{A}^\infty_{\iota_l\xi}(\mathbf{G})$ be such that there exists a prime q split in $\mathbf{K}$ as $q = xx^c$ such that Π_y is supercuspidal for some place y of $\mathbf{F}$ dividing x. Then*

$$[R_{l,\xi,\boldsymbol{\mu}}(\iota_l^{-1}\Pi^\infty)|_{W_{\mathbf{E}_w}}] = (-1)^{q(\mathbf{G}_\mathbb{R})} \cdot a(\Pi^\infty) \cdot [(r_{\iota_p^{-1}\boldsymbol{\mu}} \circ \iota_l^{-1}\mathrm{LL}(\Pi_p))|_{W_{\mathbf{E}_w}} \otimes |\cdot|^{-\dim \mathrm{Sh}/2}]$$

for some nonzero $a(\Pi^\infty) \in \mathbb{Z}$. (The character $|\cdot|$ on $W_{\mathbf{E}_w}$ is defined as in §1.) In particular,

$$\dim[R_{l,\xi,\boldsymbol{\mu}}(\iota_l^{-1}\Pi^\infty)|_{W_{\mathbf{E}_w}}] = (-1)^{q(\mathbf{G}_\mathbb{R})} a(\Pi^\infty) \cdot \prod_{\tau \in \Phi} \binom{p_\tau + q_\tau}{p_\tau}.$$

Proof. The first identity is deduced from the results of Kottwitz and Harris-Taylor in [Far04, Thm A.7.2]. The sign $(-1)^{q(\mathbf{G}_\mathbb{R})}$ comes from the fact that $R_{l,\xi,\boldsymbol{\mu}}(\Pi^\infty)$ appears in $H^{q(\mathbf{G}_\mathbb{R})}(\mathrm{Sh},\mathscr{L}_\xi)$, as explained by Fargues. The last identity is straightforward in view of the description of $r_{\iota_p^{-1}\boldsymbol{\mu}}$ above and Remark 6.8. □

Corollary 6.12. *In* $\mathrm{Groth}(\mathbf{G}(\mathbb{A}^{\infty,p})\times J_\mathbf{b}(\mathbb{Q}_p))$,

$$[\iota_l H_c(\mathrm{Ig}_{\Sigma_b},\mathscr{L}_\xi)] = (-1)^{q(\mathbf{G}_\mathbb{R})}\sum_{\Pi^\infty\in\mathscr{A}^\infty_{\iota_l\xi}(\mathbf{G})} a(\Pi^\infty)[\Pi^{\infty,p}][\mathrm{Red}^{(\mathbf{b})}(\Pi_p)].$$

Proof. By (5.2), Theorem 6.7 and Remark 6.8, $\epsilon_\mathbf{G}\cdot d(\mathbf{G}_\mathbb{R})\cdot[\iota_l H_c(\mathrm{Ig}_{\Sigma_b},\mathscr{L}_\xi)]$ equals

$$\begin{aligned} &\textstyle\sum_{\Pi^\infty}\dim[R_{l,\xi,\boldsymbol{\mu}}(\Pi^\infty)|_{W_{\mathbf{E}_w}}]\cdot[\Pi^{\infty,p}][\mathrm{Red}^{(\mathbf{b})}(\Pi_p)]\\ =\ &(-1)^{q(\mathbf{G}_\mathbb{R})}\cdot\epsilon_\mathbf{G}\cdot d(\mathbf{G}_\mathbb{R})\cdot\textstyle\sum_{\Pi^\infty}a(\Pi^\infty)\cdot[\Pi^{\infty,p}][\mathrm{Red}^{(\mathbf{b})}(\Pi_p)].\end{aligned}$$

The corollary follows. □

7. Main theorem

Our main theorem is an identity involving $\mathrm{Mant}_{b,\mu}$ where all $b\in B(G,-\mu)$ are considered simultaneously. As before, fix a prime p and let (F,V,μ) be a partial unramified Rapoport-Zink EL datum and keep the previous notation. In particular, $n=\dim_F V$.

We would like to realize (F,V,μ) as a localization of a PEL datum for Shimura varieties. As a preparation, we observe that any PEL datum $(\mathbf{B},*,\mathbf{V},\langle\cdot,\cdot\rangle,h)$ satisfying assumptions (A1)-(A2) in §5 and (B1)-(B2) in §6 can be extended to an unramified PEL datum. Let us sketch how this can be done (cf. [HT01, p.56-57]). There are decompositions $\mathbf{V}_{\mathbb{Q}_p}=\mathbf{V}_v\oplus\mathbf{V}_{v^c}$ and $\mathbf{B}_{\mathbb{Q}_p}\simeq M_n(\mathbf{F}_v)\times M_n(\mathbf{F}_{v^c})$ as we saw in §6.2. We may assume that the induced action of $*$ on $M_n(\mathbf{F}_v)\times M_n(\mathbf{F}_{v^c})$ sends (g_v,g_{v^c}) to $(g^c_{v^c},g^c_v)$. Choose a $*$-stable $\mathbb{Z}_{(p)}$-maximal order $\mathcal{O}_\mathbf{B}$ in $\mathbf{B}$ such that $\mathcal{O}_\mathbf{B}\otimes_\mathbb{Z}\mathbb{Z}_p=M_n(\mathcal{O}_{\mathbf{F}_v})\times M_n(\mathcal{O}_{\mathbf{F}_{v^c}})$ under the decomposition of $\mathbf{B}_{\mathbb{Q}_p}$. Also choose an $M_n(\mathcal{O}_{\mathbf{F}_v})$-stable lattice Λ_v in $\mathbf{V}_v$. Since $\mathbf{V}_v$ and $\mathbf{V}_{v^c}$ are in perfect duality under $\langle\cdot,\cdot\rangle$, we may take Λ_{v^c} as the $\mathbb{Z}_p$-dual of Λ_v, and then set $\Lambda_0:=\Lambda_v\oplus\Lambda_{v^c}$. It is easy to check that $(\mathbf{B},\mathcal{O}_\mathbf{B},*,\mathbf{V},\Lambda_0,\langle\cdot,\cdot\rangle,h)$ is indeed an unramified PEL datum. Let ϵ_v be an idempotent of $M_n(\mathcal{O}_{F_v})$ such that $\epsilon_v\Lambda_v$ is a free $\mathcal{O}_{F_v}$-module of rank n. (Recall that $\mathrm{rank}_{\mathcal{O}_{F_v}}\Lambda_v=n^2$.)

