Counting points on Igusa varieties

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Abstract

Igusa varieties are smooth varieties over $\overline{\mathbb{F}}_p$ which are higher-dimensional analogues of Igusa curves. They were introduced by Harris and Taylor ([HT01]) to study the bad reduction of some PEL Shimura varieties and generalized by Mantovan ([Man04], [Man05]). The present paper gives a group-theoretic formula for the traces of certain operators on the cohomology of Igusa varieties, suitable for applications via comparison with the Arthur-Selberg trace formula. Our formula generalizes the results of [HT01, V.1-V.4] to the case of any PEL Shimura varieties of type (A) and (C) and puts it in a more natural framework in the spirit of [Kot92].

1 Introduction

This article is mainly concerned with the cohomology of Igusa varieties, which are closely related to Shimura varieties. To motivate the reader, we begin with briefly discussing what has been worked out about the cohomology of Shimura varieties in relation with the Langlands correspondence.

The cohomology of Shimura varieties has been studied for decades. Apart from the case GL_2, the case of U_3 was extensively studied in [LR92]. Kottwitz and Clozel used some "simple" PEL-type Shimura varieties of unitary type (a.k.a. type (A)) and attached n-dimensional Galois representations to automorphic representations of GL_n over CM fields satisfying the local-global compatibility of the Langlands correspondence at unramified primes. Here "simple" means that the Shimura varieties have no boundary components and that there is no endoscopy. The key inputs in their work are, among other things, the counting point formula for the good reduction fiber of the Shimura variety of type (A) or (C) ([Kot92]), to be compared with the Arthur-Selberg trace formula, and the base change result for simple unitary groups ([Clo91]). Although a stabilized version of the counting point formula was available ([Kot90]), it hinged on certain forms of the fundamental lemma, which were avoided by the use of simple unitary groups.

A new method of Harris and Taylor allows us to study the bad reduction of some simple Shimura varieties associated to the unitary (similitude) groups which arise from division algebras and are $U(1, n-1) \times U(0, n) \times \cdots \times U(0, n)$ at infinity. This had important consequences such as the proof of the local-global compatibility at ramified primes and the local Langlands conjecture for GL_n over p-adic fields. Two main ingredients in their argument are new: the first basic identity ([HT01, Thm IV.2.7]), which was generalized by Mantovan ([Man04], [Man05]), and the second basic identity ([HT01, Thm V.5.4]), which essentially follows from the counting point formula for Igusa varieties. However, their counting point formula relies heavily on the specifics of their unitary groups and is not easily generalized.

In this work we formulate and prove a natural generalization of the counting point formula for Igusa varieties which arise from any PEL Shimura varieties of type (A) or (C). Our new formulation was inspired by the work of Kottwitz ([Kot92]) and many of his arguments indeed carry over without

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much modification. Our key observation was that reasonable definitions of the analogues of the triple $(\gamma_0;\gamma,\delta)$ and the cohomological invariant $\alpha(\gamma_0;\gamma,\delta)$ introduced by Kottwitz can be made to work for Igusa varieties despite the apparent difference of the settings.

Our counting point formula is expected to lead to new applications regarding the computation of the cohomology of Shimura varieties and Rapoport-Zink spaces, if we combine our result with Mantovan's formula ([Man05, Thm 22]). Thereby we may deepen our understanding of the Langlands correspondence. We will work out these applications in future writings. In many cases our formula must be stabilized to be ready for applications, as usual in the trace formula method. The stabilization will be provided in the sequel paper ([Shi]). We merely remark that we do not need the twisted fundamental lemma for stabilization since our formula does not involve twisted orbital integrals. (Unlike in the case of Kottwitz's formula; see [Kot90, p.180].) We also add that the cohomology of Rapoport-Zink spaces was computed by Fargues ([Far04]) in several cases where the Igusa varieties are zero-dimensional, using a version of Mantovan's formula (which is simpler in those cases) and techniques from rigid analytic geometry, among others. In the cases considered by Fargues, the issue of stabilization does not arise.

We briefly explain the structure of this article. In §2-§4 we build up background materials. The readers may skip this part and come back later for references. The main discussion begins in §5 where we construct a Shimura variety X defined over the reflex field E along with an integral model, starting from an integral Shimura PEL datum $(B,\mathcal{O}_B,*,V,\Lambda_0,\langle\cdot,\cdot\rangle,h)$ of type (A) or (C). Apart from the assumptions ensuring that G is unramified at p, we do not make further restrictions. In particular X need not be proper over E. Denote by G the associated algebraic group over $\mathbb{Q}$. Let J_b be the $\mathbb{Q}_p$-group arising as the automorphism group of an isocrystal of type b, which turns out to be an inner form of a Levi subgroup of $G_{\mathbb{Q}_p}$. For each Newton polygon datum $b\in B(G,-\mu_h)$, we define the Igusa variety Ig_b as a projective system. It is worth noting that $G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p)$ acts on $H_c(\mathrm{Ig}_b,\mathscr{L}_\xi)$ while $G(\mathbb{A}^\infty)\times \mathrm{Gal}(\overline{E}/E)$ acts on $H(X,\mathscr{L}_\xi)$, where $\mathscr{L}_\xi$ is an l-adic sheaf constructed from an algebraic representation of G.

From §6 until the end is devoted to obtaining the following main result, namely the counting point formula for the cohomology of Igusa varieties.

Theorem 1 (Theorem 13.1). *If $\varphi\in C_c^\infty(G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p))$ is acceptable, then*

$$\mathrm{tr}\,(\varphi|H_c(\mathrm{Ig}_b,\mathscr{L}_\xi))=\sum_{(\gamma_0;\gamma,\delta)\in KT_b^{eff}}\mathrm{vol}(I_\infty(\mathbb{R})^1)^{-1}|A(I_0)|\ \mathrm{tr}\,\xi(\gamma_0)\cdot O_{(\gamma,\delta)}^{G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p)}(\varphi)$$

The proof is done in several steps. We use Fujiwara's trace formula in §6 to convert the computation of $\mathrm{tr}\,(\varphi|H_c(\mathrm{Ig}_b,\mathscr{L}_\xi))$ into the problem of counting $\overline{\mathbb{F}}_p$-points of Ig_b fixed under correspondences. We define the notion of acceptable functions so that this works whenever φ is acceptable. In §7 using the moduli interpretation of Ig_b, the counting point problem is essentially reduced to parametrizing the triples (A,λ,i) that appear in the moduli data and conjugacy classes $[a]$ in the automorphism group of (A,λ,i) in the isogeny category. We carry out this parametrization in terms of Kottwitz triples $(\gamma_0;\gamma,\delta)$ which have a purely group-theoretic description.

Useful lemmas for studying (A,i) and λ are provided in §8 and §9, respectively, using tools from the Honda-Tate theory and Galois cohomology. We remark that Lemma 8.6 looks simple but is important in working with conjugacy classes $[a]$. In §10 and §11 we give the definition of Kottwitz triples $(\gamma_0;\gamma,\delta)$ and an important result on the vanishing of the cohomological invariant (Corollary 11.3). In §12 we complete the proof of the reparametrization of (A,λ,i) and $[a]$ in terms of $(\gamma_0;\gamma,\delta)$. In going forward the deepest fact seems the vanishing of $\alpha(\gamma_0;\gamma,\delta)$, which is a direct consequence of Corollary 11.3. In going backward, we recover (A,i) via Honda-Tate theory by reading off the necessary data from $(\gamma_0;\gamma,\delta)$. The rationality of λ and a is proved in two steps and follows from the vanishing of $\alpha(\gamma_0;\gamma,\delta)$. At this point it is easy to deduce the main result, namely Theorem 13.1.

It is worth noting that $[a]$ did not show up in the work of Kottwitz. In some sense the conjugacy classes $[a]$ reflect the level structure at p of Igusa varieties and add one more layer to the whole argument. Among the cohomological invariants in §10 $\beta(\gamma_0;\gamma,\delta)$ is a direct analogue of $\alpha(\gamma_0;\gamma,\delta)$ defined by [Kot92] and encodes the rationality of (A,λ,i) whereas our $\alpha(\gamma_0;\gamma,\delta)$ encodes the rationality of (A,λ,i) and the conjugacy class $[a]$. As such $\beta(\gamma_0;\gamma,\delta)$ plays only an auxiliary role in the proof of Lemma 12.3.

Finally we mention that this work is largely based on Chapter 1-3 of the author's Harvard thesis ([Shi07]) but made more focused on the goal of establishing the counting point formula. We also corrected minor errors in [Shi07] and changed some convention to be more compatible with the literature. We refer to [Shi07] only once, in the proof of Lemma 6.3, for minor details which are not difficult but somewhat distracting.

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Notation

First of all, we point out the notations different from those of [Man05]. We write J_b and Ig_b for the p-adic group T_b and the Igusa variety J_b of that paper, respectively. The notation J_b seems to have been widely used (for instance in [RZ96]).

A CM field is by definition an imaginary quadratic extension of a totally real field. A CM field has a well-defined automorphism of order 2, which is a restriction of the complex conjugation c on $\mathbb{C}$ via any complex embedding.

Suppose that B is a finite dimensional $\mathbb{Q}$-semisimple algebra. For a semisimple element $\gamma\in B^\times$, define $F(\gamma)$ to be the commutative F-subalgebra generated by γ. For a $\mathbb{Q}$-algebra R, we often write B_R or $B\otimes R$ for $B\otimes_{\mathbb{Q}} R$. If a tensor product is taken over anything other than $\mathbb{Q}$, the base ring will be written out explicitly.

For a number field F, we define the following notation. When v is a place of F, denote by F_v the completion of F with respect to the metric defined by v. Write ϖ_v for a uniformizer of the integer ring $\mathcal{O}_{F_v}$ of F_v. The residue field $\mathcal{O}_{F_v}/(\varpi_v)$ is denoted $k(v)$. When S is a finite set of places of F define $\mathbb{A}^S_F$ to be the restricted product $\prod'_{v\notin S}F_v$. Define $\overline{\mathbb{A}}^S_F := \varinjlim \mathbb{A}^{S(F')}_{F'}$ where F' runs over finite extension fields over F and $S(F')$ denotes the set of places of F' over S. If $F=\mathbb{Q}$ we simply write $\mathbb{A}^S$ and $\overline{\mathbb{A}}^S$.

Now suppose that G is a connected reductive group defined over a field F. For $g\in G(F)$, let $\mathrm{Int}(g)$ denote the inner automorphism $x\mapsto gxg^{-1}$ of G and $\mathrm{Int}(G)$ the group of inner automorphisms of G. We write $H^1(F,G)$ for $H^1(\mathrm{Gal}(F^{sep}/F),G(F^{sep}))$ where F^{sep} is a separable closure of F. The dual group of G will be written as $\widehat{G}$. It is a complex Lie group with $\mathrm{Gal}(F^{sep}/F)$ action. When F is a number field, we define $\ker^1(F,G):=\ker(H^1(F,G)\to\oplus_v H^1(F_v,G))$ where v runs over all places of F.

The notation $Z(A)$ (resp. $Z_A(a)$) will be used to denote the center (resp. the centralizer of a) in A where A is either a group, an algebra, or an algebraic group.

We use symbols Γ and $\Gamma(v)$ to mean $\Gamma=\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\Gamma(v)=\mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$ starting from §5. Here v can be any place of $\mathbb{Q}$, including the infinite place ∞.

When G is a topological group, $C_c^\infty(G)$ denotes the space of locally constant compactly supported functions with values in a fixed characteristic 0 field, which is often $\overline{\mathbb{Q}}_l$. (In this paper G will be a p-adic Lie group or a restricted product of such.)

We use the notation $\mathrm{Groth}(\cdot)$ for the Grothendieck group of admissible representations of topological groups. For precise definition, refer to [HT01, I.2].

2 Hermitian modules

Let C be a finite dimensional $\mathbb{Q}$-algebra with an involution $*$. For any $\mathbb{Q}$-algebra A, the involution $*$ extends to $C \otimes_{\mathbb{Q}} R$, acting as the identity on A.

Definition 2.1. Consider a finite free $C \otimes_{\mathbb{Q}} A$-module V equipped with a non-degenerate A-bilinear pairing $\langle \cdot, \cdot \rangle : V \times V \to A$. We say that $(V, \langle \cdot, \cdot \rangle)$ is a *$*$-Hermitian $C \otimes_{\mathbb{Q}} A$-module* if

$$\langle \gamma x, y \rangle = \langle x, \gamma^* y \rangle \quad \text{for all } x, y \in V \text{ and all } \gamma \in C \otimes_{\mathbb{Q}} A.$$

In this case, $\langle \cdot, \cdot \rangle$ is called a *$*$-Hermitian pairing* (with respect to $C \otimes_{\mathbb{Q}} A$).

When $\langle \cdot, \cdot \rangle$ and $*$ are understood, we simply say that V is a Hermitian $C \otimes_{\mathbb{Q}} A$-module. Two $*$-Hermitian $C \otimes_{\mathbb{Q}} A$-modules $(V_1, \langle \cdot, \cdot \rangle_1)$ and $(V_2, \langle \cdot, \cdot \rangle_2)$ are said to be *equivalent* if there exist an isomorphism of $C \otimes_{\mathbb{Q}} A$-modules $\delta : V_1 \xrightarrow{\sim} V_2$ and an element $\mu \in A^\times$ such that $\langle x, y \rangle_1 = \mu \langle \delta x, \delta y \rangle_2$ for all $x, y \in V_1$.

Given a $*$-Hermitian pairing $\langle \cdot, \cdot \rangle_0$ on V with respect to $C \otimes_{\mathbb{Q}} A$, define an algebraic group H over $\operatorname{Spec} A$ consisting of self-equivalences by

$$H(A) = \{ h \in \mathrm{End}_{C \otimes_{\mathbb{Q}} A}(V \otimes_{\mathbb{Q}} A) \mid \exists \varpi(h) \in A^\times,\ \langle h w_1, h w_2 \rangle_0 = \varpi(h) \langle v_1, v_2 \rangle_0 \text{ for all } v_1, v_2 \in V \otimes_{\mathbb{Q}} A \}$$

for any $\mathbb{Q}$-algebra A.

Let F be a field of characteristic 0. Fix a $*$-Hermitian $C \otimes_{\mathbb{Q}} F$-module V. Define $St(V)$ to be the set of equivalence classes of $*$-Hermitian $C \otimes_{\mathbb{Q}} F$-modules W which are isomorphic to V as $C \otimes_{\mathbb{Q}} F$-modules (without pairing). We view $St(V)$ as a pointed set where the equivalence class of V is distinguished. It is possible to construct a natural map $St(V) \to H^1(F, H)$ as follows. For $W \in St(V)$, choose an equivalence $h : V \otimes_F \overline{F} \xrightarrow{\sim} W \otimes_F \overline{F}$ as $C \otimes_{\mathbb{Q}} \overline{F}$-Hermitian modules. (Use the fact that any two $C \otimes_{\mathbb{Q}} \overline{F}$-Hermitian pairings on a $C \otimes_{\mathbb{Q}} \overline{F}$-module are equivalent.) Then $h^{-1} h^\sigma \in H(\overline{F})$ where $h^\sigma = (1 \otimes \sigma) h (1 \otimes \sigma)^{-1}$ for $\sigma \in \mathrm{Gal}(\overline{F}/F)$. The cocyle $\sigma \mapsto h^{-1} h^\sigma$ is the desired element of $H^1(F, H)$ associated to W. The following lemma is easy.

Lemma 2.2. *The above map defines a natural isomorphism of pointed sets between $St(V)$ and $H^1(F, H)$.*

Now let F be a *number field* and v denote a place of F. Let

$$\Gamma := \mathrm{Gal}(\overline{F}/F) \quad \text{and} \quad \Gamma(v) := \mathrm{Gal}(\overline{F}_v/F_v).$$

Let G be a connected reductive group over F. We define

$$A(G) := \pi_0(Z(\widehat{G})^\Gamma)^D, \quad A_v(G) := \pi_0(Z(\widehat{G})^{\Gamma(v)})^D \tag{1}$$

where D means the Pontryagin dual. Note that there is a natural restriction map $A_v(G) \to A(G)$. In terms of the algebraic fundamental group $\pi_1(G)$, we have canonical isomorphisms $A(G) \simeq (\pi_1(G)_\Gamma)_{\mathrm{tor}}$ and $A_v(G) \simeq (\pi_1(G)_{\Gamma(v)})_{\mathrm{tor}}$ ([Bor98, Prop 1.10]). (The subscript "tor" denotes the torsion subgroup therein.) Since the construction of $\pi_1(G)$ is functorial in G with respect to *any* F-morphism of connected reductive groups over F, it is easy to see that the groups $A_v(G)$, $A(G)$ and the map $A_v(G) \to A(G)$ are functorial in G.

Lemma 2.3. *For every place v of F, there is a canonical map*

$$\alpha_{G,v} : H^1(F_v, G) \to A_v(G)$$

which is an isomorphism if v is a finite place. This map $\alpha_{G,v}$ is functorial in G (with respect to any F-morphism). When composed with the natural map $A_v(G) \to A(G)$, the maps $\alpha_{G,v}$ induce a map $H^1(F, G(\overline{\mathbb{A}}_F)) \to A(G)$ fitting into an exact sequence

$$1 \to \ker^1(F, G) \to H^1(F, G) \to H^1(F, G(\overline{\mathbb{A}}_F)) \to A(G)$$

Proof. The lemma is proved in [Kot86, Thm 1.2, Prop 2.6] except that the functoriality of $\alpha_{G,v}$ is proved only for normal morphisms of reductive groups over F. (Some more cases are covered in Lemma 4.3 of that paper.) However functoriality is easily extended to all cases. When G^{der} is simply connected, write $D_G := G/G^{der}$. Recall from the same paper that $\alpha_{G,v}$ is the same as the composite map

$$H^1(F_v, G) \to H^1(F_v, D_G) \xrightarrow{\sim} A_v(D_G) = A_v(G) \xrightarrow{\sim} (\pi_1(G)_{\Gamma(v)})_{\mathrm{tor}}$$

where the isomorphisms are canonical. From this it is easy to prove functoriality with respect to any F-morphism $G_1 \to G_2$ granted that G_1^{der} and G_2^{der} are simply connected. The general case of functoriality is proved as in [Kot86, p.369] using z-extensions.

Alternatively, functoriality can be established in full generality using the following canonical functorial maps ([Lab99, Prop 1.6.7, Prop 1.7.3], also [Bor98, Cor 5.5]) in the context of abelianized cohomology

$$H^1(F_v, G) \twoheadrightarrow H^1_{\mathrm{ab}}(F_v, G) \hookrightarrow A_v(G) \xrightarrow{\sim} (\pi_1(G)_{\Gamma(v)})_{\mathrm{tor}},$$

whose composite is $\alpha_{G,v}$. (Of course the surjection and the injection above are isomorphisms when v is non-archimedean.) □

In the statement of the lemma, the map $\alpha_{G,v}$ is canonical in the sense that it is uniquely determined by two conditions: (i) $\alpha_{G,v}$ is the canonical map induced by Tate-Nakayama duality when G is a torus and (ii) $\alpha_{G,v}$ is functorial in G. The meaning of canonicality for β_G is taken to be the same in the following lemma.