Definition 7.1. Let (F,V,μ) be a partial unramified Rapoport-Zink EL datum. Suppose that a PEL datum $(\mathbf{B},*,\mathbf{V},\langle\cdot,\cdot\rangle,h)$ satisfies (A1)-(A2) and (B1)-(B2). Let v be a place of $\mathbf{F}$ dividing p. We say that (F,V,μ) is a *localization* of $(\mathbf{B},*,\mathbf{V},\langle\cdot,\cdot\rangle,h)$ at v if

- $F\simeq\mathbf{F}_v$,
- $V\simeq\epsilon_v\mathbf{V}_v$ and
- μ is $G(\overline{\mathbb{Q}}_p)$-conjugate to $\iota_p^{-1}\boldsymbol{\mu}_v$.

As before, let E be the reflex field associated to (F,V,μ). Lemma 6.2 shows that $\mathbf{E}_w=E$ as subfields of $\overline{\mathbb{Q}}_p$. Given this setup, an urgent question is whether a partial Rapoport-Zink datum always arises from a PEL datum.

Lemma 7.2. *For any partial unramified Rapoport-Zink EL datum (F,V,μ), there exists a Shimura PEL datum $(\mathbf{B},*,\mathbf{V},\langle\cdot,\cdot\rangle,h)$ which satisfies (A1)-(A2) and (B1)-(B2) and localizes to (F,V,μ) at a place of $\mathbf{F}$ in the sense of Definition 7.1.*

Proof. [Far04, Prop 8.1.3]. □

Definition 7.3. Let (F,V,μ) be a partial unramified Rapoport-Zink datum. We say that $\pi\in\mathrm{Irr}(GL_n(F))$ is *accessible* with respect to (F,V,μ) if there exist

- a PEL datum $(\mathbf{B},*,\mathbf{V},\langle\cdot,\cdot\rangle,h)$ with associated $\mathbb{Q}$-group $\mathbf{G}$, which

  – satisfies (A1)-(A2) and (B1)-(B2), and
  – localizes to (F, V, μ) at a place v of $\mathbf{F}$,
- an unramified character $\omega : F^\times \to \mathbb{C}^\times$,
- an irreducible algebraic representation ξ of $\mathbf{G}$ over $\overline{\mathbb{Q}}_l$,
- a representation $\Pi^\infty \in \mathscr{A}^\infty_{\iota_l \xi}(\mathbf{G})$, and
- a prime $q(\neq p)$ split in $\mathbf{K}$ as $q = xx^c$ such that $\mathbf{B}_{\mathbb{Q}_q} \simeq M_n(\mathbf{F}_q)$ (note that there are infinitely many q having this property)

such that $\Pi_v \simeq \pi \otimes (\omega \circ \det)$ and Π_y is supercuspidal for some place y of $\mathbf{F}$ dividing x.

Lemma 7.4. *An accessible representation $\pi \in \mathrm{Irr}(G(\mathbb{Q}_p))$ is tempered. The set of accessible π (with respect to (F, V, μ)) is Zariski-dense in $\mathfrak{z}_2^{G(\mathbb{Q}_p)}$. Every $\pi \in \mathrm{Irr}^2(G(\mathbb{Q}_p))$ is accessible.*

Proof. To show the first assertion, it suffices to prove that Π_v as above is tempered, which follows from [HL04, Thm 3.1.5]. Let us show the second assertion. Fix an auxiliary prime q a representation ξ as in Definition 7.3. Take $S = \{p, q\}$ and choose any $(L_q, D_q) \in \mathfrak{S}_2(\mathbf{G}(\mathbb{Q}_q))$ such that $L_q = \mathbf{G}_{\mathbb{Q}_q}$ and that D_q is an orbit of supercuspidal representations of $\mathbf{G}(\mathbb{Q}_q)$. (We are adopting the notation of §2 and §3.) Now Proposition 3.1 implies the second assertion of the lemma if we check (i)-(iv) in the beginning of §3. Parts (i) and (iii) are satisfied by (C1), (6.4) and the assumption on q. Part (ii) follows from the fact that $\mathbf{G}$ is a unitary similitude group. Let us check part (iv). The existence of a supercuspidal component for Π implies that the quadratic base change is unconditional for Π ([HL04]) and that the base change image is cuspidal, thus generic everywhere. So Π_z is generic at $z \in T$. (Compare with [HT01, Cor VI.2.4].)

The last assertion of the lemma is an immediate consequence of the second one. Indeed, let $(G, D) \in \mathfrak{S}_2(G)$ be the component containing π. (Since π is square-integrable, we have $L = G$ in the notation of §2.1.) Every $\pi' \in V(G, D)$ has the form $\pi \otimes (\omega \circ \det)$ for an unramified character $\omega : F^\times \to \mathbb{C}^\times$. But the second assertion tells us that there exists such a π' which is accessible. □

Let $\pi \in \mathrm{Irr}_l(G(\mathbb{Q}_p))$. Recall that $\mathrm{LL}(\iota_l \pi) : W_{\mathbb{Q}_p} \to {}^L G$ is defined in §6.4. Define $\mathrm{LL}_l(\pi) : W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{Q}}_l)$ by $\mathrm{LL}_l(\pi) := \iota_l^{-1} \mathrm{LL}(\iota_l \pi)$.

Theorem 7.5. *Let $\pi \in \mathrm{Irr}_l(G(\mathbb{Q}_p))$. Suppose that $\iota_l \pi$ is accessible with respect to (F, V, μ). Then*

$$\sum_{b \in B(G, -\mu)} \mathrm{Mant}_{b,\mu}(\mathrm{Red}^{(b)}(\pi)) = [\pi] \left[(r_\mu \circ \mathrm{LL}_l(\pi)|_{W_E}) \otimes |\cdot|^{-\dim \mathrm{Sh}/2} \right] \tag{7.1}$$

in $\mathrm{Groth}(G(\mathbb{Q}_p) \times W_E)$.

Remark 7.6. The author speculates that Theorem 7.5 is true for any $\pi \in \mathrm{Irr}_l(G(\mathbb{Q}_p))$ without the accessibility condition. See Conjecture 8.1.

Proof of Theorem 7.5. Choose $(\mathbf{B}, *, \mathbf{V}, \langle \cdot, \cdot \rangle, h)$ and ω, ξ, Π^∞ as in Definition 7.3. Using the notation of §6.4, we write $\Pi_p = \Pi_{p,0} \otimes \Pi_v$. We only treat the case $\omega = 1$, namely $\Pi_v \simeq \iota_l \pi$. The general case is easily reduced to this case by Lemma 4.9.

By Proposition 6.11, the following holds in $\mathrm{Groth}(G(\mathbb{A}^\infty) \times W_{\mathbf{E}_w})$.

$$\begin{aligned} & (-1)^{q(\mathbb{G}_\mathbb{R})} [R_{l,\xi,\boldsymbol{\mu}}(\iota_l^{-1} \Pi^\infty)|_{W_{\mathbf{E}_w}}] \\ = \; & a(\Pi^\infty) \cdot [(r_{\iota_p^{-1} \boldsymbol{\mu}} \circ \iota_l^{-1} \mathrm{LL}(\Pi_p))|_{W_{\mathbf{E}_w}} \otimes |\cdot|^{-\dim \mathrm{Sh}/2}] \\ = \; & a(\Pi^\infty) \cdot [r_{\boldsymbol{\mu}_0} \circ \iota_l^{-1} \mathrm{LL}(\Pi_{p,0})|_{W_{\mathbf{E}_w}}][r_\mu \circ \iota_l^{-1} \mathrm{LL}(\Pi_v)|_{W_{\mathbf{E}_w}} \otimes |\cdot|^{-\dim \mathrm{Sh}/2}] \\ = \; & a(\Pi^\infty) \cdot [r_{\boldsymbol{\mu}_0} \circ \iota_l^{-1} \mathrm{LL}(\Pi_{p,0})|_{W_E}][r_\mu \circ \mathrm{LL}_l(\pi)|_{W_E} \otimes |\cdot|^{-\dim \mathrm{Sh}/2}] \end{aligned} \tag{7.2}$$

On the other hand, by Proposition 5.4 and Corollary 6.12, $(-1)^{q(\mathbb{G}_\mathbb{R})}H(\mathrm{Sh},\mathscr{L}_\xi)|_{W_{\mathbf{E}_w}}$ equals

$$\begin{aligned}&\sum_{\mathbf{b}\in B(\mathbf{G}_{\mathbb{Q}_p},-\boldsymbol{\mu})}\sum_{\Xi^\infty} a(\Xi^\infty)\mathrm{Mant}_{\mathbf{b},\boldsymbol{\mu}}\left([\iota_l^{-1}\Xi^{\infty,p}]\mathrm{Red}^{(\mathbf{b})}[\iota_l^{-1}\Xi_p]\right)\\ =&\sum_{b\in B(G,-\mu)}\sum_{\Xi^\infty} a(\Xi^\infty)[\iota_l^{-1}\Xi^{\infty,p}][\mathrm{Mant}_{\mathbf{b}_0,\boldsymbol{\mu}_0}(\iota_l^{-1}\Xi_{p,0})|_{W_E}][\mathrm{Mant}_{b,\mu}(\mathrm{Red}^{(b)}(\iota_l^{-1}\Xi_v))]\end{aligned}$$

where the second sum in each row runs over $\Xi^\infty\in\mathscr{A}^\infty_{\iota_l\xi}(\mathbf{G})$. From both sides we take the parts on which $\mathbf{G}(\mathbb{A}^{\infty,p})$ acts via $\Pi^{\infty,p}$. Then

$$\begin{aligned}&[\iota_l^{-1}\Pi^\infty][R_{l,\xi,\boldsymbol{\mu}}(\iota_l^{-1}\Pi^\infty)|_{W_E}]\\ =&\sum_{b\in B(G,-\mu)} a(\Pi^\infty)[\iota_l^{-1}\Pi^{\infty,p}][\mathrm{Mant}_{\mathbf{b}_0,\boldsymbol{\mu}_0}(\iota_l^{-1}\Pi_{p,0})|_{W_E}][\mathrm{Mant}_{b,\mu}(\mathrm{Red}^{(b)}(\pi))].\end{aligned}\tag{7.3}$$

Here we used the fact that any $\Xi^\infty\in\mathscr{A}^\infty_{\iota_l\xi}(\mathbf{G})$ such that $\Xi^{\infty,p}\simeq\Pi^{\infty,p}$ must be isomorphic to Π^∞. Indeed, after applying quadratic base change ([HL04, Thm 3.1.3]), we deduce that $\Xi_p\simeq\Pi_p$ from the strong multiplicity one for inner forms of general linear groups after quadratic base change. (cf. [HT01, VI.2.3])

Note that $[\iota_l^{-1}\Pi_{p,0}][r_{\boldsymbol{\mu}_0}\circ\iota_l^{-1}\mathrm{LL}(\Pi_{p,0})]=[\mathrm{Mant}_{\mathbf{b}_0,\boldsymbol{\mu}_0}(\iota_l^{-1}\Pi_{p,0})]$ is a rephrase of the classical Lubin-Tate theory (over $\mathbb{Q}_p$) for formal groups (cf. $n=1$ case of [Shic, Prop 2.2.(i)]). Thus the desired equality follows from (7.2) and (7.3).

□

Corollary 7.7. *Let (F,V,μ,b) be an unramified Rapoport-Zink EL datum. Suppose that $b\in B(G,-\mu)$ is basic (Definition 4.3). For a representation $\rho\in\mathrm{Irr}_l(J_b(\mathbb{Q}_p))$ such that $\mathrm{JL}(\rho)$ is supercuspidal, the following holds in $\mathrm{Groth}(G(\mathbb{Q}_p)\times W_E)$.*

$$\mathrm{Mant}_{b,\mu}(\rho)=e(J_b)[\mathrm{JL}(\rho)]\left[r_\mu\circ\mathrm{LL}(\mathrm{JL}(\rho))|_{W_E}\ |\cdot|^{-\sum_\tau p_\tau q_\tau/2}\right]\tag{7.4}$$