Lemma 2.4. *There is a canonical map $\beta_G : H^1(F, G(\overline{\mathbb{A}}_F)/Z(G)(\overline{F})) \to A(G)$ which is functorial in G. When β_G is composed with the natural map $H^1(F, G(\overline{\mathbb{A}}_F)) \to H^1(F, G(\overline{\mathbb{A}}_F)/Z(G)(\overline{F}))$, the resulting map $H^1(F, G(\overline{\mathbb{A}}_F)) \to A(G)$ is identical to the map induced by $\alpha_{G,v}$. Moreover, the map β_G fits into an exact sequence*

$$1 \to H^1(F, G(\overline{F})/Z(G)(\overline{F})) \to H^1(F, G(\overline{\mathbb{A}}_F)/Z(G)(\overline{F})) \to A(G)$$

Proof. Everything in the lemma is proved in [Kot86, Thm 2.2, Cor 2.5] except that the functoriality of β_G is verified only for normal morphisms. The general case of functoriality is proved as in the first paragraph of the proof of Lemma 2.3, noting that ([Kot86, p.374]) if G^{der} is simply connected, β_G is the composition

$$H^1(F, G(\overline{\mathbb{A}}_F)/Z(G)(\overline{F})) \to H^1(F, D_G(\overline{\mathbb{A}}_F)/D_G(\overline{F})) \to A(D) = A(G).$$

□

From here until the end of this section, let $(B, *, V, \langle\cdot,\cdot\rangle)$ be a partial Shimura datum in Definition 5.1. By linearly extending $\langle\cdot,\cdot\rangle$, we have a $*$-Hermitian $B_{\mathbb{Q}_v}$-module $V_{\mathbb{Q}_v}$ for each place v of $\mathbb{Q}$ and a $*$-Hermitian $B \otimes_{\mathbb{Q}} \mathbb{A}^S$-module $V \otimes_{\mathbb{Q}} \mathbb{A}^S$. Here $\mathbb{A}^S$ is the ring of adeles with trivial entries at the places contained in the finite set S. Note that we will often write $B_{\mathbb{Q}_v}$ and $V_{\mathbb{Q}_v}$ for $B \otimes_{\mathbb{Q}} \mathbb{Q}_v$ and $V \otimes_{\mathbb{Q}} \mathbb{Q}_v$. By Lemma 2.2, we have horizontal bijections in the following commutative diagram.

$$\begin{array}{ccc} H^1(\mathbb{Q}, G) & \xrightarrow{1-1} & St(V) \\ \downarrow & & \downarrow \\ H^1(\mathbb{Q}_v, G) & \xrightarrow{1-1} & St(V_{\mathbb{Q}_v}) \end{array}$$

3 Conjugacy classes and Galois cohomology

In this subsection, we summarize various results concerning conjugacy classes and Galois cohomology of reductive groups from [Kot84b] and [Kot86]. We assume the reader is familiar with the dual groups and L-groups, for which one can see [Bor79].

Let F be a perfect field. Let $\overline{F}$ be an algebraic closure of F. Let G be a connected reductive group over F and assume that its derived subgroup G^{der} is simply connected.

Definition 3.1. We say that $\gamma, \gamma' \in G(F)$ are *(F-)conjugate*, or *stably conjugate* if $\gamma' = g\gamma g^{-1}$ for some g in $G(F)$ or $G(\overline{F})$, respectively. Write $\gamma \sim \gamma'$ or $\gamma \sim_{st} \gamma'$ (equivalently $\gamma \sim_{\overline{F}} \gamma'$) in each case. If F is a number field and $\gamma, \gamma' \in G(\mathbb{A}_F^S)$, then $\gamma \sim_{\mathbb{A}_F^S} \gamma'$ and $\gamma \sim_{\overline{\mathbb{A}}_F^S} \gamma'$ will have obvious meaning.

Write $St_{G(F)}(\gamma)$ for the set of F-conjugacy classes in the stable conjugacy class of $\gamma \in G(F)$. There is a natural isomorphism of pointed sets

$$St_{G(F)}(\gamma) \xrightarrow{\sim} \ker(H^1(F, Z_G(\gamma)) \to H^1(F, G)) \tag{2}$$

defined as follows: if $\gamma' = g\gamma g^{-1}$ for $g \in G(\overline{F})$, then the conjugacy class of γ' is mapped to the cocycle $\sigma \mapsto g^{-1}g^{\sigma}$.

Now let F be a number field and v denote a place of F. Suppose that $\gamma \in G(F)$ is semisimple. By our assumption that G^{der} is simply connected, $I := Z_G(\gamma)$ is a connected reductive group over F. As there is a canonical Γ-equivariant embedding $Z(\widehat{G}) \hookrightarrow Z(\widehat{I})$, we may consider the exact sequence of Γ-modules

$$1 \to Z(\widehat{G}) \to Z(\widehat{I}) \to Z(\widehat{I})/Z(\widehat{G}) \to 1$$

which gives us a long exact sequence ([Kot84b, Cor 2.3]), part of which is

$$X_*(Z(\widehat{I})/Z(\widehat{G}))^{\Gamma} \to \pi_0(Z(\widehat{G})^{\Gamma}) \to \pi_0(Z(\widehat{I})^{\Gamma}) \to \pi_0((Z(\widehat{I})/Z(\widehat{G}))^{\Gamma}) \xrightarrow{\xi} H^1(F, Z(\widehat{G})) \to H^1(F, Z(\widehat{I}))$$

At each place v of F, we have a similar sequence and in particular a homomorphism $\xi_v : \pi_0((Z(\widehat{I})/Z(\widehat{G}))^{\Gamma(v)}) \to H^1(F_v, Z(\widehat{G}))$. We define

$$\mathfrak{K}(I/F) := \xi^{-1}(\ker^1(F, Z(\widehat{G}))) \quad \text{and} \quad \mathfrak{K}(I/F_v) := \ker \xi_v.$$

Since the following diagram commutes with the horizontal maps being obvious ones, we have a canonical map $\mathfrak{K}(I/F) \to \mathfrak{K}(I/F_v)$.

$$\begin{array}{ccc} \pi_0((Z(\widehat{I})/Z(\widehat{G}))^{\Gamma}) & \longrightarrow & \pi_0((Z(\widehat{I})/Z(\widehat{G}))^{\Gamma(v)}) \\ \downarrow \xi & & \downarrow \xi_v \\ H^1(F, Z(\widehat{G})) & \longrightarrow & H^1(F_v, Z(\widehat{G})) \end{array}$$

From the definition of $\mathfrak{K}(I/F)$ and $\mathfrak{K}(I/F_v)$ the following exact sequences are immediate.

$$\pi_0(Z(\widehat{G})^{\Gamma}) \to \pi_0(Z(\widehat{I})^{\Gamma}) \to \mathfrak{K}(I/F) \to \ker^1(F, Z(\widehat{G})) \to \ker^1(F, Z(\widehat{I})) \tag{3}$$

$$\pi_0(Z(\widehat{G})^{\Gamma(v)}) \to \pi_0(Z(\widehat{I})^{\Gamma(v)}) \to \mathfrak{K}(I/F_v) \to 1 \tag{4}$$

It is well known that the last arrow in (3) is an isomorphism if G is an algebraic group arising from the PEL-type moduli problem of Shimura varieties of type A or C (use [Kot92, §7]). We remark that if γ is elliptic in $G(F)$ (resp. in $G(F_v)$), then the first arrow in (3) (resp. (4)) is injective since the ellipticity means that $X_*(Z(\widehat{I})/Z(\widehat{G}))^{\Gamma}$ (resp. $X_*(Z(\widehat{I})/Z(\widehat{G}))^{\Gamma(v)}$) is trivial.

Suppose that semisimple elements γ_v and γ'_v of $G(F_v)$ are stably conjugate to each other. Lemma 2.3 implies that we have a canonical map

$$\ker(H^1(F_v, I) \to H^1(F_v, G)) \to \ker(A_v(I) \to A_v(G)) \simeq \mathfrak{K}(I/F_v)^D$$

where the last natural isomorphism comes from the dual of the sequence (4). Choose an element $g \in G(F_v)$ such that $\gamma' = g\gamma g^{-1}$. We denote by $\mathrm{inv}_v(\gamma_v, \gamma'_v)$ the image of the 1-cocycle $\sigma \mapsto g^{-1}g^{\sigma}$ in $\mathfrak{K}(I/F_v)^D$ under the above map. By abuse of notation, $\mathrm{inv}_v(\gamma_v, \gamma'_v)$ will also be viewed as an element of $\mathfrak{K}(I/F)^D$ via the canonical map $\mathfrak{K}(I/F) \to \mathfrak{K}(I/F_v)$.

Let $\gamma = (\gamma_v)$ and $\gamma' = (\gamma'_v)$ be semisimple elements of $G(\mathbb{A}_F)$ that are $\overline{\mathbb{A}}_F$-conjugate to each other. We define the following element of $\mathfrak{K}(I/F)^D$

$$\mathrm{inv}(\gamma, \gamma') := \sum_v \mathrm{inv}_v(\gamma_v, \gamma'_v).$$

The following is an important result regarding rationality of conjugacy classes.

Lemma 3.2. *([Kot86, Thm 6.6]) Suppose that two semisimple elements $\gamma \in G(F)$ and $\gamma' \in G(\mathbb{A}_F)$ are conjugate in $G(\overline{\mathbb{A}}_F)$. The element $\gamma' \in G(\mathbb{A}_F)$ is $G(\mathbb{A}_F)$-conjugate to an element of $G(F)$ if and only if $\mathrm{inv}(\gamma, \gamma')$ is trivial.*

We can relate the conjugacy classes to Hermitian modules. Suppose that $(B, *, V, \langle\cdot,\cdot\rangle)$ comes from a partial Shimura PEL datum and G is the associated group (§5). Put $F := Z(B)$. Each semisimple element $\gamma \in G(\mathbb{Q})$ generates a F-subalgebra $F(\gamma)$ in B and naturally induces a Hermitian $B \otimes_F F(\gamma)$-module structure on V, which we call V_γ. Let us write $St(\gamma, V)$ for the set of equivalence classes of Hermitian $B \otimes_F F(\gamma)$-modules which are equivalent to V as Hermitian B-modules. $St(\gamma, V)$ is naturally a pointed set with the equivalence class of V_γ distinguished. From the definition, we have $St(\gamma, V) = \ker(St(V_\gamma) \to St(V))$. A local analogue $St(\gamma_v, V_{\mathbb{Q}_v})$ is defined in the exactly same way.

Recall that we earlier had an isomorphism of pointed sets $St_{G(\mathbb{Q})}(\gamma) \xrightarrow{\sim} \ker(H^1(\mathbb{Q}, Z_G(\gamma)) \to H^1(\mathbb{Q}, G))$. Lemma 2.2 implies that there is a natural map $St(\gamma, V) \to \ker(H^1(\mathbb{Q}, Z_G(\gamma)) \to H^1(\mathbb{Q}, G))$. There is also a natural map $St_{G(\mathbb{Q})}(\gamma) \to St(\gamma, V)$, which we now describe. The element γ maps to the Hermitian $B \otimes_F F(\gamma)$-module V_γ. If $\gamma' = g\gamma g^{-1}$ for $g \in G(\overline{\mathbb{Q}})$, then γ' endows V with a Hermitian $B \otimes_F F(\gamma')$-module structure, which we call $V_{\gamma'}$. As $\gamma \mapsto \gamma'$ induces an isomorphism $F(\gamma) \xrightarrow{\sim} F(\gamma')$, we see that $V_{\gamma'}$ is naturally an element of $St(\gamma, V)$. The map $St_{G(\mathbb{Q})}(\gamma) \to St(\gamma, V)$ given by $\gamma' \mapsto V_{\gamma'}$ is well-defined.

Lemma 3.3. *For $\gamma \in G(\mathbb{Q})$, the three pointed sets $St_{G(\mathbb{Q})}(\gamma)$, $St(\gamma, V)$ and $\ker(H^1(\mathbb{Q}, Z_G(\gamma)) \to H^1(\mathbb{Q}, G))$ are isomorphic to each other via the natural maps defined above. Moreover, these maps form a commutative diagram.*

$$\begin{array}{ccc} St_{G(\mathbb{Q})}(\gamma) & \longrightarrow & St(\gamma, V) \\ & \searrow \quad \swarrow & \\ & \ker(H^1(\mathbb{Q}, Z_G(\gamma)) \to H^1(\mathbb{Q}, G)) & \end{array}$$

For $\gamma_v \in G(\mathbb{Q}_v)$ there is a similar commutative diagram of isomorphisms among $St_{G(\mathbb{Q}_v)}(\gamma_v)$, $St(\gamma_v, V_{\mathbb{Q}_v})$ and $\ker(H^1(\mathbb{Q}_v, Z_G(\gamma_v)) \to H^1(\mathbb{Q}_v, G))$.

Proof. Immediate from the construction of maps.

□

4 Isocrystals and Barsotti-Tate groups with additional structure

Here we study isocrystals with additional structure following mainly [RR96], [Kot85] and [Kot97]. We mostly keep their convention but note that some other conventions are also used in the literature. It is worth noting that a Barsotti-Tate group over $\overline{\mathbb{F}}_p$ of pure slope λ corresponds to an isocrystal of pure slope $-\lambda$ under our covariant Dieudonné functor $\mathbb{V}$ introduced later in this section. Because of this the elements $b \in B(G)$ parametrizing the Newton polygon strata of Shimura varieties will not lie in $B(G, \mu)$ but in $B(G, -\mu)$. (See §5 and also Example 4.3).

We set up the notation for this section.

- $W := W(\overline{\mathbb{F}}_p)$ is the ring of Witt vectors.
- L is the fraction field of W.
- $\Gamma := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (In later sections Γ usually denotes $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.)
- G is a connected reductive group over $\mathbb{Q}_p$
- $T \subset G$ is a maximal torus defined over $\mathbb{Q}_p$ with Weyl group Ω.
- $A \subset G$ is a maximal $\mathbb{Q}_p$-split torus with Weyl group $\Omega_{\mathbb{Q}_p}$.
- $\mathbb{D}$ is the pro-algebraic torus with character group $\mathbb{Q}$.
- σ is the Frobenius element in $\mathrm{Gal}(L/\mathbb{Q}_p)$ inducing $x \mapsto x^p$ on the residue field.
- L_s is the fixed field of L under σ^s $(s > 0)$.

We introduce two set-valued functors on the category of connected reductive groups over $\mathbb{Q}_p$.

$$\begin{aligned} B(G) &= G(L)/\sim, \qquad x \sim y \;\Leftrightarrow\; \exists g \in G(L),\; x = g^{-1} y g^{\sigma} \\ N(G) &= (\mathrm{Int}\, G(L) \backslash \mathrm{Hom}_L(\mathbb{D}, G))^{\langle \sigma \rangle} \simeq (X_*(T)_{\mathbb{Q}}/\Omega)^{\Gamma} \simeq X_*(A)_{\mathbb{Q}}/\Omega_{\mathbb{Q}_p} \end{aligned}$$

The last isomorphism is (1.1.3.1) of [Kot84a]. For each connected reductive group G over $\mathbb{Q}_p$, there is a map

$$\nu_G : G(L) \to \mathrm{Hom}_L(\mathbb{D}, G)$$

characterized by various properties ([RR96, Thm 1.8]). Moreover, $\nu_{(\cdot)}$ induces a natural transformation of functors $\bar{\nu} : B(\cdot) \to N(\cdot)$, yielding the Newton map $\bar{\nu}_G : B(G) \to N(G)$ for each connected reductive group G.

We have the following commutative diagram which is functorial in G. (Cf. [RR96, 1.15], noting that $\pi_1(G) = X^*(Z(\widehat{G}))$.) The first row is exact as pointed sets and the second row is exact as abelian groups. The map δ_G and the horizontal arrow in the lower right corner are explained in [RR96]. The left top arrow is basically sending a cocycle in $H^1(\mathbb{Q}_p, G)$ to its evaluation at σ. (See [Kot85, 1.8] for careful definition.)

$$\begin{CD} H^1(\mathbb{Q}_p, G) @>>> B(G) @>{\bar{\nu}_G}>> N(G) \\ @V{\alpha_{G,p}}VV @V{\kappa_G}VV @V{\delta_G}VV \\ A_p(G) @>>> X^*(Z(\widehat{G})^{\Gamma}) @>>> X^*(Z(\widehat{G})_{\Gamma}) \otimes_{\mathbb{Z}} \mathbb{Q} \end{CD} \tag{5}$$

Given a cocharacter $\mu \in X_*(T) = X^*(\widehat{T})$, the finite subset $B(G,\mu)$ of $B(G)$ is defined in [Kot97, §6] (cf. Example 4.3). Let $\mu_1 \in X^*(Z(\widehat{G})^\Gamma)$ be the restriction of μ. Then every element in $B(G,\mu)$ maps to μ_1 under κ_G.

Definition 4.1. An element $\widetilde{b} \in G(L)$ is called *decent* if for some $s \in \mathbb{Z}_{>0}$, $s\nu_G(\widetilde{b})$ arises from a genuine morphism $\mathbb{G}_m \to G$ and

$$\widetilde{b}\sigma(\widetilde{b}) \cdots \sigma^{s-1}(\widetilde{b}) = s\nu_G(\widetilde{b})(p). \tag{6}$$

From here until the end of the current section, assume that G is quasi-split over $\mathbb{Q}_p$. Given $b \in B(G)$, choose a decent representative $\widetilde{b} \in G(L)$ of b, which is always possible by [Kot85, 4.3]. In fact, we can choose $\widetilde{b}$ such that the centralizer of $\nu_G(\widetilde{b})$ is defined over $\mathbb{Q}_p$ ([Kot85, p.219]). Write $M_{\widetilde{b}}$ for this Levi subgroup of G. On the other hand, define an algebraic group $J_{\widetilde{b}}$ over $\mathbb{Q}_p$ by the relation

$$J_{\widetilde{b}}(R) = \{g \in G(L \otimes_{\mathbb{Q}_p} R) \mid g = \widetilde{b}\sigma(g)\widetilde{b}^{-1}\}$$

for any $\mathbb{Q}_p$-algebra R. The group functor $J_{\widetilde{b}}$ is shown to be representable in [RZ96, 1.12].

For different representatives $\widetilde{b}$, the $\mathbb{Q}_p$-groups $J_{\widetilde{b}}$ are canonically isomorphic to each other over $\mathbb{Q}_p$. For any two $\widetilde{b}$ such that $M_{\widetilde{b}}$ is defined over $\mathbb{Q}_p$, the pairs $(\widetilde{b}, M_{\widetilde{b}})$ are conjugate to each other by an element of $G(\mathbb{Q}_p)$ ([Kot85, Prop 6.3]). For future convenience we may and will arrange that $\widetilde{b}$ is a decent element of $M_{\widetilde{b}}(L)$ (not just $G(L)$) using [Kot85, Prop 6.2]. In practice, we will write J_b (resp. M_b) for $J_{\widetilde{b}}$ (resp. $M_{\widetilde{b}}$) by agreeing that a choice of a decent representative $\widetilde{b}$ in the σ-conjugacy class b will be fixed. We remark that the fibers of the map $\bar{\nu}_G$ can be described using J_b. For each $b \in B(G)$, the set $\{b' \in B(G) | \bar{\nu}_G(b') = \bar{\nu}_G(b)\}$ is a principal homogeneous space for $H^1(\mathbb{Q}_p, J_b)$. (See [RR96, Prop 1.17].)

Lemma 4.2. *If $\widetilde{b} \in G(L)$ is decent for $s \in \mathbb{Z}_{>0}$, then $\widetilde{b}$ belongs to $G(L_s)$ and there is an isomorphism $J_b \simeq M_b$ over L_s by which J_b is an inner form of M_b over $\mathbb{Q}_p$. In case $\widetilde{b} \in M_b(L)$, this inner form is represented by the cocycle $\sigma \mapsto \mathrm{Int}(\widetilde{b})$ in $H^1(L_s/\mathbb{Q}_p, \mathrm{Int}(M_b))$.*

Proof. Corollary 1.9 and 1.14 (and the proof for the latter) in [RZ96]. □

It is easy to see that the embedding $J_b \times_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \hookrightarrow G \times_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ given by $J_b \times_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \simeq M_b \times_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ and the natural embedding $M_b \hookrightarrow G$ is canonical up to $G(\overline{\mathbb{Q}}_p)$-conjugacy.