Proof. Put $\pi=\mathrm{JL}(\rho)$. Recall that b is non-basic if and only if P_b is a proper parabolic subgroup of G. Thus $\mathrm{Red}^{(b)}(\pi)=0$ if b is not basic. When b is basic, $e(J_b)\cdot\mathrm{Red}^{(b)}(\pi)=\mathrm{LJ}(\pi)=\rho$. Since π is accessible with respect to (F,V,μ) by Lemma 7.4, the corollary is an easy consequence of Theorem 7.5. □

Remark 7.8. Since the Ext groups that appear in the definition of $\mathrm{Mant}_{b,\mu}$ vanish in positive degrees when $\mathrm{JL}(\rho)$ is supercuspidal (in which case ρ is a supercuspidal representation of $J_b(\mathbb{Q}_p)$, so there is no non-split extension of ρ as a smooth $J_b(\mathbb{Q}_p)$-representation), Corollary 7.7 recovers one of the main results of Fargues (Theorem 8.1.4 of [Far04]). In fact we deal with slightly more cases in that we obtain results even when $J_b(\mathbb{Q}_p)$ is not the unit group of a division algebra, namely when ($r=1$ and) $m_1>1$ in the notation of (4.4). Although we use the local-global compatibility (Proposition 6.11) as Fargues, our proof is different from his in that we deduce the result from a study of Igusa varieties and the first basic identity (Proposition 5.4), whereas he relates Rapoport-Zink spaces directly to the basic strata of Shimura varieties via p-adic uniformization and uses techniques in rigid analytic geometry.

8. Towards a complete description of $\mathrm{Mant}_{b,\mu}$ in the EL case

We would like to put Theorem 7.5 into context by explaining how $\mathrm{Mant}_{b,\mu}$ may be completely determined in inductive steps (§8.2) if we can prove a strengthening of Theorem 7.5 as well as a conjecture of Harris on non-basic Rapoport-Zink spaces. These are formulated as Conjectures 8.1 and 8.6 below. Actually Conjecture 8.1 is implied by Theorem 7.5 and Conjecture 8.3. The latter conjecture was proposed by Taylor in private communication. Roughly speaking, it predicts that

$\mathrm{Mant}_{b,\mu}$ is algebraic in nature. In §8.3, we explain how a few simple cases (which were not known before) of Harris's conjecture can be derived from Theorem 7.5.

In this section (F, V, μ, b) and (F', V', μ', b') will always denote unramified Rapoport-Zink data of EL type. Test functions in $C_c^\infty(G(\mathbb{Q}_p))$, $C_c^\infty(J_b(\mathbb{Q}_p))$ and so on, will take values in $\overline{\mathbb{Q}}_l$ rather than $\mathbb{C}$. The results of §2 carry over if the coefficient field $\mathbb{C}$ is replaced by $\overline{\mathbb{Q}}_l$ everywhere.

8.1. **Conjectures.** The goal of §8.1 is to state a few conjectures that would lead us to understand $\mathrm{Mant}_{b,\mu}$. In addition, relationships among those conjectures will be examined.

For $\phi \in C_c^\infty(G(\mathbb{Q}_p))$, $w \in W_E$ and $R \in \mathrm{Groth}(G(\mathbb{Q}_p) \times W_E)$, we will often consider $\mathrm{tr}\,(\phi \times w|R)$ defined as follows. Write $R = \sum_{i\in I} n_i[\pi_i][\sigma_i]$ where $n_i \in \mathbb{Z}$, $\pi_i \in \mathrm{Irr}_l(G(\mathbb{Q}_p))$, σ_i is a finite dimensional l-adic representation of W_E, and I is a finite index set. Then define

$$\mathrm{tr}\,(\phi \times w|R) := \sum_{i\in I} n_i \mathrm{tr}\,(\phi|\pi_i)\mathrm{tr}\,(w|\sigma_i).$$

It is easy to check that this value is independent of the expansion of R.

Conjecture 8.1. *Theorem 7.5 holds for all* $\pi \in \mathrm{Irr}_l(G(\mathbb{Q}_p))$ *(without the assumption on accessibility).*

Remark 8.2. When $\dim(\mu) = 1$, the conjecture follows from the results of Harris and Taylor. (See the last paragraph of §8.1.) In the Drinfeld case (cf. Remark 4.2), which is not covered in this paper, the analogue of Conjecture 8.1 is known by [Dat07, Thm A]. In the latter case $B(G, -\mu)$ has only one element, which is basic.

Conjecture 8.3. *(Taylor) For every* $\phi \in C_c^\infty(G(\mathbb{Q}_p))$ *and* $w \in W_E$*,*

(i) $\rho \mapsto \mathrm{tr}\,(\phi \times w|\mathrm{Mant}_{b,\mu}(\rho))$ *is a trace function on* $\mathrm{Groth}(J_b(\mathbb{Q}_p))$ *and*

(ii) $\pi \mapsto \mathrm{tr}\,(\phi \times w|\mathrm{Mant}_{b,\mu}(\mathrm{Red}^{(b)}(\pi)))$ *is a trace function on* $\mathrm{Groth}(G(\mathbb{Q}_p))$*.*

Lemma 8.4. *In Conjecture 8.3, part (i) implies part (ii). If b is basic, the converse is true.*

Proof. We begin with the implication (i)⇒(ii). Part (i) of Conjecture 8.3 tells us that there exists a function $f' \in C_c^\infty(J_b(\mathbb{Q}_p))$ such that for every ϕ and w as above and $\rho \in \mathrm{Groth}(J_b(\mathbb{Q}_p))$,

$$\mathrm{tr}\,(\phi \times w|\mathrm{Mant}_{b,\mu}(\rho)) = \mathrm{tr}\,\rho(f'). \tag{8.1}$$

On the other hand, Lemma 6.10 allows us to choose $f \in C_c^\infty(G(\mathbb{Q}_p))$ such that

$$\mathrm{tr}\,\mathrm{Red}^{(b)}(\pi)(f') = \mathrm{tr}\,\pi(f)$$

for every $\pi \in \mathrm{Groth}(G(\mathbb{Q}_p))$. This identity together with (8.1) for $\rho = \mathrm{Red}^{(b)}(\pi)$ shows part (ii) of Conjecture 8.3.

Now assume that b is basic and that (ii) of the conjecture is true. Then there exists $f \in C_c^\infty(G(\mathbb{Q}_p))$ such that

$$\mathrm{tr}\,(\phi \times w|\mathrm{Mant}_{b,\mu}(LJ(\pi))) = \mathrm{tr}\,\pi(f)$$

for every $\pi \in \mathrm{Groth}(G(\mathbb{Q}_p))$. (We simply write LJ for LJ_{M_b,J_b}.) We deduce that

$$\mathrm{tr}\,(\phi \times w|\mathrm{Mant}_{b,\mu}(\rho))) = \mathrm{tr}\,(\mathrm{JL}(\rho))(f)$$

for every $\rho \in \mathrm{Groth}(J_b(\mathbb{Q}_p))$ by plugging in $\pi = \mathrm{JL}(\rho)$. By Lemma 2.15, there exists $f' \in C_c^\infty(J_b(\mathbb{Q}_p))$ such that $\mathrm{tr}\,(\mathrm{JL}(\rho))(f) = \mathrm{tr}\,\rho(f')$ for every $\rho \in \mathrm{Groth}(J_b(\mathbb{Q}_p))$. Hence (i) of Conjecture 8.3 follows. □

Lemma 8.5. *Part (ii) of Conjecture 8.3 implies Conjecture 8.1.*

Proof. For $\pi \in \mathrm{Irr}_l(G(\mathbb{Q}_p))$, let $X_1(\pi)$ (resp. $X_2(\pi)$) denote the left (resp. right) hand side of (7.1). Our goal is to show $X_1(\pi) = X_2(\pi)$. It will be helpful to define $X_1(\pi)$ and $X_2(\pi)$ for every $\pi \in \mathfrak{z}_2^G(\mathbb{C}) = \mathfrak{z}_2^{GL_n(F)}(\mathbb{C})$ (π may not be irreducible) and check $X_1(\pi) = X_2(\pi)$ for such a π. first. For $\pi \in \mathfrak{z}_2^G(\mathbb{C})$, we still take $X_1(\pi)$ as the left side of (7.1) but $X_2(\pi)$ is defined by

$$X_2(\pi) = [\pi]\left[r_\mu \circ \mathrm{LL}_l(\pi_0)|_{W_E}\ |\cdot|^{-\sum_\tau p_\tau q_\tau/2}\right]$$

where π_0 is any irreducible subquotient of π. We see that $X_2(\pi)$ is well-defined since $\mathrm{LL}_l(\pi_0)$ depends only on the supercuspidal support of π_0. (Recall that LL_l is the semisimplified local Langlands map.)

We verify that $X_1(\pi) = X_2(\pi)$ for every $\pi \in \mathfrak{z}_2^G(\mathbb{C})$. To prove the claim, note that for any $f \in C_c^\infty(G(\mathbb{Q}_p))$, $\pi \mapsto \mathrm{tr}\,(f|X_1(\pi))$ is a regular function on $\mathfrak{z}_2^G(\mathbb{C})$ by Conjecture 8.3.(ii). The same is true for $\pi \mapsto \mathrm{tr}\,(f|X_2(\pi))$ ([BD84, Prop 2.11]). Since $\mathrm{tr}\,(f|X_1(\pi)) = \mathrm{tr}\,(f|X_2(\pi))$ when π belongs to a Zariski dense subset of $\mathfrak{z}_2^G$ (Lemma 7.4, Theorem 7.5), it follows that the same equality holds for every $\pi \in \mathfrak{z}_2^G(\mathbb{C})$. Therefore $X_1(\pi) = X_2(\pi)$.

It remains to deduce $X_1(\pi) = X_2(\pi)$ for $\pi \in \mathrm{Irr}_l(G(\mathbb{Q}_p))$. It suffices to consider a non-generic π (i.e. a subquotient of a reducible point of $\mathfrak{z}_2^G(\mathbb{C})$); for other π the equality was already proved in the last paragraph. But any non-generic π can be written as a finite linear combination $\pi = \sum_i n_i \pi_i$ in $\mathrm{Groth}(G(\mathbb{Q}_p))$, where $n_i \in \mathbb{Z}$ and $\pi_i \in \mathfrak{z}_2^G(\mathbb{C})$, such that all π_i have the same supercuspidal support as π. Now it is elementary to deduce $X_1(\pi) = X_2(\pi)$ from $X_1(\pi_i) = X_2(\pi_i)$ for all i.

□

To introduce another conjecture, we set up some notation. Let (F, V_i, μ_i, b_i) be an unramified Rapoport-Zink datum of EL-type for $1 \le i \le r$ and define $G_i := R_{F/\mathbb{Q}_p} GL_F(V_i)$. We can construct (F, V, μ, b) by putting them together as follows. Set $V := \oplus_{i=1}^r V_i$ and $G := R_{F/\mathbb{Q}_p} GL_F(V)$. By composing with the obvious embedding $\prod_{i=1}^r G_i \hookrightarrow G$, we obtain $\mu : \mathbb{G}_m \to G$ from $(\mu_1, \ldots, \mu_r)$. The same embedding induces a map $\prod_{i=1}^r B(G_i, -\mu_i) \to B(G, -\mu)$, and we take b as the image of $(b_1, \ldots, b_r)$. In this situation, we write

$$b = \prod_{i=1}^r b_i \quad \text{and} \quad \mu = \prod_{i=1}^r \mu_i. \tag{8.2}$$

Note that $\dim(\mu) = \sum_{i=1}^r \dim(\mu_i)$.

We will be interested in the case where $b_1, \ldots, b_r$ are basic elements with mutually distinct slopes. (See Remark 4.4 for the notion of slope.) Denote by E_i the reflex field for (F, V_i, μ_i, b_i). It is easy to check that $E_i \subset E$ as subfields of $\overline{\mathbb{Q}}_p$. The following conjecture is due to Harris ([Har01, Conj 5.2]).

Conjecture 8.6. *(Harris) In the situation of* (8.2)*, suppose that* $b_1, \ldots, b_r$ *are basic elements with mutually distinct slopes. Then*

$$\mathrm{Mant}_{b,\mu}(\otimes_{i=1}^r \rho_i) = \mathrm{Ind}_{P_b}^G(\otimes_{i=1}^r \mathrm{Mant}_{b_i,\mu_i}(\rho_i)|_{W_E}). \tag{8.3}$$

(Recall that Ind *denotes the non-normalized induction.)*

Remark 8.7. If true, the conjecture implies that supercuspidal representations of $G(\mathbb{Q}_p)$ appear in the image of $\mathrm{Mant}_{b,\mu}$ only when b is basic.