Example 4.3. Consider the case $G = \mathrm{Res}_{K/\mathbb{Q}_p} GL_n$ where K is a finite extension of $\mathbb{Q}_p$.

First observe that $N(G) \simeq (\mathbb{Q}^n)^{S_n}$, where S_n denotes the symmetry group of n variables, can be identified with the set of the following data:

$$(r, \{\lambda_i\}_{1\le i\le r}, \{m_i\}_{1\le i\le r}) \quad \text{such that} \quad \lambda_i \in \mathbb{Q},\ r, m_i \in \mathbb{Z}_{>0},\ \lambda_1 < \cdots < \lambda_r,\ \sum_{i=1}^{r} m_i = n.$$

We exhibit $B(G)$ as the image of the map $\bar{\nu}_G : B(G) \to N(G)$, which turns out to be injective. Each image $\bar{\nu}_G(b)$ is given by rational numbers $\lambda_1 < \cdots < \lambda_r$ and the multiplicities m_i of λ_i. Our normalization is that if $G = GL_1$ then $\bar{\nu}_G$ sends the uniformizer of K to 1. The image of $\bar{\nu}_G$ is characterized by the condition

$$\forall_i,\ m_i\lambda_i \in \mathbb{Z}. \tag{7}$$

Suppose that $\bar{\nu}_G(b) = (r, \{\lambda_i\}, \{m_i\})$. Then $\delta_G(r, \{\lambda_i\}, \{m_i\}) = \sum_i \lambda_i m_i$, $\kappa_G(b) = \sum_i \lambda_i m_i$ and

$$J_b = \mathrm{Res}_{K/\mathbb{Q}_p} \prod_{i=1}^{r} GL_{m_i/h_i}(D_{-\lambda_i}), \qquad M_b = \mathrm{Res}_{K/\mathbb{Q}_p} \prod_{i} GL_{m_i}. \tag{8}$$

where $D_{-\lambda_i}$ denotes the division algebra with center K and invariant $-\lambda \in \mathbb{Q}/\mathbb{Z}$, and $h_i := [D_{-\lambda_i} : K]^{1/2}$. Noting that $G \times_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \simeq (GL_n)_{\overline{\mathbb{Q}}_p}^{\mathrm{Hom}_{\mathbb{Q}_p}(K,\overline{\mathbb{Q}}_p)}$, consider a cocharacter $\mu : \mathbb{G}_m \to G$ over $\overline{\mathbb{Q}}_p$ represented (up to conjugacy) by

$$z \mapsto \prod_{\tau \in \mathrm{Hom}_{\mathbb{Q}_p}(K,\overline{\mathbb{Q}}_p)} (\mathrm{diag}(\underbrace{z, \dots, z}_{p_\tau}, \underbrace{1, \dots, 1}_{q_\tau})$$

for nonnegative integers $(p_\tau, q_\tau)_{\tau \in \mathrm{Hom}_{\mathbb{Q}_p}(K,\overline{\mathbb{Q}}_p)}$ such that $p_\tau + q_\tau = n$. Let $n' := [K : \mathbb{Q}_p]n$. Given μ as above, set

$$(y_1, \dots, y_{n'}) := (\underbrace{1, \dots, 1}_{\sum_\tau p_\tau}, \underbrace{0, \dots, 0}_{\sum_\tau q_\tau}).$$

For an element $b \in B(G)$ corresponding to $(r, \{\lambda_i\}_{1 \le i \le r}, \{m_i\}_{1 \le i \le r})$ as above, set

$$(x_1, \dots, x_{n'}) := (\underbrace{-\lambda_1, \dots, -\lambda_1}_{[K:\mathbb{Q}_p]m_1}, \dots, \underbrace{-\lambda_r, \dots, -\lambda_r}_{[K:\mathbb{Q}_p]m_r}).$$

Then $b \in B(G, \mu)$ if and only if

$$\sum_{i=1}^{j} x_i \le \sum_{i=1}^{j} y_i \quad \text{for} \quad 1 \le j < n' \qquad \text{and} \qquad \sum_{i=1}^{n'} x_i = \sum_{i=1}^{n'} y_i. \tag{9}$$

In particular the condition (9) implies that $1 \ge -\lambda_1 > \dots > -\lambda_r \ge 0$.

Definition 4.4. By an *isocrystal*, we mean a pair (V, Φ) where V is a finite-dimensional L-vector space and $\Phi : V \to V$ is a bijection such that $\Phi(lv) = \sigma(l)\Phi(v)$ for all $l \in L, v \in V$. The *height* of (V, Φ) is the dimension of V as an L-vector space. The height 1 isocrystal $(L, p^n\sigma)$ for $n \in \mathbb{Z}$ will be denoted $L(n)$. A morphism of isocrystals from (V_1, Φ_1) to (V_2, Φ_2) is a morphism $f : V_1 \to V_2$ of L-vector spaces such that $f \circ \Phi_1 = \Phi_2 \circ f$. We denote by Isoc the category of isocrystals.

Definition 4.5. By an *isocrystal with G-structure* or simply a *G-isocrystal*, we mean an exact faithful $\mathbb{Q}_p$-linear tensor functor

$$\mathfrak{M} : \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Isoc}.$$

Here $\mathrm{Rep}_{\mathbb{Q}_p} G$ denotes the category consisting of (ρ, V) where V is a finite dimensional $\mathbb{Q}_p$-vector space and $\rho : G \to GL(V)$ is a morphism of algebraic groups over $\mathbb{Q}_p$. Denote by G-Isoc the groupoid of isocrystals with G-structure. In other words, morphisms in G-Isoc are *isomorphisms* of $\mathbb{Q}_p$-linear tensor functors.

For each σ-conjugacy class $b \in B(G)$, choose a representative $\widetilde{b} \in G(L)$. Define a functor $\mathfrak{M}_{\widetilde{b}}$ by setting $\mathfrak{M}_{\widetilde{b}}(\rho, V) = (V \otimes_{\mathbb{Q}_p} L, \rho_L(\widetilde{b}) \cdot (1 \otimes \sigma))$. We will call $\mathfrak{M}_{\widetilde{b}}$ an isocrystal with G-structure of *type* b. Note that if $\widetilde{b}'$ is another representative of b, then $\mathfrak{M}_{\widetilde{b}'}$ is canonically isomorphic to $\mathfrak{M}_{\widetilde{b}}$.

Lemma 4.6. *The association $b \mapsto \mathfrak{M}_{\widetilde{b}}$ defines a natural bijection from $B(G)$ onto the set of isomorphism classes of isocrystals with G-structure.*

Proof. The inverse map to $b \mapsto \mathfrak{M}_{\widetilde{b}}$ is given in [RR96, Rem 3.5], using Steinberg's theorem on vanishing of H^1-cohomology. □

Lemma 4.7. *There is a natural isomorphism* $\mathrm{Aut}_{G\text{-Isoc}}(\mathfrak{M}_{\widetilde{b}}) \simeq J_{\widetilde{b}}(\mathbb{Q}_p)$.

Proof. Define $\mathfrak{M}_{\widetilde{b}}^{fib} : \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Vect}_L$ to be the composition of $\mathfrak{M}_{\widetilde{b}}$ with the forgetful functor where Vect_L denotes the category of L-vector spaces. Then the automorphisms of the functor $\mathfrak{M}_{\widetilde{b}}$ are those automorphisms of $\mathfrak{M}_{\widetilde{b}}^{fib}$ preserving isocrystal structures. Namely,

$$\mathrm{Aut}(\mathfrak{M}_{\widetilde{b}}) = \{\mathscr{F} \in \mathrm{Aut}(\mathfrak{M}_{\widetilde{b}}^{fib}) : \mathscr{F}(\rho, V)\rho_L(\widetilde{b})(1 \otimes \sigma) = \rho_L(\widetilde{b})(1 \otimes \sigma)\mathscr{F}(\rho, V) \quad \text{on } V_L \quad \text{for all } (\rho, V)\}.$$

Using the isomorphism $\mathrm{Aut}(\mathfrak{M}_{\widetilde{b}}^{fib}) \simeq G(L)$, we have

$$\begin{aligned} \mathrm{Aut}(\mathfrak{M}_{\widetilde{b}}) &= \{g \in G(L) : \rho_L(g)\rho_L(\widetilde{b})(1 \otimes \sigma) = \rho_L(\widetilde{b})(1 \otimes \sigma)\rho_L(g) \quad \text{for all } (\rho, V)\} \\ &= \{g \in G(L) : g\widetilde{b}\sigma = \widetilde{b}\sigma g\} = J_{\widetilde{b}}(\mathbb{Q}_p) \end{aligned}$$

□

Remark 4.8. Define a category $\mathscr{B}(G)$ whose objects are elements of $G(L)$ and $\mathrm{Mor}(\widetilde{b}_1, \widetilde{b}_2) := \{g \in G(L) : g\widetilde{b}_1\sigma = \widetilde{b}_2\sigma g\}$. Lemma 4.6 and Lemma 4.7 mean that $\mathscr{B}(G)$ is equivalent to G-Isoc via $\widetilde{b} \mapsto \mathfrak{M}_{\widetilde{b}}$.

Example 4.9. Consider $(B, *, V, \langle\cdot,\cdot\rangle)$ and the associated $\mathbb{Q}_p$-group G as below.

- B is a finite dimensional semisimple algebra over $\mathbb{Q}_p$ with involution $*$ such that $F := Z(B)$ is a product of unramified extensions over $\mathbb{Q}_p$.
- V is a finite B-module with a $*$-Hermitian pairing $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{Q}_p$.
- G is the $\mathbb{Q}_p$-group such that for any $\mathbb{Q}_p$-algebra R,

$$G(R) = \{g \in \mathrm{End}_B(V) \otimes_{\mathbb{Q}_p} R \mid \langle gv, gw\rangle = \varpi(g)\langle v, w\rangle \text{ for some } \varpi(g) \in R^\times, \forall v, w \in V\}.$$

For instance, the datum as above is obtained by taking $\otimes_{\mathbb{Q}}\mathbb{Q}_p$ of a Shimura datum (see §5). Define a category G-Isoc$'$ whose objects are tuples $(V', \Phi', C', \langle\cdot,\cdot\rangle', i')$ where

- (V', Φ') is an isocrystal such that $V' \simeq V_L$ as L-vector spaces.
- C' is a height 1 isocrystal.
- $i' : B \to \mathrm{End}_{\mathrm{Isoc}}(V', \Phi')$ is a $\mathbb{Q}_p$-algebra map.
- $\langle\cdot,\cdot\rangle' : V' \otimes V' \to C'$ is a map of isocrystals such that the underlying map on L-vector spaces defines an L-linear nondegenerate and alternating $*$-Hermitian pairing (with respect to B-action).
- A morphism from $(V_1', \Phi_1', C_1', \langle\cdot,\cdot\rangle_1', i_1')$ to $(V_2', \Phi_2', C_2', \langle\cdot,\cdot\rangle_2', i_2')$ is a pair of isomorphisms of isocrystals $\alpha : V_1' \xrightarrow{\sim} V_2'$ and $\beta : C_1' \xrightarrow{\sim} C_2'$ such that $\langle\cdot,\cdot\rangle_2' \circ (\alpha, \alpha) = \beta \circ \langle\cdot,\cdot\rangle_1'$.

Denote by ρ the standard representation $G \hookrightarrow GL(V)$, which is defined over $\mathbb{Q}_p$. It can be shown that $\mathfrak{M} \mapsto \mathfrak{M}(\rho)$ gives an equivalence of categories G-Isoc $\xrightarrow{\sim}$ G-Isoc$'$. (cf. [RR96, Rem 3.4.(v)]) If b belongs to $B(G, -\mu)$ where G and μ arise from a Shimura datum (§5), then we may take $C' = L(-1)$ for isocrystals of type b.

We will use the terminology of Barsotti-Tate groups (or simply BT-groups) following [Mes72, Ch1].

Definition 4.10. Let Σ_1 and Σ_2 be BT-groups over S. A morphism $f : \Sigma_1 \to \Sigma_2$ is called an *isogeny* if f is an epimorphism and $\ker f$ is a finite locally free group scheme over S. A *quasi-isogeny* from Σ_1 to Σ_2 is a global section f of the sheaf $\underline{\mathrm{Hom}}_S(\Sigma_1, \Sigma_2) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that any point of S has a Zariski neighborhood where $p^n f$ is an isogeny for some positive integer n.

Definition 4.11. A *polarization* (resp. *quasi-polarization*) of a BT-group Σ over S is an isogeny (resp. a quasi-isogeny) $\lambda : \Sigma \to \Sigma^\vee$ over S such that $\lambda^\vee = [-1]\lambda$ (via the canonical isomorphism $\Sigma \simeq \Sigma^{\vee\vee}$).

Denote by BT_S^0 the category whose objects are BT-groups over S and morphisms are given by $\mathrm{Hom}_S(\Sigma_1, \Sigma_2) \otimes_{\mathbb{Z}} \mathbb{Q}$. An endomorphism algebra in BT_S^0, written as $\mathrm{End}_S^0(\Sigma)$, is a $\mathbb{Q}_p$-algebra.

Consider the case $S = \mathrm{Spec}\ \overline{\mathbb{F}}_p$. Let D be the contravariant Dieudonné functor from the category of BT-groups over $\overline{\mathbb{F}}_p$ to the category of finite free W-modules equipped with F and V actions which are semilinear for σ and σ^{-1}, respectively, such that $FV = VF = p$. This is an anti-equivalence of categories. (For instance, see [Dem72].) In this paper, we will use a covariant version of the Dieudonné functor by taking dual vector spaces. The functor $\mathbb{V}$ sends a BT-group Σ over $\overline{\mathbb{F}}_p$ to the isocrystal

$$\mathbb{V}(\Sigma) := (\mathrm{Hom}_L(D(\Sigma) \otimes_W L, L), F^*)$$

where F^* is induced by the F action on $D(\Sigma)$. We see that $\mathbb{V}$ is a fully faithful functor from $BT^0_{\overline{\mathbb{F}}_p}$ to Isoc. Observe that $\mathbb{V}(\boldsymbol{\mu}_{p^\infty}) = L(-1)$ as an isocrystal and that more generally $\mathbb{V}(\Sigma)$ has the set of slopes $\{-\lambda_i\}$ if Σ has $\{\lambda_i\}$. The usual height of a BT-group Σ is equal to the height of the isocrystal $\mathbb{V}(\Sigma)$.

Remark 4.12. The categories Isoc and BT_S^0 are $\mathbb{Q}_p$-linear categories. In particular, a morphism need not have an inverse morphism. However when it comes to isocrystals or BT-groups with G-structure, we will restrict our attention to invertible morphisms in the categories.

Now we consider BT-groups with PEL structure. Recall the notation B, F, V, G from Example 4.9. Fix a maximal order $\mathcal{O}_B$ of B.

Definition 4.13. The category $BT_S^{0,G}$ has as objects the triples (Σ, λ, i) such that

- Σ is a BT-group over S,
- $\lambda : \Sigma \to \Sigma^\vee$ is a quasi-polarization, and
- $i : \mathcal{O}_B \to \mathrm{End}_S(\Sigma)$ is a $\mathbb{Z}_p$-algebra morphism such that $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$ for all $b \in \mathcal{O}_B$.

The morphisms from $(\Sigma_1, \lambda_1, i_1)$ to $(\Sigma_2, \lambda_2, i_2)$ are the quasi-isogenies $f : \Sigma_1 \xrightarrow{\sim} \Sigma_2$ satisfying two conditions: $f \circ i_1(b) = i_2(b) \circ f$ for all $b \in \mathcal{O}_B$ and $\lambda_1 = \gamma f \circ \lambda_2 \circ f^\vee$ for some $\gamma \in \mathbb{Q}_p^\times$. The automorphism group of (Σ, λ, i) is denoted by $\mathrm{Aut}^0(\Sigma, \lambda, i)$.

The previous functor $\mathbb{V}$ can be made to incorporate the G-structure in this case. Given (Σ, λ, i), we have the isocrystal $(V', \Phi') := \mathbb{V}(\Sigma)$. By functoriality, the map i induces a $\mathbb{Q}_p$-algebra map $i' : B \to \mathrm{End}_{\mathrm{Isoc}}(V', \Phi')$. The map of BT-groups $\Sigma \times \Sigma \to \mu_{p^\infty}$ coming from λ induces a map of isocrystals $\langle \cdot, \cdot \rangle' : V' \otimes V' \to L(-1)$. So $\mathbb{V}$ can be extended to $BT^{0,G}_{\overline{\mathbb{F}}_p}$ as follows, using the notation of Example 4.9. The extended functor is again fully faithful.

$$\begin{aligned} \mathbb{V} : \ BT^{0,G}_{\overline{\mathbb{F}}_p} &\to G\text{-Isoc}' \\ (\Sigma, \lambda, i) &\mapsto (V', \Phi', L(-1), \langle \cdot, \cdot \rangle', i'). \end{aligned}$$

From Lemma 4.7 we deduce the following.

Lemma 4.14. *If $\mathbb{V}(\Sigma, \lambda, i)$ is a G-isocrystal of type b then there is an isomorphism $J_b(\mathbb{Q}_p) \simeq \mathrm{Aut}^0(\Sigma, \lambda, i)$ which is canonical up to an inner automorphism.*

Remark 4.15. Using the B-action and the pairing on V, the isocrystal $(V \otimes_{\mathbb{Q}_p} L, \widetilde{b}(1 \otimes \sigma))$ naturally extends to an object of G-Isoc$'$, which is canonically isomorphic to the image of $\mathfrak{M}_{\widetilde{b}}$ under $G\text{-Isoc} \xrightarrow{\sim} G\text{-Isoc}'$. Call this object $\mathbb{V}(\widetilde{b})$. The condition that $\mathbb{V}(\Sigma, \lambda, i)$ is a G-isocrystal of type b is equivalent to the condition that $\mathbb{V}(\Sigma, \lambda, i) \simeq \mathbb{V}(\widetilde{b})$ in G-Isoc$'$.

5 PEL-type Shimura varieties and Igusa varieties

We fix the choice of the rational primes p and l. We always assume that $p \neq l$.

Definition 5.1. A *partial Shimura PEL datum* is a quadruple $(B, *, V, \langle\cdot,\cdot\rangle)$ where

- B is a finite-dimensional simple $\mathbb{Q}$-algebra.
- $*$ is an involution of B. We assume $*$ is positive, i.e. $\mathrm{tr}\,(bb^*) > 0$ for every $b \in B^\times$.
- V is a finite semisimple B-module.
- $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{Q}$ is a $*$-Hermitian pairing with respect to B-action.

We will denote the center of B by F. We associate a $\mathbb{Q}$-group G to $(B, *, V, \langle\cdot,\cdot\rangle)$ by

$$G(R) = \{g \in \mathrm{End}_{B\otimes R}(V \otimes R) \mid \exists \varpi(g) \in R^\times,\ \langle gv_1, gv_2\rangle = \varpi(g)\langle v_1, v_2\rangle \text{ for all } v_1, v_2 \in V \otimes R\}$$

for any $\mathbb{Q}$-algebra R.

Given $(B, *, V, \langle\cdot,\cdot\rangle)$ as above, we can define a simple algebra $C := \mathrm{End}_B(V)$ with an involution $\#$. The involution $\#$ is uniquely determined by the following relation: for each $c \in C$, $\langle cv, w\rangle = \langle v, c^\# w\rangle$ for all $v, w \in V$.