Remark 8.8. One should make the right choice of parabolic subgroup in (8.3) as the non-normalized parabolic induction depends on that choice. Let $-\lambda_i$ be the slope for b_i (Remark 4.5). If $-\lambda_i$ are ordered so that $-\lambda_1 > \cdots > -\lambda_r$ (which can be assumed without loss of generality), then P_b consists of block upper triangular matrices. Compare with the definition of P_b in §4.1.

Lemma 8.9. *Suppose that Conjecture 8.6 is known. If (i) of Conjecture 8.3 is true whenever* b *is basic, then it is also true for every non-basic* b.

Proof. Let $b = \coprod_{i=1}^{r} b_i$ be non-basic ($r > 1$). Let ϕ and w be as in Conjecture 8.3. Recall the standard construction of the constant term $\phi^{(M_b)} \in C_c^\infty(M_b(\mathbb{Q}_p))$ which has the property that for every $\pi_{M_b} \in \mathrm{Groth}(M_b(\mathbb{Q}_p))$,

$$\mathrm{tr}\,\pi_{M_b}(\phi^{(M_b)}) = \mathrm{tr}\,\text{n-ind}_{P_b}^G(\phi) = \mathrm{tr}\,\left(\mathrm{Ind}_{P_b}^G(\pi_{M_b} \otimes \delta_P^{1/2})\right)(\phi).$$

(See [vD72, p.237], for instance.) Set $\phi^{[M_b]} := \phi^{(M_b)}\delta_P^{-1/2}$. Then

$$\mathrm{tr}\,\pi_{M_b}(\phi^{[M_b]}) = \mathrm{tr}\,\left(\mathrm{Ind}_{P_b}^G(\pi_{M_b})\right)(\phi).$$

As $M_b(\mathbb{Q}_p) = \prod_{i=1}^{r} M_{b_i}(\mathbb{Q}_p)$, we may write

$$\phi^{[M_b]} = \sum_{\alpha \in A} \phi_{\alpha,1}\phi_{\alpha,2}\cdots\phi_{\alpha,r}$$

for a finite index set A and $\phi_{\alpha,i} \in C_c^\infty(M_{b_i}(\mathbb{Q}_p))$. (Note that $M_{b_i} = G_i$.) It follows from Conjecture 8.6 that for every $\rho = \otimes_{i=1}^{r}\rho_i \in \mathrm{Irr}_l(J_b(\mathbb{Q}_p))$,

$$\begin{aligned}\mathrm{tr}\,(\phi \times w|\mathrm{Mant}_{b,\mu}(\otimes_{i=1}^{r}\rho_i)) &= \mathrm{tr}\,\left(\phi^{[M_b]} \times w| \otimes_{i=1}^{r} \mathrm{Mant}_{b_i,\mu_i}(\rho_i)\right) \\ &= \sum_{\alpha\in A}\prod_{i=1}^{r} \mathrm{tr}\,\left(\phi_{\alpha,i} \times w|\mathrm{Mant}_{b_i,\mu_i}(\rho_i)\right).\end{aligned}$$

By applying Conjecture 8.3.(i) to each b_i, we can choose $f'_{\alpha,i} \in C_c^\infty(J_{b_i}(\mathbb{Q}_p))$ (depending on $\phi_{\alpha,i}$ and w but not on ρ_i) such that for every $\rho_i \in \mathrm{Irr}(J_{b_i}(\mathbb{Q}_p))$,

$$\mathrm{tr}\,(\phi_{\alpha,i} \times w|\mathrm{Mant}_{b_i,\mu_i}(\rho_i)) = \mathrm{tr}\,\rho_i(f'_{\alpha,i}).$$

Set $f' := \sum_{\alpha\in A} f'_{\alpha,1}f'_{\alpha,2}\cdots f'_{\alpha,r}$. We see that

$$\mathrm{tr}\,(\phi \times w|\mathrm{Mant}_{b,\mu}(\rho)) = \mathrm{tr}\,\rho(f')$$

for every $\rho \in \mathrm{Irr}_l(J_b(\mathbb{Q}_p))$. Hence Conjecture 8.3.(i) holds for b. □

Corollary 8.10. *Suppose that Conjecture 8.6 is known. If either (i) or (ii) of Conjecture 8.3 is true for every basic b, then Conjecture 8.1 is true.*

Proof. Immediate from Lemmas 8.4, 8.5 and 8.9. □

We close this subsection with remarks on the known cases of the conjectures (for unramified Rapoport-Zink spaces of EL-type). If $\dim(\mu) \le 1$ then $\mathrm{Mant}_{b,\mu}$ is well understood thanks to [HT01] (cf. [Har05]) even without the unramifiedness assumption and every conjecture in §8.1 is easily verified from this. (See [Shic, §2] for precise reference points in [HT01] and [Har05], and a review of their results.) In case $\dim(\mu) > 1$, apart from the results of Fargues on the basic case ([Far04]) and our present article, there was a progress on Harris's conjecture by Mantovan ([Man08]). Also see §8.3.

8.2. **Inductive steps.** Assuming Conjectures 8.1 and 8.6, we will sketch the inductive steps to determine $\mathrm{Mant}_{b,\mu}$ arising from (F, V, μ, b). The set $B(G, -\mu)$ is equipped with the partial ordering $\prec$ of [RR96, §2]. In particular, the unique basic element b of $B(G, -\mu)$ satisfies $b \prec b'$ for every $b' \in B(G, -\mu)$. The basic idea is to employ induction on $n = \dim_F V$, $\dim(\mu)$ and b.

Step 1. If $\dim(\mu) \le 1$ then $\mathrm{Mant}_{b,\mu}$ is known, as we mentioned in the last paragraph of §8.1. (For this it is not necessary to assume Conjectures 8.1 and 8.6.)

Step 2. Let (F, V, μ, b) be given. As an induction hypothesis, suppose that every $\mathrm{Mant}_{b',\mu'}$ is determined for (F', V', μ', b') satisfying either

- $\dim_{F'} V' < \dim_F V$ and $\dim(\mu') \le \dim(\mu)$, or
- $(F', V', \mu') = (F, V, \mu)$ and $b \prec b'$.

We divide into two cases according as b is basic or not.
Step 2-1. If b is basic, note that $\mathrm{Red}^{(b)} = e(J_b) \cdot LJ_{M_b,J_b}$. For any $\rho \in \mathrm{Groth}(J_b(\mathbb{Q}_p))$, we can choose $\pi \in \mathrm{Groth}(G(\mathbb{Q}_p))$ such that $\rho = \mathrm{Red}^{(b)}(\pi)$ since LJ_{M_b,J_b} is surjective (Lemma 2.14). In view of Conjecture 8.1, formula (7.1) allows us to compute

$$\begin{aligned}\mathrm{Mant}_{b,\mu}(\rho) \quad &= \quad [\pi]\left[r_\mu \circ \mathrm{LL}_l(\pi)|_{W_E}\ |\cdot|^{-\sum_\tau p_\tau q_\tau/2}\right] \\ &\quad - \sum_{b' \in B(G,-\mu),\ b' \neq b} e(J_{b'})\mathrm{Mant}_{b',\mu}(\mathrm{Red}^{(b')}(\pi)).\end{aligned}$$

Observe that the right hand side is understood by the induction hypothesis.
Step 2-2. If b is not basic, $\mathrm{Mant}_{b,\mu}$ is easily described by Conjecture 8.6 and the induction hypothesis.

Remark 8.11. In principle the above inductive steps can be used to give an explicit combinatorial recipe for $\mathrm{Mant}_{b,\mu}(\rho)$ (conditional on Conjectures 8.1 and 8.6), but we have not attempted to do so. Such a recipe would be quite complicated, as can be seen already in the case considered by Harris and Taylor ([HT01, Thm VII.1.5]).

8.3. **An evidence for Harris's conjecture.** We consider the case $r = 2$ of Conjecture 8.6 under the following assumptions. Set $n_i := \dim_F(V_i)$ for $i = 1, 2$.

- $\dim(\mu_1) = \dim(\mu_2) = 1$,
- $JL(\rho_1)$ and $JL(\rho_2)$ are supercuspidal and
- $\pi := \mathrm{Ind}_{P_b}^G(JL(\rho_1) \otimes JL(\rho_2))$ is irreducible and accessible with respect to (F, V, μ).

We briefly indicate how to verify Conjecture 8.6 in this case. It is left to the reader to fill out the details, which involve elementary computation with the Jacquet-Langlands map, Jacquet module and parabolic induction.

The argument is as follows. The above assumptions imply that the Newton polygon for b has two slopes $1/n_1$ and $1/n_2$ each of which corresponds to b_1 and b_2. Consider $b', b'' \in B(G, -\mu)$ such that b' (resp. b'') has exactly two slopes 0 and $2/n_1$ (resp. 0 and $2/n_2$). Applying Theorem 7.5 to our (F, V, μ) and π, an easy computation with LJ and the Jacquet module shows that the summand in the left hand side of (7.1) survives for only three elements in $B(G, -\mu)$, which are b, b' and b''. But by [Man08, Cor 5], Conjecture 8.6 is verified for b' and b''. Combining this fact with Corollary 7.7, the summands for b' and b'' can be computed without difficulty. Therefore $\mathrm{Mant}_{b,\mu}(\mathrm{Red}^{(b)}(\pi)) = \mathrm{Mant}_{b,\mu}(\rho_1 \otimes \rho_2)$ can be computed in terms of all the other terms in (7.1), which are known to us. Comparing the result with the right hand side of (8.3), which can be computed by Corollary 7.7, we finish verifying Conjecture 8.6.

To our knowledge the above case of Harris's conjecture was not known before. For instance, it is not covered by the results of [Man08] (if $n_1, n_2 > 1$). There do exist many π as above in view of Lemma 7.4.

Remark 8.12. Combining Theorem 7.5 with Mantovan's result on Conjecture 8.6 cited above ([Man08]), we can check a few more cases of Conjecture 8.6, but certainly not much. Since the general case would require new ideas, we have not tried to optimize our evidence but limited ourselves to a simple case which might be still illuminating.

8.4. **Scope of generalization.** So far we have been concerned with the unramified datum of EL-type as in Definition 4.1 where F is a finite unramified extension of $\mathbb{Q}_p$. For the general datum of EL-type, F needs to be replaced by a finite dimensional simple $\mathbb{Q}_p$-algebra whose center may be ramified over $\mathbb{Q}_p$. It would be nice to extend our results to this case, but our method does not apply. Unless we assume $\dim(\mu) \leq 1$, Proposition 5.4 is not available and Igusa varieties are not defined in that generality. An effort to extend Proposition 5.4 to the ramified case with $\dim(\mu) > 1$ should probably be preceded by a good understanding of integral models of Shimura varieties in that situation. Despite recent progress in the theory of such integral models, it does not seem that we know enough yet.

One may also ask whether our method (for the EL case) generalizes to the unramified PEL case ([Far04, 2.2.2, 2.2.3], cf. [RZ96, 3.82]), in which case the analogue of the $\mathbb{Q}_p$-group G is either an unramified unitary or symplectic similitude group. Some solid results would be proved in the unitary case, just as [Far04, Thm 8.2.2] was proved by Fargues, but not in the symplectic case. In the symplectic case, there are two serious problems in addition to the problem of local endoscopy. First, since the relevant PEL Shimura varieties are not compact, Proposition 5.4 fails. Second, the trace formula for a global symplectic similitude group $\mathbf{G}$ has more than the elliptic part. The fact that our counting point formula for Igusa varieties accounts for the elliptic part suggests that there should be analogous formulas matching the boundary terms in the trace formula for $\mathbf{G}$. The author does not know how to formulate those formulas.

Sug Woo Shin
Department of Mathematics, University of Chicago,
5734 S University Ave, Chicago, IL 60610, USA

References

[Art89] J. Arthur, *The L^2-Leftschetz numbers of Hecke operators*, Invent. Math. **97** (1989), 257–290.

[Bad07] A. I. Badulescu, *Jacquet-Langlands et unitarisabilité*, J. Inst. Math. Jussieu **6** (2007), 349–379.

[BC79] Borel and Casselman (eds.), *Automorphic forms, representations, and L-functions*, Proc. of Symp. in Pure Math., vol. 33.2, Providence, RI, AMS, Amer. Math. Soc., 1979.

[BD84] J. Bernstein and P. Deligne, *Le "centre" de Bernstein*, pp. 1–32, Hermann, Paris, 1984.