Definition 5.2. An *(unramified) integral Shimura PEL datum* is a septuple $(B, \mathcal{O}_B, *, V, \Lambda_0, \langle\cdot,\cdot\rangle, h)$ where

- $(B, *, V, \langle\cdot,\cdot\rangle)$ is a partial Shimura PEL datum where $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is isomorphic to a product of matrix algebras over unramified extension fields of $\mathbb{Q}_p$.
- $h : \mathbb{C} \to C_{\mathbb{R}}$ is an $\mathbb{R}$-algebra homomorphism with involution (i.e. $\forall z \in \mathbb{C},\ h(z^c) = h(z)^*$) such that the bilinear pairing $(v, w) \mapsto \langle v, h(\sqrt{-1})w\rangle$ is symmetric and positive definite.
- $\mathcal{O}_B$ is a $\mathbb{Z}_{(p)}$-maximal order in B that is preserved by $*$ such that $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a maximal order in $B_{\mathbb{Q}_p}$.
- Λ_0 is a $\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p}$ that is preserved by $\mathcal{O}_B$ and self-dual for $\langle\cdot,\cdot\rangle$.

Put $F := Z(B)$ and $F^+ := F^{*=1}$. The above definition implies that for an integral Shimura PEL datum, F is a finite extension of $\mathbb{Q}$ unramified at p and G is unramified over $\mathbb{Q}_p$. Let us define a $\mathbb{Q}$-group G_0 by the exact sequence $1 \to G_0 \to G \xrightarrow{\varpi} \mathbb{G}_m \to 1$. Let $n := [B : F]^{1/2}$ and $d := [F^+ : \mathbb{Q}]$.

The map h in the datum induces a group homomorphism $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \to G_{\mathbb{R}}$, again written as h by abuse of notation. Define a map $\mu_h : \mathbb{G}_m \to G$ defined over $\mathbb{C}$ by the composition

$$\mathbb{C}^\times \hookrightarrow \mathbb{C}^\times \times \mathbb{C}^\times \simeq (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times \xrightarrow{(h,id)} (C \otimes_{\mathbb{R}} \mathbb{C})^\times.$$

The first arrow is the embedding $z \mapsto (z, 1)$ and the inverse of the second map is induced by the map $z_1 \otimes z_2 \mapsto (z_1 z_2, z_1 \bar{z}_2)$ on the underlying $\mathbb{C}$-algebras. We often write μ for μ_h when there is no confusion. We obtain a decomposition $V_{\mathbb{C}} = V^0 \oplus V^1$ where V^0 (resp. V^1) is the $\mathbb{C}$-vector space on which $\mu(z)$ acts by 1 (resp. z). On the other hand, as $B_{\mathbb{C}} \simeq \prod_{\tau \in \mathrm{Hom}_{\mathbb{Q}}(F,\mathbb{C})} M_n(\mathbb{C})$, we have a corresponding decomposition $V_{\mathbb{C}} = \oplus_\tau V_\tau$. The two decompostions are compatible in the sense that we have further decomposition $V_\tau = V_\tau^0 \oplus V_\tau^1$ as $B \otimes_{F,\tau} \mathbb{C}$-modules such that $V^0 = \oplus_\tau V_\tau^0$ and $V^1 = \oplus_\tau V_\tau^1$ for each $\tau \in \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{C})$. We define an integer $p_\tau := (\dim_{\mathbb{C}} V_\tau^1)/n$ for each τ. The field of definition E for the $B_{\mathbb{C}}$-module V^1 is called the reflex field. Note that E is naturally a subfield of $\mathbb{C}$. As F is unramified at p, so is the reflex field E.

An (integral) Shimura PEL datum falls into types (A), (C) or (D). Note that the existence of the map h tells us that $N := [F : F^+](\dim_F C)^{1/2}/2$ is an integer. When $*$ is of the second kind (i.e. $[F : F^+] = 2$), G_0 defines a unitary group and the PEL datum is said to be of type (A). When $*$ is of the first kind (i.e. $F = F^+$), $C_{\mathbb{R}}$ is isomorphic to either a product of $[F : \mathbb{Q}]$-copies of $M_{2N}(\mathbb{R})$ or a product of $[F : \mathbb{Q}]$ copies of $M_N(\mathbb{H})$. The former case is called type (C) and the latter type (D). We will discard type (D) throughout this paper. For type (C) we have $G_0(\mathbb{R}) \simeq \prod_{\tau \in \mathrm{Hom}(F,\mathbb{R})} Sp_{2N}(\mathbb{R})$. Note that $N = n \cdot \mathrm{rank}_B V$ for type (A) and $N = n \cdot \mathrm{rank}_B V/2$ for type (C).

Now we give the description of the Shimura variety associated to an integral Shimura PEL datum $(B, \mathcal{O}_B, *, V, \Lambda_0, \langle\cdot,\cdot\rangle, h)$ of type (A) or (C) as a moduli space of abelian schemes with additional structures. Let $U = U^p \times U_p^{max} \subseteq G(\mathbb{A}^\infty)$ where U^p is an open compact subgroup of $G(\mathbb{A}^{\infty,p})$ and U_p^{max} is the stabilizer of Λ_0 in $G(\mathbb{Q}_p)$, which is a hyperspecial subgroup. Consider the following moduli problem, which extends to the category of arbitrary schemes over $\mathcal{O}_{E,(p)}$.

$$\begin{pmatrix} \text{connected locally noetherian} \\ \text{schemes over } \mathcal{O}_{E,(p)} \end{pmatrix} \longrightarrow (\text{Sets})$$
$$S \mapsto \{(A, i, \lambda, \bar{\eta}^p)\}/\sim$$

where the quadruples on the right consist of

- A is an abelian scheme over S.
- $\lambda : A \to A^\vee$ is a prime-to-p polarization.
- $i : \mathcal{O}_B \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ such that $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$, $\forall b \in \mathcal{O}_B$.
- $\bar{\eta}^p$ is a $\pi_1(S, s)$-invariant U^p-orbit of isomorphisms of $B \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules $\eta^p : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{\sim} V^p A_s$ which take the pairing $\langle\cdot,\cdot\rangle$ to the λ-Weil pairing up to $(\mathbb{A}^{\infty,p})^\times$-multiples. Here s is any geometric point of S. (For any two geometric points s and s', $\bar{\eta}^p$ may be canonically identified.)
- (Determinant condition) An equality of polynomials $\det_{\mathcal{O}_S}(b \,|\, \mathrm{Lie}\, A) = \det_E(b \,|\, V^1)$ holds for all $b \in \mathcal{O}_B$, in the sense of [Kot92, §5].
- Two quadruples $(A_1, \lambda_1, i_1, \bar{\eta}_1^p)$ and $(A_2, \lambda_2, i_2, \bar{\eta}_2^p)$ are equivalent if there is a prime-to-p isogeny $A_1 \to A_2$ taking $\lambda_1, i_1, \bar{\eta}_1^p$ to $\gamma\lambda_2, i_2, \bar{\eta}_2^p$ for some $\gamma \in \mathbb{Z}_{(p)}^\times$.

If U^p is sufficiently small, this functor is representable by a quasi-projective smooth scheme over $\mathcal{O}_{K,(p)}$ of finite type, which we call X_U. For the proof of representability, see the comment in [Kot92, p.391]. Henceforth, we will write $(\mathscr{A}, \lambda^{univ}, i^{univ}, (\bar{\eta}^p)^{univ})$ for the universal object. (So $\mathscr{A}$ is an abelian scheme over X_U.)

Before considering the special fiber of X_U, fix an isomorphism $\iota_p : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$, which determines an embedding $E \hookrightarrow \overline{\mathbb{Q}}_p$ and thus a place w of E over p. We also fix a reduction map $\bar{\iota}_p : \mathcal{O}_{\mathbb{Q}_p^{ur}} \twoheadrightarrow \overline{\mathbb{F}}_p$.

Observe that the maps $E \hookrightarrow \overline{\mathbb{Q}}_p$ and $\bar{\iota}_p$ pin down the composite map $\mathcal{O}_E \hookrightarrow \mathcal{O}_{\mathbb{Q}_p^{ur}} \twoheadrightarrow \overline{\mathbb{F}}_p$. We choose an embedding $k(w) \hookrightarrow \overline{\mathbb{F}}_p$ so that the reduction map $\mathcal{O}_E \twoheadrightarrow k(w)$ followed by $k(w) \hookrightarrow \overline{\mathbb{F}}_p$ coincides with the last composite map.

Using ι_p, we may view $\mu = \mu_h$ as a map $\mathbb{G}_m \to G$ defined over $\overline{\mathbb{Q}}_p$. Define $\mu_1 \in X^*(Z(\widehat{G})^{\Gamma(p)})$ as follows. Choosing a maximal torus $\widehat{T}$ of $\widehat{G}$, obtain a Weyl group orbit of $\widehat{\mu}$ in $X^*(\widehat{T})$. Then μ_1 is the restriction of $\widehat{\mu}$ to $Z(\widehat{G})^{\Gamma(p)}$. It is easily seen that μ_1 is independent of the choice of h (in its $G(\mathbb{R})$-conjugacy class), $\widehat{T}$ and $\widehat{\mu}$. Clearly this definition of μ_1 is compatible with the one in the last section.

Put $\overline{X}_U := X_U \times_{\mathcal{O}_{E,(p)}} k(w)$. Consider the Newton polygon stratification

$$\overline{X}_U = \coprod_{b \in B(G)} \overline{X}_U^{(b)}$$

where each stratum is set-theoretically given by

$$\overline{X}_U^{(b)} := \{x \in \overline{X}_U : \ (\mathscr{A}_x[p^\infty], \lambda_x^{univ}, i_x^{univ}) \simeq (\Sigma, \lambda_\Sigma, i_\Sigma) \ \text{ in } \ BT^{0,G}_{\overline{\mathbb{F}}_p}\}.$$

Note that any $(\mathscr{A}_x[p^\infty], \lambda_x^{univ}, i_x^{univ})$ is a BT-group of type b for some $b \in B(G, -\mu)$ by the moduli problem. Thus $\overline{X}_U^{(b)} = \emptyset$ if $b \notin B(G, -\mu)$. As $\overline{X}_U^{(b)}$ is a locally closed subset of $\overline{X}_U$, we give $\overline{X}_U^{(b)}$ the reduced subscheme structure.

From now on, we fix $b \in B(G, -\mu)$ and focus on a single stratum $\overline{X}_U^{(b)}$. Fix once and for all a decent representative $\widetilde{b} \in G(L)$ of b (Definition 4.1). We will keep writing $\overline{X}_U^{(b)}$ and $\mathscr{A}$ for $\overline{X}_U^{(b)} \times_{k(w)} \overline{\mathbb{F}}_p$ and $\mathscr{A} \times_{k(w)} \overline{\mathbb{F}}_p$ by abuse of notation.

The result of [Win05, Thm 2] ensures the existence of a BT-group $\Sigma = \Sigma_b$ over $\overline{\mathbb{F}}_p$ equipped with a polarization $\lambda_\Sigma : \Sigma \to \Sigma^\vee$ and a $\mathbb{Z}_p$-algebra map $i_\Sigma : \mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p \hookrightarrow \mathrm{End}_{\overline{\mathbb{F}}_p}(\Sigma)$ such that

(i) $\mathbb{V}(\Sigma, \lambda_\Sigma, i_\Sigma)$ is a G-isocrystal of type b. (See §4.)

(ii) $\Sigma = \oplus_{i=1}^r \Sigma^i$ where Σ^i has slope λ_i, and $1 \geq \lambda_1 > \cdots > \lambda_r \geq 0$.

(iii) (Determinant condition) An equality of polynomials $\det_{\overline{\mathbb{F}}_p}(a \,|\, \mathrm{Lie}\,\Sigma) = \det_E(a \,|\, V^1)$ holds for all $a \in \mathcal{O}_B$, in the sense of [Kot92, §5]. (The polynomial on the left (resp. right) hand side has coefficients in $\overline{\mathbb{F}}_p$ (resp. $\mathcal{O}_{E,(p)}$). The two are compared via $\bar{\iota}_p : \mathcal{O}_{E,(p)} \to \overline{\mathbb{F}}_p$.)

(iv) The degree of λ_Σ is prime to p.

We fix a choice of such $(\Sigma, \lambda_\Sigma, i_\Sigma)$. Define the $\overline{\mathbb{F}}_p$-subscheme C_{b,U^p} of $\overline{X}_U^{(b)}$ as follows. Set-theoretically

$$C_{b,U^p} := \{x \in \overline{X}_U : \ (\mathscr{A}_x[p^\infty], \lambda_x^{univ}, i_x^{univ}) \simeq (\Sigma, \lambda_\Sigma, i_\Sigma) \ \text{ in } \ BT^{G}_{\overline{\mathbb{F}}_p}\}.$$

We give C_{b,U^p} the reduced closed subscheme structure, which makes sense since C_{b,U^p} is Zariski closed in $\overline{X}_U^{(b)}$. Recall that $\overline{X}_U^{(b)}$ is smooth over $\overline{\mathbb{F}}_p$. In fact, C_{b,U^p} is also smooth over $\overline{\mathbb{F}}_p$ ([Man05, prop 1]). Note that C_{b,U^p} could be an empty set without the condition (iii) on $(\Sigma, i_\Sigma, \lambda_\Sigma)$.

As we have a natural immersion of C_{b,U^p} into $\overline{X}_U = \overline{X}_{U^p \times U_p^{max}}$, we may pull back the BT-group (with additional structure) of the universal abelian scheme $\mathscr{A}$ to define a BT-group $\mathscr{G}$ over C_{b,U^p} with additional structure.

$$\begin{array}{ccccc} \mathscr{G} & & \longrightarrow & & \mathscr{A}[p^\infty] \\ \downarrow & & & & \downarrow \\ C_{b,U^p} & \longrightarrow & \overline{X}_U^{(b)} & \longrightarrow & \overline{X}_U \end{array}$$

Then $\mathscr{G}$ is completely slope divisible ([Man04, 3.2.3]), which means that $\mathscr{G}$ has a slope filtration $(0) = \mathscr{G}_0 \subset \mathscr{G}_1 \subset \cdots \subset \mathscr{G}_r = \mathscr{G}$ such that each quotient $gr^i\mathscr{G} := \mathscr{G}_i/\mathscr{G}_{i-1}$ has a pure slope λ_i and that $\mathscr{G}_i$ is slope divisible with respect to λ_i where $1 \geq \lambda_1 > \cdots > \lambda_r \geq 0$. Of course, the numbers r and λ_i are the same as the ones for Σ in the above. The subquotients $gr^i\mathscr{G}$ inherit the additional structure i and λ from $\mathscr{G}$. For more detail, see [Man05, §3].

Now we are ready to define Igusa varieties. They depend on our choice of $(\Sigma, i_\Sigma, \lambda_\Sigma)$ as C_{b,U^p} does.

Definition 5.3. Let m be a positive integer. The Igusa variety $\mathrm{Ig}_{b,U^p,m}$ is defined to be the moduli space of the set of the following isomorphisms of finite flat group schemes over C_{b,U^p}

$$j^{univ}_{m,i} : \Sigma^i[p^m] \times_{\overline{\mathbb{F}}_p} C_{b,U^p} \xrightarrow{\sim} gr^i\mathscr{G}[p^m], \quad 1 \leq i \leq r$$

where $j^{univ}_{m,i}$ extends étale locally to all higher levels $m' \geq m$ and preserves $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$-actions and polarizations, the latter up to $(\mathbb{Z}/p^m\mathbb{Z})^\times$-multiples.

The moduli problem for Igusa varieties is proved to be representable. (See the remark following Definition 3 of [Man05].) Note that there is a natural projection map from $\mathrm{Ig}_{b,U^p,m}$ to C_{b,U^p} forgetting the data $j^{univ}_{m,i}$. This map $\mathrm{Ig}_{b,U^p,m} \to C_{b,U^p}$ is finite étale and Galois ([Man05, prop.5]). Thus $\mathrm{Ig}_{b,U^p,m}$ is smooth over $\overline{\mathbb{F}}_p$. However $\mathrm{Ig}_{b,U^p,m}$ is usually not proper over $\overline{\mathbb{F}}_p$.

Choose an irreducible algebraic representation ξ of G on a finite dimensional $\overline{\mathbb{Q}}_l$-vector space. It naturally defines a lisse $\overline{\mathbb{Q}}_l$-sheaf $\mathscr{L}_\xi$ on $X_{U^p \times U_p^{max}}$ whenever U^p is small enough. (See [Kot92, §6] for instance.) The pullback of $\mathscr{L}_\xi$ to $\mathrm{Ig}_{b,U^p,m}$ is again denoted $\mathscr{L}_\xi$ by abuse of notation. Let Ig_b denote the projective system $\varprojlim_{U^p,m} \mathrm{Ig}_{b,U^p,m}$ and define

$$H^k_c(\mathrm{Ig}_b, \mathscr{L}_\xi) := \varinjlim_{U^p,m} H^k_c(\mathrm{Ig}_{b,U^p,m}, \mathscr{L}_\xi), \qquad \mathrm{Ig}_b(\overline{\mathbb{F}}_p) := \varprojlim_{U^p,m} \mathrm{Ig}_{b,U^p,m}(\overline{\mathbb{F}}_p).$$

We describe the action of $G(\mathbb{A}^{\infty,p})$ on the projective system Ig_b. In terms of the moduli data, $g \in G(\mathbb{A}^{\infty,p})$ acts as

$$(A, \lambda, i, \bar{\eta}^p, \{j_{m,i}\}) \mapsto (A, \lambda, i, \bar{\eta}^p \circ g, \{j_{m,i}\}).$$

We remark that g maps $\mathrm{Ig}_{b,U^p,m}$ to $\mathrm{Ig}_{b,g^{-1}U^pg,m}$.

Defining the action of $J_b(\mathbb{Q}_p)$ is more subtle. Recall from Lemma 4.14 that

$$J_b(\mathbb{Q}_p) \simeq \mathrm{Aut}^0(\Sigma, \lambda_\Sigma, i_\Sigma).$$

We fix this isomorphism and define another group consisting of genuine automorphisms (not quasi-isogenies) in the group $J_b(\mathbb{Q}_p)$:

$$\Gamma_b := \mathrm{Aut}(\Sigma) \cap J_b(\mathbb{Q}_p).$$

Choose a positive integer s such that $s\lambda_i \in \mathbb{Z}$ for all i. Define an element in $\mathrm{End}^0(\Sigma)$, formally written as fr^{-s}, which acts as $p^{-s\lambda_i}$ on Σ^i for each i. Observe that fr^{-s} belongs to the center of $J_b(\mathbb{Q}_p)$. Denote by $fr^s \in J_b(\mathbb{Q}_p)$ the inverse of fr^{-s} in $J_b(\mathbb{Q}_p)$.

We recall from [Man05, §4] the definition of a submonoid S_b of $J_b(\mathbb{Q}_p)$. For $\delta \in J_b(\mathbb{Q}_p)$, suppose that δ^{-1} is an isogeny. Any δ may be written as $\delta = (\delta_i)_{i=1}^r$ with $\delta_i \in \mathrm{End}^0(\Sigma^i)$. For each $i \in [1, r]$, we define $e(\delta_i)$ and $f(\delta_i)$ to be the minimal and maximal integers such that $\ker[p^{f(\delta_i)}] \subset \ker[\delta^{-1}] \subset \ker[p^{e(\delta_i)}]$. The monoid S_b is defined by

$$S_b := \{\delta \in J_b(\mathbb{Q}_p) \,|\, \delta^{-1} \text{ is an isogeny}, \ f(\delta_{i-1}) \geq e(\delta_i), \ \forall\, 2 \leq i \leq r\}$$

We list some properties of S_b. First, the relation $\Gamma_b \subset S_b \subset J_b(\mathbb{Q}_p)$ holds. Second, S_b contains p^{-1} and fr^{-s}. Finally, J_b is generated by S_b and the two elements p, fr^s as a monoid.