[BDK86] J. Bernstein, P. Deligne, and D. Kazhdan, *Trace Paley-Wiener theorem for reductive p-adic groups*, J. Analyse Math. **47** (1986), 180–192.

[Ber93] V. Berkovich, *Étale cohomology for non-archimedean analytic spaces*, Pub. Math. IHES **78** (1993), 5–161.

[Bor79] A. Borel, *Automorphic L-functions*, in Borel and Casselman [BC79], pp. 27–61.

[BZ77] I. Bernstein and A. Zelevinskii, *Induced representations of reductive p-adic groups. I*, Ann. Sci. École Norm. Sup. **10** (1977), 441–472.

[CD90] L. Clozel and P. Delorme, *Le théorème de Paley-Wiener invariant pour les groupes de lie réductifs. II*, Ann. Sci. École Norm. Sup. **23** (1990), 193–228.

[CL99] L. Clozel and J.-P. Labesse, *Changement de base pour les représentations cohomologiques des certaines groupes unitaires*, appendix to [Lab99] (1999).

[Clo86] L. Clozel, *On limit multiplicities of discrete series representations in spaces of automorphic forms*, Invent. Math. **83** (1986), 265–284.

[Dat07] J.-F. Dat, *Théorie de Lubin-Tate non-abélienne et représentations elliptiques*, Invent. Math. **169** (2007), 75–152.

[Del71] P. Deligne, *Travaux de Shimura, Sém. Bourbaki Février 1971, Expoé 389*, LNM, no. 244, Springer, 1971.

[DKV84] P. Deligne, D. Kazhdan, and M.-F. Vigneras, *Représentations des algèbres centrales simples p-adiques*, pp. 33–117, Hermann, 1984.

[Far04] L. Fargues, *Cohomologie des espaces de modules de groupes p-divisibles et correspondances de Langlands locales*, Astérisque **291** (2004), 1–200.

[Har01] M. Harris, *Local Langlands correspondences and vanishing cycles on Shimura varieties*, Prog. in math. **201** (2001), 407–427.

[Har05] ———, *The local Langlands correspondence: notes of (half) a course at the IHP Spring 2000.*, Astérisque **298** (2005), 17–145.

[HL04] M. Harris and J.-P. Labesse, *Conditional base change for unitary groups*, Asian J. Math. **8** (2004), 653–683.

[HT01] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Princeton University Press, no. 151, Annals of Mathematics Studies, Princeton, New Jersey, 2001.

[Kaz86] D. Kazhdan, *Cuspidal geometry of p-adic groups*, J. Analyse Math. **47** (1986), 1–36.

[Kot] R. Kottwitz, *unpublished*.

[Kot83] ———, *Sign changes in harmonic analysis on reductive groups*, Trans. of the Amer. Math. Soc. **278** (1983), 289–297.

[Kot84] ———, *Stable trace formula: Cuspidal tempered terms*, Duke Math. **51** (1984), 611–650.

[Kot85] ———, *Isocrystals with additional structure*, Comp. Math **56** (1985), 201–220.

[Kot86] ———, *Stable trace formula: Elliptic singular terms*, Math. Ann. **275** (1986), 365–399.

[Kot88] ———, *Tamagawa numbers*, Annals of Math. **127** (1988), 629–646.

[Kot90] ———, *Shimura varieties and λ-adic representations*, Perspectives in Math., vol. 10, Academic Press, 1990, pp. 161–209 (vol I).

[Kot92a] ———, *On the λ-adic representations associated to some simple Shimura variaties*, Invent. Math. **108** (1992), 653–665.

[Kot92b] ———, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), 373–444.

[Kot97] ———, *Isocrystals with additional structure II*, Comp. Math **109** (1997), 255–339.

[KR03] R. Kottwitz and M. Rapoport, *On the existence of F-crystals*, Comment. Math. Helv. **78** (2003), 153–184.

[Lab99] J.-P. Labesse, *Cohomologie, stabilisation et changement de base*, Astérisque, no. 257, 1999.

[Lan79] R. Langlands, *Stable conjugacy: definitions and lemmas*, Can. J. Math. **31** (1979), 700–725.

[LS87] R. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann. **278** (1987), 219–271.

[Man04] E. Mantovan, *On certain unitary group Shimura varieties*, Astérisque **291** (2004), 201–331.

[Man05] ———, *On the cohomology of certain PEL type Shimura varieties*, Duke Math. **129** (2005), 573–610.

[Man08] ———, *On non-basic Rapoport-Zink spaces*, Annales scientifiques de l'ENS **41** (2008), 671–716.

[Mes72] W. Messing, *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*, LNM, no. 264, Springer, 1972.

[Ngo] B. C. Ngo, *Le lemme fondamental pour les algèbres de Lie*, arXiv:0801.0446v1 [math.AG].

[Rap95] M. Rapoport, *Non-archimedean period domains*, Proc. of ICM (Zürich 1994) (1995), 423–434.

[RR96] M. Rapoport and M. Richartz, *On the classification and specialization of F-isocrystals with additional structure*, Comp. Math. **103** (1996), 153–181.

[RZ96] M. Rapoport and T. Zink, *Period spaces for p-divisible groups*, Princeton University Press, no. 141, Annals of Mathematics Studies, Princeton, New Jersey, 1996.

[Shia] S. W. Shin, *Automorphic Plancherel density theorem*, submitted.

[Shib] ———, *Counting points on Igusa varieties*, to appear in J. Inst. Math. Jussieu.

[Shic] ———, *Galois representations arising from some compact Shimura varieties*, to appear in Annals of Math.

[Shi09] ———, *Counting points on Igusa varieties*, Duke Math. **146** (2009), 509–568.

[Tad90] M. Tadić, *Induced representations of $GL(n, A)$ for p-adic division algebras a*, J. reine angew. Math. **405** (1990), 48–77.

[Tat79] J. Tate, *Number theoretic background*, in Borel and Casselman [BC79], pp. 3–26.

[vD72] G. van Dijk, *Computation of certain induced characters of p-adic groups*, Math. Ann. **199** (1972), 229–240.

[Zel80] A. Zelevinsky, *Induced representations of reductive p-adic groups. II. On irreducible representations of $GL(n)$.*, Ann. Sci. École Norm. Sup. **13** (1980), no. 2, 165–210.