An element $\gamma \in \Gamma_b$ acts on $\mathrm{Ig}_{b,U^p,m}$ as

$$(A, \lambda, i, \bar{\eta}^p, \{j_{m,i}\}) \mapsto (A, \lambda, i, \bar{\eta}^p, \{j_{m,i} \circ \gamma\})$$

and this action extends to the projective system Ig_b. It is possible to extend this to an action of S_b on Ig_b (see Lemma 5 and the paragraph below it in [Man05]), but not to an action of $J_b(\mathbb{Q}_p)$. Nevertheless, the S_b-action on the cohomology space $H^k_c(\mathrm{Ig}_b, \mathscr{L}_\xi)$ does extend to a $J_b(\mathbb{Q}_p)$-action since the actions of $p^{-1}, fr^{-s} \in S_b$ on $H^k_c(\mathrm{Ig}_b, \mathscr{L}_\xi)$ are invertible ([Man05, Lem 6]). We define

$$U_p(m) := \ker(\Gamma_b \to \mathrm{Aut}(\Sigma[p^m], \lambda_\Sigma, i_\Sigma))$$

which are subgroups of Γ_b. They form an open basis around the identity in the group Γ_b. We know from ([Man05, Prop 4, Prop 7]) that

(i) $H^k_c(\mathrm{Ig}_{b,U^p,m}, \mathscr{L}_\xi) \simeq H^k_c(\mathrm{Ig}_b, \mathscr{L}_\xi)^{U^p \times U_p(m)}$.

(ii) The natural map $\mathrm{Ig}_{b,U^p,m} \to C_{b,U^p}$ is finite and Galois with Galois group $\Gamma_b/U_p(m)$.

In particular, the action of $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ on $H^k_c(\mathrm{Ig}_b, \mathscr{L}_\xi)$ is continuous and admissible. Define

$$H_c(\mathrm{Ig}_b, \mathscr{L}_\xi) := \sum_k (-1)^k H^k_c(\mathrm{Ig}_b, \mathscr{L}_\xi).$$

as an object of $\mathrm{Groth}(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$. Our primary goal is to study this space via a counting point formula.

6 From trace to counting points

Denote by char_H the function which has the value 1 on H and 0 outside H. Set $U^p(m) := U^p \times U_p(m)$ for any $m \in \mathbb{Z}_{>0}$. Any function $\varphi \in C^\infty_c(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$ can be written as

$$\varphi = \Sigma_{g \in I} \alpha_g \mathrm{char}_{U^p(m) g U^p(m)}$$

for some $\alpha_g \in \mathbb{C}$, $g \in I$ where I is a finite subset of $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$. So the computation of $\mathrm{tr}\,(\varphi | H_c(\mathrm{Ig}_b, \mathscr{L}_\xi))$ comes down to the case where φ is of the form $\mathrm{char}_{U^p(m) g U^p(m)}$.

Write $U^p(m) g U^p(m) = \coprod_i g_i U^p(m)$, which is a finite union. Then the following double coset action is well-defined.

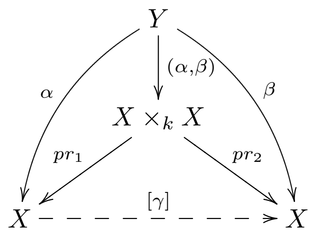
$$\mathrm{tr}\,([U^p(m) g U^p(m)] | H_c(\mathrm{Ig}_{b,U^p,m}, \mathscr{L}_\xi)) := \sum_i \sum_{k \geq 0} (-1)^k \mathrm{tr}\,(g_i | H^k_c(\mathrm{Ig}_{b,U^p,m}, \mathscr{L}_\xi)) \qquad (10)$$

It is an elementary matter to check that

$$\mathrm{tr}\,(\mathrm{char}_{U^p(m) g U^p(m)} | H_c(\mathrm{Ig}_b, \mathscr{L}_\xi)) = \mathrm{vol}(U^p(m)) \mathrm{tr}\,([U^p(m) g U^p(m)] | H_c(\mathrm{Ig}_{b,U^p,m}, \mathscr{L}_\xi)). \qquad (11)$$

We recall the notion of fixed points of an algebraic correspondence in general. Let $\alpha, \beta : Y \to X$ be morphisms of k-varieties where k is an algebraically closed field. The correspondence induced by

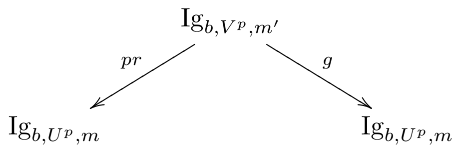
α and β will be denoted $[\gamma]$. The maps pr_1 and pr_2 are the projections onto the first and the second components.



Then we have an induced map $Y(k) \overset{(\alpha,\beta)}{\to} X(k) \times X(k)$. Define the set of fixed points

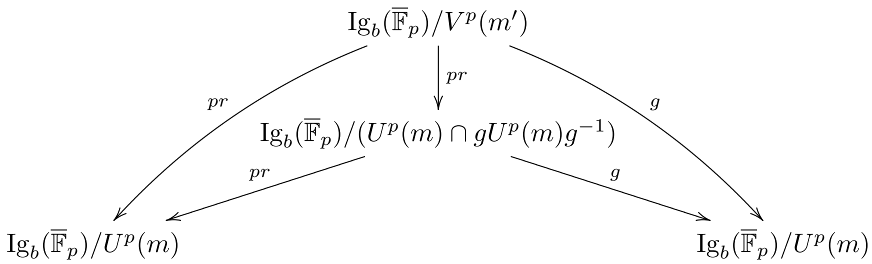
$$\mathrm{Fix}([\gamma]) := \{y \in Y(k) | \alpha(y) = \beta(y)\}. \tag{12}$$

Now we consider correspondences on Igusa varieties. We would like to interpret the action of $[U^p(m)gU^p(m)]$ in (10) as an algebro-geometric correspondence to which Fujiwara's trace formula can be applied. For this interpretation, we need to assume that $g \in G(\mathbb{A}^{\infty,p}) \times S_b$. Then we may choose a small enough subgroup $V^p(m')$ contained in $U^p(m) \cap gU^p(m)g^{-1}$ so that the map given by g below is well-defined ([Man05, Lem 6]). The correspondence $[U^p(m)gU^p(m)]$ is understood as in the following diagram where pr means the natural projection.



(13)

In practice, we may often regard $[U^p(m)gU^p(m)]$ as a set-theoretic correspondence. Recall that $\mathrm{Ig}_b(\overline{\mathbb{F}}_p) = \varprojlim_{U^p,m} \mathrm{Ig}_{b,U^p,m}(\overline{\mathbb{F}}_p)$ and $\mathrm{Ig}_{b,U^p,m}(\overline{\mathbb{F}}_p) = \mathrm{Ig}_b(\overline{\mathbb{F}}_p)/U^p(m)$ as sets with right $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-action. On the level of $\overline{\mathbb{F}}_p$-points, (13) fits into the following diagram. Note that in general the map $g : \mathrm{Ig}_b(\overline{\mathbb{F}}_p)/(U^p(m) \cap gU^p(m)g^{-1}) \to \mathrm{Ig}_b(\overline{\mathbb{F}}_p)/U^p(m)$ does not come from a map of algebraic varieties.



(14)

When we consider $[U^p(m)gU^p(m)]$ as an algebraic correspondence on $\mathrm{Ig}_{b,U^p,m}$, we need to think of it a priori as (13). When dealing with $\overline{\mathbb{F}}_p$-points, $[U^p(m)gU^p(m)]$ may also be understood as (14). The set of fixed points under $[U^p(m)gU^p(m)]$ as a set-theoretic correspondence will be understood as

$$\mathrm{Fix}([U^p(m)gU^p(m)]) = \{x \in \mathrm{Ig}_b(\overline{\mathbb{F}}_p)/(U^p(m) \cap gU^p(m)g^{-1}) \,|\, x = xg \text{ in } \mathrm{Ig}_b(\overline{\mathbb{F}}_p)/U^p(m)\}. \tag{15}$$

The virtue of the algebro-geometric interpretation is that we may apply Fujiwara's trace formula to compute the trace. (See the proof of Lemma 6.3.) The formula that we would like to have is

$$\mathrm{tr}\left([U^p(m)gU^p(m)] | H_c(\mathrm{Ig}_{b,U^p,m}, \mathscr{L}_\xi)\right) = \sum_{x \in \mathrm{Fix}([U^p(m)gU^p(m)])} \mathrm{tr}\left([U^p(m)gU^p(m)] | (\mathscr{L}_\xi)_x\right). \tag{16}$$

where $\mathrm{Fix}([U^p(m)gU^p(m)])$ is the set of fixed points in the sense of (12) under the algebro-geometric correspondence $[U^p(m)gU^p(m)]$ understood as (13) (for a chosen subgroup $V^p(m')$ there). But once we know the validity of (16), it is an easy exercise to check that the same identity still holds if $[U^p(m)gU^p(m)]$ is interpreted as the double coset action in (10) and $\mathrm{Fix}([U^p(m)gU^p(m)])$ as in (15).

The following definitions are motivated in two ways. On one hand, we want to allow a twist by high powers of Frobenius so that the fixed point formula is available. On the other hand, we want to separate slope components of elements in $J_b(\mathbb{Q}_p)$ in terms of p-adic valuation, which will play a role in harmonic analysis later.

Definition 6.1. An element $\delta \in J_b(\mathbb{Q}_p)$ is called *acceptable* if $\delta = (\delta_i)$ viewed inside $\prod_{i=1}^r \mathrm{End}^0(\Sigma_i)^\times$ verifies the following condition: if $\lambda_i > \lambda_j$ (i.e. if $i < j$), any eigenvalue e_i of δ_i and e_j of δ_j satisfy $v_p(e_i) < v_p(e_j)$. Here $v_p : \overline{\mathbb{Q}}_p^\times \to \mathbb{Q}$ is an additive p-adic valuation.

Definition 6.2. A function $\varphi \in C_c^\infty(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$ is called *acceptable* if

(i) For any $(g, \delta) \in G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ in supp φ, the element δ is acceptable and belongs to S_b, and there exist a finite subset $I \subset G$, a sufficiently small subgroup $U^p(m)$ (in particular, having no finite torsion elements) and $(\alpha_g)_{g\in I} \in \mathbb{C}$ satisfying $\varphi = \Sigma_{g\in I}\alpha_g \mathrm{char}_{U^p(m)gU^p(m)}$ such that

(ii) $\mathrm{Fix}([U^p(m)gU^p(m)])$ is a finite set for every $g \in I$, and

(iii) For every $g \in I$, the formula (16) holds.

We will show in Lemma 6.4 that acceptable functions are abundant enough to establish an identity of representations in $\mathrm{Groth}(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$. So it is harmless to assume that the test function φ is acceptable when computing the trace. First we prove that any given test function becomes acceptable after enough twists.

Lemma 6.3. *For each $\varphi \in C_c^\infty(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$, there exists a positive integer M such that whenever $N > M$, the function $\varphi^{(N)}$ defined by $\varphi^{(N)}(g) = \varphi(g \cdot (fr^s)^N)$ is acceptable.*

Proof. The proof is easily reduced to the case $\varphi = \mathrm{char}_{U^p(m)gU^p(m)}$ where $U^p(m)$ is small enough. From the definition of acceptable elements and the set S_b, there is clearly an integer M such that every $\varphi^{(N)}$ for $N > M$ satisfies (i) of Definition 6.2.

The conditions (ii) and (iii) can be verified using Fujiwara's trace formula (a.k.a. Deligne's conjecture). For this purpose, we choose a particular model $\mathcal{J}_{b,U^p,m}$ over some finite field $\mathbb{F}_{p^s}$ such that $\mathcal{J}_{b,U^p,m} \times_{\mathbb{F}_{p^s}} \overline{\mathbb{F}}_p \simeq \mathrm{Ig}_{b,U^p,m}$ and $F_{ab}^s \times 1 = fr^{-s}$ under this isomorphism. Here F_{ab}^s is the absolute Frobenius on $\mathcal{J}_{b,U^p,m}$ and $fr^{-s} \in J_b(\mathbb{Q}_p)$ acts on $\mathrm{Ig}_{b,U^p,m}$ as described in §5. (That we can choose $\mathcal{J}_{b,U^p,m}$ is explained in [Shi07, §2.3] in more detail. For this we assume that $\mathbb{F}_{p^s}$ contains $k(w)$ by enlarging s if necessary.)

According to Fujiwara's formula([Fuj97, Cor 5.4.5], [Var07, Thm 2.3.2]), the following is true: there exists an integer $M' > 0$ such that whenever $N > M'$, $\mathrm{Fix}((Fr_{ab}^s \times 1)^N \circ [U^p(m)gU^p(m)])$ is a finite set and the identity (16) holds with $[U^p(m)gU^p(m)]$ replaced by $(Fr_{ab}^s \times 1)^N \circ [U^p(m)gU^p(m)]$. The number M' can be chosen independently of the coefficient sheaf.

By the identity of correspondences on $\mathrm{Ig}_{b,U^p,m}$

$$[U^p(m)g(fr^{-s})^N U^p(m)] = ((F_{ab}^s \times 1)^N) \circ [U^p(m)gU^p(m)],$$

the conditions (ii) and (iii) are verified by $\varphi^{(N)}$ for every $N > M'$. Finally increase M, if necessary, to ensure $M \geq M'$. □

Lemma 6.4. *Suppose that* Π_1 *and* Π_2 *belong to* $\mathrm{Groth}(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$*. If* $\mathrm{tr}\,\Pi_1(\varphi) = \mathrm{tr}\,\Pi_2(\varphi)$ *for every acceptable function* φ*, then* $\Pi_1 \simeq \Pi_2$ *in* $\mathrm{Groth}(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$*.*

Proof. For simplicity of notation, let $H := G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$. Let us choose an arbitrary function $\varphi \in C_c^\infty(H)$ (which is not necessarily acceptable). If we show $\mathrm{tr}\,\Pi_1(\varphi) = \mathrm{tr}\,\Pi_2(\varphi)$, then the proof will be complete.

There are only finitely many irreducible representations $\{\pi_i\}_{i\in I}$ of H contributing to Π_1 or Π_2 such that $\pi_i(\varphi)$ is nontrivial. Let m_i (resp. n_i) be the multiplicity of π_i in Π_1 (resp. Π_2) and set

$$\Pi_1' := \sum_{i\in I} m_i\pi_i, \qquad \Pi_2' := \sum_{i\in I} n_i\pi_i.$$

Set $t := fr^s$. Note that t belongs to the center of H. Consider the map $\theta : H \times \mathbb{Z} \to H$ given by $(h, z) \mapsto (h \cdot t^z)$. Write Π_1'' and Π_2'' for pullbacks of Π_1' and Π_2' along θ. For each $i \in I$, the pullback of π_i by θ has the form $\pi_i \otimes \chi_i$ for a character χ_i of $\mathbb{Z}$. Define $\varphi^{(z)} \in C_c^\infty(H)$ by $\varphi^{(z)}(h) := \varphi(ht^z)$. By Lemma 6.3, there exists a constant $C > 0$ (depending only on φ) such that $\varphi^{(z)}$ is acceptable for all $z > C$. By assumption,

$$\mathrm{tr}\,\Pi_1'(\varphi^{(z)}) = \mathrm{tr}\,\Pi_2'(\varphi^{(z)}), \quad \forall z > C. \tag{17}$$

We claim that for any $\psi \in C_c^\infty(\mathbb{Z}_{<-C})$

$$\mathrm{tr}\,\Pi_1''(\varphi \times \psi) = \mathrm{tr}\,\Pi_2''(\varphi \times \psi). \tag{18}$$

Once we prove the claim, since there are finitely many characters $\{\chi_i\}_{i\in I}$ (not necessarily distinct), it follows that (18) is true for any $\psi \in C_c^\infty(\mathbb{Z})$. In particular, we choose ψ to be a function supported on $0 \in \mathbb{Z}$ to deduce that $\mathrm{tr}\,\Pi_1'(\varphi) = \mathrm{tr}\,\Pi_2'(\varphi)$, or $\mathrm{tr}\,\Pi_1(\varphi) = \mathrm{tr}\,\Pi_2(\varphi)$.

It remains to prove the above claim. It suffices to prove that (18) holds for every ψ_y ($y < -C$) such that $\psi_y(z)$ equals 1 if $z = y$ and 0 if $z \neq y$. For any w in the representation space of Π_j', computation with respect to a Haar measure on H shows

$$\begin{aligned}
\Pi_j''(\varphi \times \psi_y)w &= \sum_{z\in\mathbb{Z}} \int_H (\varphi(h)\psi_y(z)) \cdot \Pi_j''(h, z)w \cdot dh \\
&= \int_H \varphi(h)\Pi_j'(ht^y)w \cdot dh \\
&= \int_H \varphi(ht^{-y})\Pi_j'(h)w \cdot dh = \Pi_j'(\varphi^{(-y)}) \cdot w
\end{aligned}$$

Combining with (17), we deduce that (18) holds for $\psi = \psi_y$. This proves our claim.

□

7 $\overline{\mathbb{F}}_p$-points of Igusa varieties

In order to describe the set of fixed points on Igusa varieties under correspondences, we give here a moduli-theoretic description of $\overline{\mathbb{F}}_p$-points on Igusa varieties.

As Ig_b has a moduli interpretation, we can describe its $\overline{\mathbb{F}}_p$-points in terms of abelian varieties over $\overline{\mathbb{F}}_p$ with additional structure. We can see from the construction of Ig_b in §5 that $\mathrm{Ig}_b(\overline{\mathbb{F}}_p)$ is identified with the following set $\widetilde{\mathrm{Ig}}_b^p$.

$\widetilde{\mathrm{Ig}}_b^p = \{(A, \lambda, i, \eta^p, \{j_i\})\}/\sim$, where

- A is an abelian variety over $\overline{\mathbb{F}}_p$ such that there exists an isomorphism $A[p^\infty] \xrightarrow{\sim} \oplus_{i=1}^r gr^i A[p^\infty]$.
- $\lambda : A \to A^\vee$ is a *prime-to-p* polarization.
- $i : \mathcal{O}_B \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a map of $\mathbb{Z}_{(p)}$-algebras such that $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$, $\forall b \in \mathcal{O}_B$.
- $\eta^p : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{\sim} V^p A$ is an isomorphism of $B \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules sending $\langle \cdot, \cdot \rangle$ to the λ-Weil pairing up to $(\mathbb{A}^{\infty,p})^\times$-multiple.
- $\{j_i\}_{1 \le i \le r} : \Sigma^i \to gr^i A[p^\infty]$ is an isomorphism in the category $BT^G_{\overline{\mathbb{F}}_p}$. (i.e. preserving $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$-actions and polarizations, the latter up to $\mathbb{Z}_p^\times$-multiple.)
- $(A, \lambda, i, \eta^p, \{j_i\})$ and $(A', \lambda', i', \eta^{p\prime}, \{j_i'\})$ are equivalent if there is a *prime-to-p* isogeny $A \to A'$ sending $(\lambda, i, \eta^p, \{j_i\})$ to $(\gamma\lambda', i', \eta^{p\prime}, \{j_i'\})$ where $\gamma \in \mathbb{Z}_{(p)}^\times$.

We will see that the set $\widetilde{\mathrm{Ig}}_b^p$ is in natural bijection with the following set $\widetilde{\mathrm{Ig}}_b$ which is simpler to describe. Note that the prime-to-p condition is removed below.

$\widetilde{\mathrm{Ig}}_b = \{(A, \lambda, i, \eta^p, \{j_i\})\}/\sim$, where

- A is an abelian variety over $\overline{\mathbb{F}}_p$.
- $\lambda : A \to A^\vee$ is a polarization.
- $i : B \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a map of $\mathbb{Q}$-algebras such that $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$, $\forall b \in B$.
- $\eta^p : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{\sim} V^p A$ is an isomorphism of $B \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules sending $\langle \cdot, \cdot \rangle$ to the λ-Weil pairing up to $(\mathbb{A}^{\infty,p})^\times$-multiple.
- $\{j_i\}_{1 \le i \le r} : \Sigma^i \to gr^i A[p^\infty]$ is a quasi-isogeny, which is an isomorphism in the category $BT^{0,G}_{\overline{\mathbb{F}}_p}$. (i.e. preserving $B_{\mathbb{Q}_p}$-actions and polarizations, the latter up to $\mathbb{Q}_p^\times$-multiple.)
- $(A, \lambda, i, \eta^p, \{j_i\})$ and $(A', \lambda', i', \eta^{p\prime}, \{j_i'\})$ are equivalent if there is an isogeny $A \to A'$ sending $(\lambda, i, \eta^p, \{j_i\})$ to $(\gamma\lambda', i', \eta^{p\prime}, \{j_i'\})$ where $\gamma \in \mathbb{Q}^\times$.

Lemma 7.1. *(Compare [HT01, Lem V.1.1.1])*
There is a natural bijection between $\mathrm{Ig}_b(\overline{\mathbb{F}}_p)$ *and* $\widetilde{\mathrm{Ig}}_b^p$. *The natural map* $\widetilde{\mathrm{Ig}}_b^p \to \widetilde{\mathrm{Ig}}_b$ *is also a bijection.*

Proof. The first sentence is clear from the moduli description of the variety Ig_b. We will prove the second sentence of the lemma.

We first prove that the map $\widetilde{\mathrm{Ig}}_b^p \to \widetilde{\mathrm{Ig}}_b$ is injective. In other words, if $(A, \lambda, i, \eta^p, \{j_i\})$ and $(A', \lambda', i', \eta^{p\prime}, \{j_i'\})$ in $\widetilde{\mathrm{Ig}}_b^p$ become equivalent in $\widetilde{\mathrm{Ig}}_b$ by an isogeny $f : A \to A'$, then we need to find a prime-to-p isogeny which identifies the two data in $\widetilde{\mathrm{Ig}}_b^p$. But f itself has to be a prime-to-p isogeny since $j_i' = f \circ j_i$ for every i where both j_i and j_i' are isomorphisms. Also if $\gamma \in \mathbb{Q}^\times$ is such that λ is sent to $\gamma\lambda'$, then γ should belong to $\mathbb{Z}_{(p)}^\times$ since both λ and λ' are prime-to-p polarizations. Therefore f gives an equivalence in $\widetilde{\mathrm{Ig}}_b^p$.

We prove the map is surjective. We start from an element $(A, \lambda, i, \eta^p, \{j_i\})$ in $\widetilde{\mathrm{Ig}}_b$ and obtain an element in $\widetilde{\mathrm{Ig}}_b^p$ using the equivalence in $\widetilde{\mathrm{Ig}}_b$. By changing A by an isogeny if necessary, we may assume that the condition $A[p^\infty] \xrightarrow{\sim} \oplus_{i=1}^r gr^i A[p^\infty]$ is satisfied. Keeping the last condition, all j_i can be arranged to be isomorphisms. We explain the last point in more detail. First we may assume that each quasi-isogeny $\{j_i^{-1}\}$ is an isogeny by applying p-power multiplication map to A if necessary. Let $H_i := \ker j_i^{-1}$, $A' := A/(\oplus_i H_i)$ and $f : A \to A/(\oplus_i H_i)$ be the natural quotient map. Then f sends j_i to $f \circ j_i$, but $f \circ j_i : \Sigma^i \xrightarrow{\sim} gr^i A'[p^\infty]$ is an isomorphism for each i by construction.

Next we want λ to be prime-to-p. This is easy because the equivalence in $\widetilde{\mathrm{Ig}}_b$ allows multiplying a scalar in $\mathbb{Q}_p^\times$ to λ. First observe that the following diagram commutes for some $\gamma \in \mathbb{Q}_p^\times$ where maps are allowed to be quasi-isogenies.

$$\begin{CD} \Sigma @>{j}>{\sim}> A[p^\infty] \\ @V{\lambda_\Sigma}VV @VV{\gamma\lambda}V \\ \Sigma^\vee @<{\sim}<{j^\vee}< A^\vee[p^\infty] \end{CD}$$

Write $\gamma = p^a u$ for $a \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^\times$. Then we simply replace λ with $p^a\lambda$ to get a prime-to-p polarization. (Recall that λ_Σ is already a prime-to-p polarization.) At this point, it only remains to check that the image of $\mathcal{O}_B$ under i lies in $\mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$, but this is automatic since $\mathcal{O}_B$ is a maximal $\mathbb{Z}_{(p)}$-order of B. Now our new $(A, \lambda, i, \eta^p, \{j_i\})$ belongs to $\widetilde{\mathrm{Ig}}_b^p$, completing the proof of surjectivity. □

In [Man05, §5], it was shown that the action of $G(\mathbb{A}^{\infty,p}) \times S_b$ on $\widetilde{\mathrm{Ig}}_b^p$ extends to an action of $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$. When this action is transported to $\widetilde{\mathrm{Ig}}_b$ via the bijection in Lemma 7.1, the action of each element $(\alpha, \beta) \in G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ on $\widetilde{\mathrm{Ig}}_b$ can be described as

$$(A, \lambda, i, \eta^p, \{j_i\}) \mapsto (A, \lambda, i, \eta^p \circ \alpha, \{j_i \circ \beta\}).$$

In view of Lemma 7.1, the right $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-set $\mathrm{Ig}_b(\overline{\mathbb{F}}_p)$ will be described in terms of $\widetilde{\mathrm{Ig}}_b$ from now on. To further analyze $\mathrm{Ig}_b(\overline{\mathbb{F}}_p)$, we consider the fibration of this set over the set of the triples (A, λ, i).

Definition 7.2. We define the set $\mathrm{PIC}_b = \{(A, \lambda, i)\}/\sim$ whose representatives are those (A, λ, i) that appear in the description of $\mathrm{Ig}_b(\overline{\mathbb{F}}_p)$ (i.e. $\exists\ \eta^p, \{j_i\}$ such that $(A, \lambda, i, \eta^p, \{j_i\}) \in \mathrm{Ig}_b(\overline{\mathbb{F}}_p)$). We consider (A, λ, i) and (A', λ', i') equivalent if there is an isogeny $A \to A'$ sending λ, i to $\gamma\lambda'$ and i' for some $\gamma \in \mathbb{Q}^\times$.

By construction, we have a natural $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-equivariant (with trivial action on PIC_b) surjection of sets

$$\pi : \mathrm{Ig}_b(\overline{\mathbb{F}}_p) \to \mathrm{PIC}_b \quad \text{defined by} \quad (A, \lambda, i, \eta^p, \{j_i\}) \mapsto (A, \lambda, i).$$

Before we give a group theoretic expression of the fibers of π, we set up some notation. Let $z = [(A, \lambda, i)]$ be an equivalence class in PIC_b. We define the following.

- $C_{(A,\lambda,i)} := \mathrm{End}_B^0(A)$,
- $M_{(A,\lambda,i)} := Z(C_{(A,\lambda,i)})$
- $\ddagger_{(A,\lambda,i)}$ is the Rosati involution $f \mapsto \lambda^{-1}f^\vee\lambda$ on $C_{(A,\lambda,i)}$,
- $H_{(A,\lambda,i)}$ is the $\mathbb{Q}$-group scheme such that $H_{(A,\lambda,i)}(R) := \{g \in C_{(A,\lambda,i)} \otimes_{\mathbb{Q}} R | gg^{\ddagger_{(A,\lambda,i)}} \in R^\times\}$

Suppose that $[(A, \lambda, i)] = [(A', \lambda', i')]$ and let $f : A \to A'$ be an isogeny providing the equivalence of triples. Then the induced identification $M_{(A,\lambda,i)} = M_{(A',\lambda',i')}$ is independent of the choice of f. But f induces an isomorphism of the pairs $(C_{(A,\lambda,i)}, \ddagger_{(A,\lambda,i)})$ and $(C_{(A',\lambda',i')}, \ddagger_{(A',\lambda',i')})$ and an isomorphism of $\mathbb{Q}$-groups $H_{(A,\lambda,i)} \simeq H_{(A',\lambda',i')}$ which are canonical only up to $H_{(A,\lambda,i)}(\mathbb{Q})$-conjugacy. Keeping this in mind, we may sometimes write $C_{(A,\lambda,i)}$, $M_{(A,\lambda,i)}$, $\ddagger_{(A,\lambda,i)}$ and $H_{(A,\lambda,i)}$ as C_z, M_z, $\ddagger_z$ and H_z when $H_{(A,\lambda,i)}(\mathbb{Q})$-conjugacy is harmless.

Let us give an embedding $\iota_{(A,\lambda,i)} : H_{(A,\lambda,i)}(\mathbb{A}^\infty) \hookrightarrow G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$. For this we need to choose some $(\eta_0^p, \{j_{0,i}\})$ which defines a point in $\mathrm{Ig}_b(\overline{\mathbb{F}}_p)$ together with (A, λ, i). First consider the composite map

$$\mathrm{End}_B^0(A) \otimes \mathbb{A}^{\infty,p} \hookrightarrow \mathrm{End}_B(V^p A) \simeq \mathrm{End}_B(V \otimes \mathbb{A}^{\infty,p})$$

where the latter isomorphism is $g \mapsto (\eta_0^p)^{-1} g \eta_0^p$. Thereby we get an embedding of groups

$$H_{(A,\lambda,i)}(\mathbb{A}^{\infty,p}) \hookrightarrow G(\mathbb{A}^{\infty,p}).$$

On the other hand, we have maps

$$\mathrm{End}_B^0(A) \otimes \mathbb{Q}_p \hookrightarrow \mathrm{End}_B^0(A[p^\infty]) \simeq \mathrm{End}_B^0(\Sigma)$$

where the second map is induced from the quasi-isogeny $\Sigma \to A[p^\infty]$ given by $\{j_{0,i}\}$. By restricting to the elements preserving polarizations on both sides, obtain

$$H_{(A,\lambda,i)}(\mathbb{Q}_p) \hookrightarrow J_b(\mathbb{Q}_p).$$

Putting these together, we obtain an embedding $\iota_{(A,\lambda,i)} : H_{(A,\lambda,i)}(\mathbb{A}^\infty) \hookrightarrow G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$, which is canonical up to $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-conjugacy. The next lemma, whose proof is straightforward, gives a description of the fibers of π.

Lemma 7.3. *See [HT01, Lem V.1.2]*

Choose of a base point $\widetilde{x}$ in $\pi^{-1}[(A, \lambda, i)]$. This gives a bijection of sets with right $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-action

$$\begin{array}{ccc} \pi^{-1}[(A, \lambda, i)] & \simeq & \iota_{(A,\lambda,i)}(H_{(A,\lambda,i)}(\mathbb{Q})) \backslash (G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)) \\ \widetilde{x}g & \leftrightarrow & g \end{array}$$

If we choose $\widetilde{x}' = \widetilde{x}h$ ($h \in G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$) as a base point, the above isomorphism changes by multiplication by h while $\iota_{(A,\lambda,i)}$ changes by conjugation by h.

Suppose that g is an element of $G(\mathbb{A}^{\infty,p}) \times S_b$ and that $[U^p(m) g U^p(m)]$ satisfies the conditions (ii) and (iii) of Definition 6.2. For simplicity let us write U for $U^p(m)$ in this section. We deduce from (11) and (16) that

$$\mathrm{tr}\,(\mathrm{char}_{UgU} | H_c(\mathrm{Ig}_b, \mathscr{L}_\xi)) = \mathrm{vol}(U) \sum_{x \in \mathrm{Fix}([UgU])} \mathrm{tr}\,([UgU] | (\mathscr{L}_\xi)_x) \qquad (19)$$

The sum has finitely many nonzero terms and is finite. Our next task is to analyze the set $\mathrm{Fix}([UgU])$, which is given by (15). Let us write G_b for $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$. Arguing as in [HT01, p.153-155], the expression in (15) can be rewritten as

$$\mathrm{Fix}([UgU]) = \coprod_{z \in \mathrm{PIC}_b} \coprod_{[a] \in H_z(\mathbb{Q})/\sim} \iota_z(H_z(\mathbb{Q})) \backslash \{y \in G_b | y^{-1} \iota_z(a) y \in gU\} / U \cap gUg^{-1} \qquad (20)$$

where the equivalence relation in $H_z(\mathbb{Q})$ is given by $H_z(\mathbb{Q})$-conjugacy action. Again proceeding as in [HT01, Lem V.1.4], we obtain the first form of the counting point formula.

Lemma 7.4. *Suppose that $\varphi \in C_c^\infty(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$ is an acceptable function. Then*

$$\mathrm{tr}\,(\varphi | H_c(\mathrm{Ig}_b, \mathscr{L}_\xi)) = \sum_{z \in \mathrm{PIC}_b} \sum_{[a] \in H_z(\mathbb{Q})/\sim} \mathrm{vol}(\iota_z(Z_{H_z}(a)(\mathbb{Q})) \backslash Z_{G_b}(\iota_z(a)))\ O_{\iota_z(a)}^{G_b}(\varphi) \cdot \mathrm{tr}\,\xi(\iota_z(a)) \qquad (21)$$

The sum has finitely many nonzero terms and is finite. The measure on $\iota_z(Z_{H_z}(a)(\mathbb{Q}))$ is chosen such that every point has measure 1. Haar measures on other groups are chosen to be compatible with each other.

Remark 7.5. Since the group $H_z(\mathbb{R})$ is compact modulo center, any element $a \in H_z(\mathbb{Q})$ is semisimple and elliptic in $H_z(\mathbb{R})$.

8 Honda-Tate theory

In this section we use a version of Honda-Tate theory to parametrize the pairs (A, i) by p-adic types over F. We also give a necessary condition for (A, i) to appear in the set PIC_b. We generalize the notion of p-adic types in [HT01, V.2] in order to classify isogeny classes of abelian varieties over $\overline{\mathbb{F}}_p$ which are not necessarily simple. Before defining p-adic types, we will set up some notation.

- Let I be a finite index set.
- Let M_t be a CM field or a totally real field for each $t \in I$. Note that M_t has a well-defined complex conjugation c, an automorphism of order 2 or 1, respectively.
- Let $\mathfrak{P}_{M_t}$ be the set of places of M_t over p, also written as $\mathfrak{P}_t$ for simplicity.
- $\mathbb{Q}[\mathfrak{P}_t] := \oplus_{x \in \mathfrak{P}_t} \mathbb{Q} \cdot x$ is the $\mathbb{Q}$-vector space with basis $\mathfrak{P}_t$.
- If $\mathfrak{a}$ is a fractional ideal of M_t, we define $[\mathfrak{a}] := \sum_{x \in \mathfrak{P}_t} x(\mathfrak{a}) \cdot x \in \mathbb{Q}[\mathfrak{P}_t]$.
- If $i : M \hookrightarrow N$ is a finite extension of fields where M, N are totally real or CM, we define a $\mathbb{Q}$-linear map $i_* : \mathbb{Q}[\mathfrak{P}_M] \to \mathbb{Q}[\mathfrak{P}_N]$ by $x \mapsto \sum_{y|x} e_{y/x} y$.

Definition 8.1. Let F_0 be a number field. A *p-adic type* over F_0 is a quadruple $(M, \vec{\eta}, \vec{n}, \kappa)$ where

- $M = \prod_{t \in I} M_t$ is a product of totally real or CM fields for a nonempty index set I,
- $\vec{\eta} = (\eta_t)_{t \in I}$ where $\eta_t = \sum_{x \in \mathfrak{P}_t} \eta_{t,x} x \in \mathbb{Q}[\mathfrak{P}_t]$,
- $\vec{n} = (n_t)_{t \in I}$ is a collection of positive integers and
- $\kappa : F_0 \to M$ is a $\mathbb{Q}$-algebra homomorphism.

such that for all $t \in I$, $\eta_t + c_* \eta_t = [p]$ and $\forall x \in \mathfrak{P}_t,\ \eta_{t,x} \geq 0$. We will often drop κ from the data when κ is well understood as the F_0-algebra structure map of M.

Definition 8.2. A p-adic type $(M, \vec{\eta}, \vec{n}, \kappa)$ is called *simple* if M is a field and $\vec{n} = (1)$. Such a p-adic type will often be written as (M, η) when κ is understood.

Remark 8.3. The p-adic types defined in [HT01, V.2] correspond to our simple p-adic types.

We say $(M', \vec{\eta}', \vec{n}', \kappa')$ and $(M'', \vec{\eta}'', \vec{n}'', \kappa'')$ are *equivalent* over F_0 if there exist a p-adic type $(M, \vec{\eta}, \vec{n}, \kappa)$ and F_0-algebra embeddings $i' : M' \hookrightarrow M$, $i'' : M'' \hookrightarrow M$ (F_0-structure given by $\kappa, \kappa', \kappa''$) such that

(i) Whenever $t_1' \neq t_2'$ and $t_1'' \neq t_2''$, we have $i'(M'_{t_1'}) i'(M'_{t_2'}) = 0$ and $i''(M''_{t_1''}) i''(M''_{t_2''}) = 0$.

(ii) There is a partition of the index set $I' = \coprod_{t \in I} I'_t$ ($I'_t \neq \emptyset$) for $(M', \vec{\eta}', \vec{n}')$ satisfying the following: $\forall t' \in I'_t$, i' induces $M'_{t'} \hookrightarrow M_t$, $i'_* \eta_{t'} = \eta_t$, and $\sum_{t' \in I'_t} n_{t'} = n_t$. There is a partition $I'' = \coprod_{t \in I} I''_t$ such that an exactly analogous condition holds for $(M'', \vec{\eta}'', \vec{n}'')$.

Two p-adic types $(M', \vec{\eta}', \vec{n}', \kappa')$ and $(M'', \vec{\eta}'', \vec{n}'', \kappa'')$ over F_0 are said to be *isomorphic* if there exists an F_0-algebra isomorphism $M' \xrightarrow{\sim} M''$ sending $\vec{\eta}', \vec{n}'$ to $\vec{\eta}'', \vec{n}''$.

We can define an F_0-minimal representative of any equivalence class of p-adic types over F_0. We begin with simple p-adic types first. A simple p-adic type (M, η) over F_0 is minimal if for any other simple p-adic type (M', η') equivalent to (M, η), there exists an F_0-algebra homomorphism $i' : M \to M'$ such that $\eta' = i'_* \eta$. It can be easily seen that any equivalence class of simple p-adic types over F_0 has a minimal representative: we can pushforward a given p-adic type into any big CM field $\widetilde{M}$ which is Galois over F_0 to get $(\widetilde{M}, \widetilde{\eta})$ and take the fixed field M of $\widetilde{M}$ under Galois automorphisms preserving the η. Then the descended p-adic type (M, η) is F_0-minimal.

Now we consider p-adic types that are not necessarily simple. We say that a p-adic type $(M, \vec{\eta}, \vec{n})$ is *minimal* over F_0 if every constituent (M_t, η_t) is minimal as a simple p-adic type over F_0 and no two constituents (M_{t_1}, η_{t_1}) and (M_{t_2}, η_{t_2}) are equivalent over F_0 for $t_1 \neq t_2$. It is easy to check that the new definition of minimality coincides with the one for simple p-adic types and that there exists a unique minimal representative over F_0 up to isomorphism in any equivalence class of p-adic types.

We are about to explain a version of Honda-Tate theory. One may see [HT01, p.158-159] for more detail and other references in the case of simple p-adic types over $\mathbb{Q}$ or F. Let A be a simple abelian variety over $\overline{\mathbb{F}}_p$, which has a model over a finite field $\mathbb{F}_{p^s}$. Let π_A be the geometric Frobenius with respect to $\mathbb{F}_{p^s}$, viewed as an element of the division algebra $\mathrm{End}^0(A)$. Then we associate the simple p-adic type $(M, \eta) := (\mathbb{Q}[\pi_A], [\pi_A]/s)$ to the abelian variety A. The equivalence class of the resulting p-adic type is independent of the choice of $\mathbb{F}_{p^s}$. According to Honda-Tate theory, this construction gives a natural bijection between the set of isogeny classes of simple abelian varieties over $\overline{\mathbb{F}}_p$ and the set of equivalence classes of simple p-adic types over $\mathbb{Q}$. The following facts are among the assertions of Honda-Tate theory.

- $M = Z(\mathrm{End}^0(A))$.
- $A[x^\infty]$ has pure slope $\eta_x / e_{x/p}$ and height $[M_x : \mathbb{Q}_p][\mathrm{End}^0(A) : M]^{1/2}$ for each place x of M over p.

We consider the category AV_B^0 whose objects are the pairs (A, i) where A is an abelian variety over $\overline{\mathbb{F}}_p$ and $i : B \to \mathrm{End}^0(A)$ is a $\mathbb{Q}$-algebra homomorphism. The morphisms from (A, i) to (A', i') are elements $f \in \mathrm{Hom}(A, A') \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $f \circ i(b) = i'(b) \circ f$ for all $b \in B$. We denote by $\mathrm{End}^0(A, i)$ or $\mathrm{End}^0_B(A)$ an endomorphism algebra in AV_B^0. By an easy extension of the Poincaré reducibility theorem, AV_B^0 is an F-linear semisimple category. We classify simple objects of AV_B^0 using [Kot92, §2]. If (A, i) is a simple object, then A is isogenous to A_0^m for a simple abelian variety A_0 over $\overline{\mathbb{F}}_p$ and $m \in \mathbb{Z}_{>0}$. Moreover, the centralizer of B in $M_m(\mathrm{End}^0(A_0))$ is a division algebra, whose center is denoted M. Let (M_0, η_0) be the minimal simple p-adic type (over $\mathbb{Q}$) associated to A_0. The field M is equipped with $\mathbb{Q}$-algebra maps $j : M_0 \to M$ and $\kappa : F \to M$ coming from i. Then

$$(A, i) \mapsto (M, j_* \eta_0, (1), \kappa)$$

is how we associate a simple p-adic type over F to each simple object (A, i) of AV_B^0. This induces a well-defined bijection between the set of isomorphism classes of simple objects of AV_B^0 and the set of equivalence classes of simple p-adic types over F.

As AV_B^0 is a semisimple category, any object (A, i) is isomorphic to $\oplus_{t \in I} (A_t, i_t)^{n_t}$ for a finite set I and simple objects (A_t, i_t) such that there is no nontrivial morphism between (A_t, i_t) and $(A_{t'}, i_{t'})$ for $t \neq t'$. If (A_t, i_t) corresponds to $(M_t, \eta_t, (1), \kappa_t)$ for each $t \in I$, we define the following association:

$$(A, i) \mapsto \left(\prod_{t \in I} M_t, (\eta_t)_{t \in I}, (n_t)_{t \in I}, (\kappa_t)_{t \in I} \right). \tag{22}$$

We construct from a simple minimal p-adic type (M_t, η_t, κ_t) a division algebra C_t with center M_t whose invariants at places x of M_t are given by the following. (Recall that $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ splits by assumption.)

$$\mathrm{inv}_x(C_t) = \begin{cases} 1/2 - \mathrm{inv}_x(B \otimes_F M_{t,x}), & x : \text{real} \\ \eta_x f_{x/p}, & x|p \\ -\mathrm{inv}_x(B \otimes_F M_{t,x}), & x \nmid p, x \nmid \infty \end{cases}. \tag{23}$$

Proposition 8.4. *The map (22) gives a natural bijection between the following two sets.*

(i) The set of isomorphism classes in AV_B^0.

(ii) The set of equivalence classes of p-adic types over F.

We can find a minimal representative $(\prod_t M_t, (\eta_t)_{t\in I}, (n_t)_{t\in I}, (\kappa_t)_{t\in I})$ corresponding to $\oplus_{t\in I}(A_t, i_t)^{n_t}$, where (A_t, i_t) are simple objects in distinct isomorphism classes of AV_B^0, such that the following are true for each $t \in I$.

- *$M_t = Z(\mathrm{End}^0(A_t, i_t))$ and $C_t \simeq \mathrm{End}^0(A_t, i_t)$.*
- *$A_t[x^\infty]$ has pure slope $\eta_x/e_{x/p}$ for each place x of M_t over p, and height $[M_{t,x} : \mathbb{Q}_p][B : F]^{1/2}[C_t : M_t]^{1/2}$.*

Proof. The bijection between (i) and (ii) is straightforward given the discussion preceding this proposition. The assertions about M_t, C_t and the slope follow from the case of simple p-adic types over $\mathbb{Q}$ using general facts in [Kot92, §3].

□

Let v be a place of F over p and (M_t, η_t) a simple p-adic type over F. For $\lambda \in \mathbb{Q}$, we define $S_{\lambda,v}(M_t)$ to be the places x of M_t over v such that $\lambda = \eta_x/e_{x/p}$. Recall that the BT-group $\Sigma = \oplus_{i=1}^r \Sigma^i$ has an action by $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \prod_{v|p} \mathcal{O}_{F,v}$. Correspondingly we have a decomposition $\Sigma^i = \oplus_{v|p}\Sigma^i[v^\infty]$ for each i.

Corollary 8.5. *The bijection in Proposition 8.4 restricts to the bijection of the following two sets.*

(i) The set of isomorphism classes of (A, i) in AV_B^0 for which there exists a quasi-isogeny $j : \Sigma \to A[p^\infty]$ compatible with the action of $B_{\mathbb{Q}_p}$.

(ii) The set of equivalence classes of p-adic types $(\prod_{t\in I} M_t, (\eta_t)_{t\in I}, (\vec{n}_t)_{t\in I})$ over F such that

- *For each $\lambda \in \mathbb{Q}$, there exist $t \in I$ and a place $v|p$ of F such that $S_{\lambda,v}(M_t) \neq \emptyset$ if and only if there exists i ($1 \le i \le r$) such that $\lambda = \lambda_i$ (i.e. λ is among the slopes of Σ.)*
- *For each $1 \le i \le r$, $\lambda = \lambda_i$ and each place $v|p$ of F,*

$$\sum_{t\in I} \sum_{x \in S_{\lambda,v}(M_t)} n_t [M_{t,x} : \mathbb{Q}_p][B : F]^{1/2}[C_t : M_t]^{1/2} = \mathrm{height}(\Sigma^i[v^\infty]).$$

Proof. Given Proposition 8.4, we only need to check that the additional conditions in (i) and (ii) match. Since $B \otimes_F F_v$ is a matrix algebra over F_v, the problem boils down to comparing the slope decompositions of $\Sigma[v^\infty]$ and $A[v^\infty]$. The first condition in (ii) means that Σ and $A[p^\infty]$ have the same set of slopes. The second condition in (ii) implies that the heights of each slope component are the same for $\Sigma[v^\infty]$ and $A[v^\infty]$.

□

The following lemma, generalizing [HT01, Lem V.2.2], is indispensable in later argument.

Lemma 8.6. *Let $z = [(A, \lambda, i)]$ be an equivalence class in PIC_b. Let $\iota_z(a) = (a^p, a_p)$ be the image of $a \in H_z(\mathbb{Q})$ in $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$. Suppose that a_p is acceptable. Then $M_z \subset F(a)$ (as F-subalgebras in $\mathrm{End}^0_B(A)$) and*

$$Z_{G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p)}(\iota_z(a)) = \iota_z((Z_{H_z}(a)(\mathbb{A}^\infty))) \tag{24}$$

Proof. We explain how the first assertion implies the second assertion. Clearly the inclusion $\supset$ always holds in (24). In case $M_z \subset F(a)$, we have the other inclusion. Indeed, if $g \in G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ centralizes $\iota_z(a)$ then g centralizes $F(a)$ so it also centralizes M_z (viewed inside $\mathrm{End}_B(V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}) \times \mathrm{End}^0_B(\Sigma)$), therefore g belongs to $H_z(\mathbb{A}^\infty)$.

We will prove the first assertion in several steps. Let $(A, i) = (\oplus_{t\in I}(A_t, i_t)^{n_t})$ where (A_t, i_t) are simple objects and no two of them are isomorphic in AV^0_B. The pair (A, i) corresponds to a minimal p-adic type $(\prod_{t\in I} M_t, (\eta_t)_{t\in I}, (n_t)_{t\in I}, (\kappa_t)_{t\in I})$ over F so that no two (M_t, κ_t) are isomorphic over F. Note that $M_z = \prod_{t\in I} M_t$. Write $a = (a_t)$ in $\prod_{t\in I} M_{n_t}(\mathrm{End}^0(A_t, i_t))$. The proof of $M_z \subset F(a)$ reduces to showing $M_t \subset F(a_t)$ for each $t \in I$.

We decompose $F(a_t)$ into a product of fields $F(a_t) = \prod_i F_{t,i}$. We have $F_{t,i} \otimes_F M_t = \prod_j M_{t,i,j}$ where $M_{t,i,j}$ are fields. To prove $M_t \subset F(a_t)$, it suffices to prove that $M_t \subset F_{t,i}$ in $M_{t,i,j}$ for all i, j. In the following, we work with fixed i,j and write M, $\widetilde{F}$, $\widetilde{M}$ for M_t, $F_{t,i}$, $M_{t,i,j}$.

We introduce some notation. Let $S := \{\lambda_1, \dots, \lambda_r\}$ be the set of all slopes of the BT-group Σ (or equivalently, $A[p^\infty]$). Let N be the Galois closure of $\widetilde{M}$ over $\widetilde{F} \cap M$ (the intersection taken in $\widetilde{M}$). Let u denote a place of $\widetilde{F} \cap M$ over p. Define $\mathfrak{P}_u(K)$ to be the set of places of K over u where K is either M, $\widetilde{M}$, $\widetilde{F}$, or N. We have obvious maps $\mathfrak{P}_u(N) \to \mathfrak{P}_u(\widetilde{M}) \to \mathfrak{P}_u(M)$ given by restriction of places.

Write $a_p \in J_b(\mathbb{Q}_p)$ as $a_p = (a_i)_{i=1}^r \in \prod_{i=1}^r \mathrm{End}^0(\Sigma)$ via $J_b(\mathbb{Q}_p) \subset \mathrm{End}^0(\Sigma)$. Since a_p is acceptable, we may find positive real numbers $\epsilon_0, \dots, \epsilon_r$ such that for any i ($1 \le i \le r$) and any eigenvalue e_i of a_i, the inequality $\epsilon_{i-1} < |e_i|_p < \epsilon_i$ holds. Then we define a map $s_{\widetilde{F},u} : \mathfrak{P}_u(\widetilde{F}) \to S$ by $v \mapsto \lambda_i$ if i is such that $\epsilon_{i-1} < |a|_v^{1/[\widetilde{F}_v:\mathbb{Q}_p]} < \epsilon_i$. We also define a map $s_{M,u} : \mathfrak{P}_u(M) \to S$ by $w \mapsto \eta_w/e_{w/p}$. This means that $A_t[w^\infty]$ has pure slope $s_{M,u}(w)$ by Honda-Tate theory. We induce maps $s_{\widetilde{M},u} : \mathfrak{P}_u(\widetilde{M}) \to S$ and $s_{N,u} : \mathfrak{P}_u(N) \to S$ from $s_{M,u}$. On the other hand, we induce a map $s'_{\widetilde{M},u} : \mathfrak{P}_u(\widetilde{M}) \to S$ from $s_{\widetilde{F},u}$.

Our first claim is that $s_{\widetilde{M},u} = s'_{\widetilde{M},u}$. To prove this, let x be a place in $\mathfrak{P}_u(\widetilde{M})$ and put $w := x|_M$. By definition, $s'_{\widetilde{M},u}(x) = \lambda_i$ means that $\epsilon_{i-1} < |a|_x^{1/[\widetilde{M}_x:\mathbb{Q}_p]} < \epsilon_i$. This is equivalent to the fact that a acts on $A[w^\infty]$ by an eigenvalue whose p-adic absolute value is between ϵ_{i-1} and ϵ_i, which means that $A[w^\infty]$ has slope λ_i. Thus the first claim follows.

We prove our second claim that $s_{M,u}$ is a constant function. Observe that $\sigma \in \mathrm{Gal}(N/\widetilde{F} \cap M)$ acts on $s_{N,u}$ by $f \mapsto f \circ \sigma$. We assert that $\mathrm{Gal}(N/\widetilde{F})$ fixes $s_{N,u}$. Indeed, if y is a place in $\mathfrak{P}_u(N)$ such that $s_{N,u}(y) \neq s_{N,u}(\sigma y)$, then $s_{\widetilde{F},u}(y|_{\widetilde{F}}) \neq s_{\widetilde{F},u}(\sigma y|_{\widetilde{F}})$, but this contradicts $y|_{\widetilde{F}} = \sigma y|_{\widetilde{F}}$. It is obvious that $\mathrm{Gal}(N/M)$ also fixes $s_{N,u}$. Since $\mathrm{Gal}(N/\widetilde{F} \cap M)$ is generated by $\mathrm{Gal}(N/\widetilde{F})$ and $\mathrm{Gal}(N/M)$, we conclude that $s_{N,u}$ is fixed under $\mathrm{Gal}(N/\widetilde{F} \cap M)$. The latter group is transitive on $\mathfrak{P}_u(N)$, implying that $s_{N,u}$ is a constant function. So $s_{M,u}$ is also a constant function.

Now we construct a simple p-adic type $(\widetilde{F} \cap M, \xi, \kappa)$ over F. The map κ is induced from $\kappa_r : F \hookrightarrow M$. Let λ_u be the common image of $s_{M,u}$. We define $\xi_u := e_{u/p}\lambda_u$ for each place u. Then we readily check that $(\widetilde{F} \cap M, \xi, \kappa)$ is equivalent to (M_t, η_t, κ_t) over F. Recall that we agreed to write M for M_t. By the minimality of the p-adic type that was originally chosen, we conclude that $M_t = \widetilde{F} \cap M_t$, namely $M_t \subset \widetilde{F}$.

□

9 Polarizations

This section serves as a preparatory step for parametrizing polarizations using Galois cohomology. In this section we do not assume that (A, λ, i) represents an element of PIC_b until Definition 9.7.

We start with a general discussion of C-polarizations. Consider $(C, *)$ where C is a finite dimensional semisimple $\mathbb{Q}$-algebra whose center $Z(C)$ is a product of CM or totally real fields and $*$ is an involution on C which acts on $Z(C)$ as complex conjugation. Let A be an abelian variety over $\overline{\mathbb{F}}_p$ and $i : C \to \mathrm{End}^0(A)$ be a $\mathbb{Q}$-algebra homomorphism. When $\lambda : A \to A^\vee$ is a polarization, denote by $\ddagger_\lambda$ the λ-Rosati involution on $\mathrm{End}^0(A)$.

Definition 9.1. We call a polarization $\lambda : A \to A^\vee$ a $(C, *)$*-polarization* for (A, i) as above if $\ddagger_\lambda(i(c)) = i(c^*)$ for all $c \in C$. Two $(C, *)$-polarizations λ_1 and λ_2 are *equivalent* if there exists $f \in \mathrm{End}^0_C(A)^\times$ such that $\lambda_2 = \gamma f^\vee \lambda_1 f$ for some $\gamma \in \mathbb{Q}^\times$. When there is no ambiguity, a $(C, *)$-polarization is simply called a C-polarization.

Lemma 9.2. *Given any (A, i) as above, there exists a C-polarization λ_0 for (A, i). If the $\mathbb{Q}$-group H_{λ_0} is defined so that for any $\mathbb{Q}$-algebra R,*

$$H_{\lambda_0}(R) = \{g \in \mathrm{End}^0_C(A) \otimes_{\mathbb{Q}} R : gg^{\ddagger_{\lambda_0}} \in R^\times\}, \tag{25}$$

then the set of equivalence classes of C-polarizations for (A, i) is in natural bijection with $\ker(H^1(\mathbb{Q}, H_{\lambda_0}) \to H^1(\mathbb{R}, H_{\lambda_0}))$.

Proof. [Kot92, Lem 9.2] for the existence of a C-polarization. For the bijection, see [HT01, Lem V.3.1]. □

We go back to the notation of the previous section. From now on, whenever we consider triples (A, λ, i) we will assume that λ is a B-polarization for (A, i). Suppose we have two such triples (A, λ, i) and (A', λ', i'). We say that they are $\overline{\mathbb{Q}}$*-isogenous* if there is an element $f \in (\mathrm{Hom}(A, A') \otimes_{\mathbb{Z}} \overline{\mathbb{Q}})^\times$ such that $f \circ i'(b) = i(b) \circ f$ for all $b \in B$ and $f^\vee \lambda' f = \gamma \lambda$ for some $\gamma \in \overline{\mathbb{Q}}^\times$. A usual isogeny (or $\mathbb{Q}$-isogeny) of two triples is defined analogously with $f \in (\mathrm{Hom}(A, A') \otimes_{\mathbb{Z}} \mathbb{Q})^\times$ and $\gamma \in \mathbb{Q}^\times$. Note that this notion of isogeny is the same as the equivalence relation in Definition 7.2.

For a triple (A, λ, i), define a $\mathbb{Q}$-group H_λ as in (25), using λ and B instead of λ_0 and C. Whenever a triple (A', λ', i') is $\overline{\mathbb{Q}}$-isogenous to (A, λ, i), it defines a cocycle $\tau \mapsto f^{-1} \circ f^\tau$ in $H^1(\mathbb{Q}, H_\lambda)$. This association defines the bijection in the following lemma.

Lemma 9.3. *Let (A, i) be an object of AV^0_B. Choose λ_0 as in Lemma 9.2. Then the set of isogeny classes of the triples (A', λ', i') which are $\overline{\mathbb{Q}}$-isogenous to (A, λ_0, i) is in natural bijection with the set* $\ker(H^1(\mathbb{Q}, H_{\lambda_0}) \to H^1(\mathbb{R}, H_{\lambda_0}))$.

Proof. [Kot92, Lem 17.1]. □

Definition 9.4. We say that (A, λ, i) and (A', λ', i') are *nearly equivalent* if there exists an isogeny $f : A \to A'$ satisfying $f \circ i(b) = i'(b) \circ f$ for all $b \in B$ such that the map f induces

(i) an equivalence of $V^p A$ and $V^p A'$ as $(* \otimes c)$-Hermitian $B \otimes_F M \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ modules with Weil pairings given by λ, λ', respectively. Here M denotes $M_{[(A,\lambda,i)]} \simeq M_{[(A',\lambda',i')]}$ (identified via f).

(ii) an isogeny $A[p^\infty] \to A'[p^\infty]$ over $\overline{\mathbb{F}}_p$ compatible with $B \otimes_F M \otimes_{\mathbb{Q}} \mathbb{Q}_p$-actions and polarizations, the latter up to $\mathbb{Q}_p^\times$-multiple.

We say that equivalence classes $[(A, \lambda, i)]$ and $[(A', \lambda', i')]$ are nearly equivalent if their representatives are nearly equivalent.

If (A, λ, i) and (A', λ', i') are nearly equivalent, then they are $\overline{\mathbb{Q}}$-isogenous. (One can use the same argument as in the first paragraph of page 436 of [Kot92].) So (A', λ', i') gives an element of $\ker(H^1(\mathbb{Q}, H_\lambda) \to H^1(\mathbb{R}, H_\lambda))$ via Lemma 9.3.

Lemma 9.5. *Given (A, λ, i) as before, the set of isogeny classes $[(A', \lambda', i')]$ that are nearly equivalent to $[(A, \lambda, i)]$ is in natural bijection with $\ker^1(\mathbb{Q}, H_\lambda)$.*

Proof. In the definition of near equivalence, the condition (i) means that the Galois cocycle for $[(A', \lambda', i')]$ belongs to $\ker(H^1(\mathbb{Q}, H_\lambda) \to H^1(\mathbb{Q}, H_\lambda(\overline{\mathbb{A}}^{\infty,p})))$ and (ii) means that the same cocycle belongs to $\ker(H^1(\mathbb{Q}, H_\lambda) \to H^1(\mathbb{Q}_p, H_\lambda))$. With this observation, the current lemma follows from Lemma 9.3.

□

Lemma 9.6. *When $[(A, \lambda, i)]$ and $[(A', \lambda', i')]$ are nearly equivalent, there is an isomorphism*

$$H_{(A,\lambda,i)} \simeq H_{(A',\lambda',i')},$$

canonical up to conjugation by an element of $H_{(A,\lambda,i)}(\mathbb{Q})$.

Proof. The triple (A', λ', i') defines a cocycle $c : \tau \mapsto f^{-1} \circ f^\tau$ in $\ker^1(\mathbb{Q}, H_\lambda)$. Since $\ker^1(\mathbb{Q}, H_\lambda) \simeq \ker^1(\mathbb{Q}, Z(H_\lambda))$ by [Kot92, §7], modifying (A', λ', i') by an isogeny if necessary, we may assume that $f^{-1} \circ f^\tau \in Z(H_\lambda) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ for all $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Composing f with an isogeny, we may also assume that f belongs to $(M_{(A,\lambda,i)} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^\times$ since the cocycle c is trivialized under the map $\ker^1(\mathbb{Q}, Z(H_\lambda)) \to H^1(\mathbb{Q}, M^\times_{(A,\lambda,i)})$, the latter being a trivial group by Hilbert 90.

It suffices to prove that $\ddagger_\lambda = \ddagger_{\lambda'}$. This follows from the basic fact that $f \in Z(\mathrm{End}^0(A) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^\times$ and thus $f^\vee \in Z(\mathrm{End}^0(A^\vee) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^\times$. Indeed, $\lambda = \gamma^{-1} f^\vee \lambda' f$ for some $\gamma \in \mathbb{Q}^\times$ by near equivalence and

$$\ddagger_\lambda(g) = \lambda^{-1} g^\vee \lambda = (f^\vee \lambda' f)^{-1} g^\vee (f^\vee \lambda' f) = \lambda'^{-1} g^\vee \lambda' = \ddagger_{\lambda'}(g)$$

□

Definition 9.7. The set FP_b^{AV} is defined to be the set of pairs $(z, [a])$ where

(i) $z = [(A, \lambda, i)]$ is a near equivalence class in PIC_b and

(ii) $[a]$ is the $H_z(\mathbb{A})$-conjugacy class of $a \in H_z(\mathbb{Q})$, where a is an acceptable element in $H_z(\mathbb{Q}_p)$. (We consider a acceptable if its image under the embedding $\iota_z : H_z(\mathbb{Q}_p) \hookrightarrow J_b(\mathbb{Q}_p)$ is acceptable. This property is unchanged if ι_z is replaced with a $J_b(\mathbb{Q}_p)$-conjugate.)

10 Kottwitz triples and Kottwitz invariants

In this section, we define Kottwitz triples which will be used to parametrize the set FP_b^{AV}. The Kottwitz invariant $\alpha(\gamma_0; \gamma, \delta)$ is associated to each Kottwitz triple $(\gamma_0; \gamma, \delta)$ and tells us exactly when a Kottwitz triple arises from the moduli data of Igusa varieties.

Definition 10.1. By a *Kottwitz triple* (of type b), we mean a triple $(\gamma_0; \gamma, \delta)$ where

- $\gamma_0 \in G(\mathbb{Q})$ is semisimple, and elliptic in $G(\mathbb{R})$

- $\gamma \in G(\mathbb{A}^{\infty,p})$ and $\gamma_0 \sim_{\overline{\mathbb{A}}^{\infty,p}} \gamma$.
- $\delta \in J_b(\mathbb{Q}_p)$ is acceptable (see Definition 6.1) and $\gamma_0 \sim_{\overline{\mathbb{Q}}_p} \delta$ in $G(\overline{\mathbb{Q}}_p)$ via any embedding in the canonical $G(\overline{\mathbb{Q}}_p)$-conjugacy class of embeddings $J_b(\overline{\mathbb{Q}}_p) \hookrightarrow G(\overline{\mathbb{Q}}_p)$ (as in the paragraph below Lemma 4.2).

Two Kottwitz triples $(\gamma_0; \gamma, \delta) \sim (\gamma_0'; \gamma', \delta')$ are said to be equivalent if $\gamma_0 \sim_{st} \gamma_0'$, $\gamma \sim_{\mathbb{A}^{\infty,p}} \gamma'$, and $\delta \sim \delta'$.

Remark 10.2. The notion of Kottwitz triples clearly depends on $b \in B(G, -\mu)$, but not on the extra choice of $\widetilde{b}$ in the following sense: the equivalence classes of Kottwitz triples for any two decent representatives $\widetilde{b}$ and $\widetilde{b}'$ of b are in canonical bijection with each other via $J_{\widetilde{b}} \simeq J_{\widetilde{b}'}$.

Definition 10.3. For each $b \in B(G, -\mu)$, we define KT_b to be the set of equivalence classes of all Kottwitz triples of type b.

We explain how to attach a p-adic type over F to a Kottwitz triple. Denote by $F(\gamma_0)$ the F-algebra generated by γ_0 in $\mathrm{End}_B(V)$. It admits a product decomposition $F(\gamma_0) \simeq \prod_{t \in I} F_t$ into fields. Let I_t be the set of places of F_t over p so that $F_t \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \prod_{y \in I_t} F_{t,y}$ is a product of fields. Since $\gamma_0 \sim_{\overline{\mathbb{Q}}_p} \delta$, we are able to choose an isomorphism $F_{\mathbb{Q}_p}(\gamma_0) \simeq F_{\mathbb{Q}_p}(\delta)$ such that $\gamma_0 \mapsto \delta$. Under this isomorphism, each $F_{t,y}$ is mapped nontrivially into $\mathrm{End}^0_B(\Sigma^{k(t,y)})$ for only one $k(t,y) \in \mathbb{Z}$ $(1 \le k(t,y) \le r)$. Since δ is acceptable, $k(t,y)$ is independent of the choice of the isomorphism. For each $t \in I$, we define a simple p-adic type $(F_t, \widetilde{\eta}_t, \widetilde{\kappa}_t)$ over F by $\widetilde{\eta}_{t,y} = e_{y/p}\lambda_{k(t,y)}$. The map $\widetilde{\kappa}_t : F \to F_t$ is the F-algebra structure map of F_t. Finally we find a minimal representative (M_t, η_t, κ_t) over F for $(F_t, \widetilde{\eta}_t, \widetilde{\kappa}_t)$.

Write $M_t \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \prod_{x \in J_t} M_{t,x}$ as a product of fields. As the BT-group Σ is acted on by $F_{\mathbb{Q}_p}(\delta) = \prod_{t \in I} F_t \otimes_{\mathbb{Q}} \mathbb{Q}_p$, it is also acted on by $\prod_{x \in J_t} M_{t,x}$ via $M_t \hookrightarrow F_t$, allowing a decomposition up to isogeny

$$\Sigma \simeq \oplus_{t \in I} \oplus_{x \in J_t} \Sigma_{t,x}.$$

There is induced a $\mathbb{Q}_p$-algebra map $B \otimes_F M_{t,x} \simeq M_n(M_{t,x}) \to \mathrm{End}^0(\Sigma_{t,x})$. Using an idempotent in $M_n(M_{t,x})$, we find a BT-group $\Sigma^{\mathrm{red}}_{t,x}$ such that $\Sigma_{t,x} \simeq (\Sigma^{\mathrm{red}}_{t,x})^{\oplus n}$ with compatible $M_{t,x}$-actions in the isogeny category. We define a rational number

$$n_t := \frac{\mathrm{height}(\Sigma^{\mathrm{red}}_{t,x})}{[M_{t,x} : \mathbb{Q}_p][C_t : M_t]^{1/2}}.$$

Lemma 10.4. *The following are true.*

(i) The number n_t is an integer and independent of x.

(ii) The map

$$(\gamma_0; \gamma, \delta) \mapsto \left(\prod_{t \in I} M_t, (\eta_t), (n_t), (\kappa_t) \right)$$

gives a well-defined map from KT_b to the set of equivalence classes of p-adic types over F.

(iii) The image of the above map lies in the set decribed in (ii) of Corollary 8.5.

Remark 10.5. Note that the p-adic type $(\prod_{t \in I} M_t, (\eta_t), (n_t), (\kappa_t))$ we constructed above is not necessarily minimal. It may happen that (M_t, η_t, κ_t) and $(M_{t'}, \eta_{t'}, \kappa_{t'})$ are equivalent for $t \neq t'$.

Proof. Assuming we have proved (i), we obtain (ii) as an easy consequence since the construction of the map does not change if we replace γ_0 with a stably conjugate element or δ by a conjugate element. To see (iii), observe that

$$\text{height}(\Sigma_{t,x}) = \text{height}(\Sigma_{t,x}^{\text{red}})[B:F]^{1/2} = n_t[M_{t,x}:\mathbb{Q}_p][B:F]^{1/2}[C_t:M_t]^{1/2}.$$

For each slope λ_k of Σ for $1 \le k \le r$ and each place $v|p$ of F,

$$\sum_{t\in I}\sum_{x\in S_{\lambda_k,v}(M_t)} \text{height}(\Sigma_{t,x}) = \sum_{(t,x)\text{ with } x|v,\ k(t,x)=\lambda_k} \text{height}(\Sigma_{t,x}) = \text{height}(\Sigma^k[v^\infty]).$$

This verifies (iii) of the lemma.

We begin the proof of (i). As we have $M_t \hookrightarrow F_t$ and $B\otimes_F F(\gamma_0) \simeq \prod_t B\otimes_F F_t$ acts on V preserving B-action and Hermitian pairing, we have a decompostion $V \simeq \oplus_t V_t$ where V_t is a Hermitian $B\otimes_F M_t$-module. Let $\widetilde{r}_t := \text{rank}_{M_t} V_t$. As $M_t \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \prod_{x|p} M_{t,x}$, we have $V_t \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \oplus_{x|p} V_{t,x}$ correspondingly. Then $V_{t,x}$ is a module over $M_n(M_{t,x})$ which is an $\widetilde{r}_t$-dimensional vector space over $M_{t,x}$. Therefore $r_t := \widetilde{r}_t/n$ is an integer. Since there is an isomorphism $\mathbb{V}(\Sigma_{t,x}) \simeq V_{t,x} \otimes_{\mathbb{Q}_p} L$ as L-vector spaces,

$$\text{height}(\Sigma_{t,x}^{\text{red}}) = r_t[M_{t,x}:\mathbb{Q}_p]. \tag{26}$$

What we need to prove is that $n_t = r_t/[C_t:M_t]^{1/2}$ belongs to $\mathbb{Z}$. As C_t is a division algebra with center M_t, it suffices to prove that $r_t \cdot \text{inv}_x C_t = 0 \in \mathbb{Q}/\mathbb{Z}$ for every place x of M_t. At this point recall that $\text{inv}_x C_t$ is given in (23) by Honda-Tate theory.

Consider the case $x|p$. Let k be $k(t,x)$ defined in the paragraph preceding Lemma 10.4. Recall that $\Sigma_{t,x}^{\text{red}}$ is a BT-group of pure slope $\lambda_k = \eta_x/e_{x/p}$ with $M_{t,x}$-action. Find m_t such that $\Sigma_{t,x}^{\text{red}}$ consists of m_t copies of simple BT-groups of slope λ_k. Write $\lambda_k = a_k/b_k$ for coprime integers a_k and b_k. We deduce that $r_t\text{inv}_x C_t = r_t[M_{t,x}:\mathbb{Q}_p]a_k/b_k$ is 0 in $\mathbb{Q}/\mathbb{Z}$ from the equalities $\text{height}(\Sigma_{t,x}^{\text{red}}) = m_t b_k$ and (26).

We deal with the case where $x \nmid p$ and $x \nmid \infty$. Suppose that x divides a rational prime $q(\neq p)$. According to $M_t \otimes_{\mathbb{Q}} \mathbb{Q}_q \simeq \prod_{x|q} M_{t,x}$, we decompose $V_t \otimes_{\mathbb{Q}} \mathbb{Q}_q \simeq \prod_{x|q} V_{t,x}$. Let $d_{t,x}$ be the denominator of $\text{inv}_x(B\otimes_F M_{t,x})$ so that $B\otimes_F M_{t,x} \simeq M_{n/d_{t,x}}(D_{t,x})$ for some central division algebra $D_{t,x}$ over $M_{t,x}$ with degree $d_{t,x}^2$. As $V_{t,x}$ is a module over $B\otimes_F M_{t,x}$, it follows that $nd_{t,x}$ divides $\text{rank}_{M_{t,x}} V_{t,x} = nr_t$. Therefore $r_t\text{inv}_x(C_t) = -r_t\text{inv}_x(B\otimes_F M_{t,x})$ is $0 \in \mathbb{Q}/\mathbb{Z}$.

A real place x occurs only when F and M_t are totally real fields, for a PEL datum of type (C). We know that $V_t \otimes_{\mathbb{Q}} \mathbb{R}$ is a Hermitian $B\otimes_F M_t \otimes_{\mathbb{Q}} \mathbb{R}$-module. We decompose $V_t \otimes_{\mathbb{Q}} \mathbb{R}$ into a product of $B\otimes_F M_{t,x}$-modules $V_{t,x}$ as x runs over real places of M_t. If $B\otimes_F M_{t,x}$ does not split, $\text{inv}_x(C_t) = 0$. If $B\otimes_F M_{t,x}$ splits, we know that r_t is even from the existence of a symplectic pairing on $V_{t,x}$ and this is enough for conclusion.

□

Corollary 10.6. *The maps in Lemma 10.4 and Corollary 8.5 induce a map from KT_b to the set of those isomorphism classes (A,i) in AV_B^0 for which there exists a quasi-isogeny $j:\Sigma \to A[p^\infty]$ compatible with $B_{\mathbb{Q}_p}$-action.*

Proof. Immediate from Lemma 10.4 and Corollary 8.5. □

As preparation for the definition of the invariants $\alpha(\gamma_0;\gamma,\delta)$ and $\beta(\gamma_0;\gamma,\delta)$, we introduce two algebraic groups I_0 and H_0 for a Kottwitz triple $(\gamma_0;\gamma,\delta)$. Using Lemma 10.4, we find an F-algebra $M = \prod_t M_t$ appearing in the minimal p-adic type for $(\gamma_0;\gamma,\delta)$. Define

$$H_0(R) = \{g \in \text{End}_{B\otimes_F M}(V)\otimes_{\mathbb{Q}} R | gg^\# \in R^\times\} \tag{27}$$

for each $\mathbb{Q}$-algebra R. Also define $I_0 := Z_G(\gamma_0)$ so that $I_0(R) = \{g \in \mathrm{End}_{B\otimes F(\gamma_0)}(V)\otimes_{\mathbb{Q}} R | gg^{\#} \in R^{\times}\}$. Clearly $I_0 \subset H_0 \subset G$.

Now we introduce local components α_v and β_v, which lead to the invariants $\alpha(\gamma_0;\gamma,\delta)$ and $\beta(\gamma_0;\gamma,\delta)$. We begin with α_v, β_v for $v \neq p, \infty$. Denoting the v-component of γ by $\gamma_v \in G(\mathbb{Q}_v)$, there exists $g \in G(\overline{\mathbb{Q}}_v)$ such that $g\gamma_0 g^{-1} = \gamma_v$. From this, we construct a cocycle $c_v := (\tau \mapsto g^{-1}g^{\tau})$ in $\ker(H^1(\mathbb{Q}_v, I_0) \to H^1(\mathbb{Q}_v, G))$. In the diagram below which is commutative by Lemma 2.3, we define $\alpha_v \in A_v(I_0)$ and $\beta_v \in A_v(H_0)$ as the image of c_v. Then α_v and β_v map to the trivial element in $A_v(G)$. We will also view α_v, β_v as elements of $X^*(Z(\widehat{I_0})^{\Gamma(v)})$ and $X^*(Z(\widehat{H_0})^{\Gamma(v)})$, respectively.

$$\begin{array}{ccccc} H^1(\mathbb{Q}_v, I_0) & \longrightarrow & H^1(\mathbb{Q}_v, H_0) & \longrightarrow & H^1(\mathbb{Q}_v, G) \\ \downarrow & & \downarrow & & \downarrow \\ A_v(I_0) & \longrightarrow & A_v(H_0) & \longrightarrow & A_v(G) \end{array} \tag{28}$$

Next we deal with α_p and β_p. We freely borrow notation from §4. Observe that the L-vector spaces $V \otimes_{\mathbb{Q}} L$ and $\mathbb{V}(\Sigma, \lambda_\Sigma, i_\Sigma)$ are Hermitian modules with respect to the natural action of $B \otimes_F F_{\mathbb{Q}_p}(\gamma_0) \simeq B \otimes_F F_{\mathbb{Q}_p}(\delta)$, where the last isomorphism is chosen so that $\gamma_0 \mapsto \delta$. As $V \otimes_{\mathbb{Q}} L$ is equivalent to $\mathbb{V}(\Sigma, \lambda_\Sigma, i_\Sigma)$ as Hermitian $B \otimes_F F_{\mathbb{Q}_p}(\gamma_0) \otimes_{\mathbb{Q}_p} L$-modules (by Steinberg's vanishing theorem of H^1), choose any such equivalence f and transport the map Φ on $\mathbb{V}(\Sigma, \lambda_\Sigma, i_\Sigma)$ to Φ_0 on $V \otimes L$. Define $\widetilde{b}_\delta$ by the relation $\Phi_0 = \widetilde{b}_\delta(1 \otimes \sigma)$. (So $\widetilde{b}_\delta = f^{-1}\widetilde{b} f^{\sigma}$.) Then $\widetilde{b}_\delta$ belongs to $I_0(L)$ and defines $b_\delta \in B((I_0)_{\mathbb{Q}_p})$, which is independent of the choice of the above isomorphisms. Define $\alpha_p := \kappa_{I_0}(b_\delta)$. (In view of Remark 4.15, we may replace $\mathbb{V}(\Sigma, \lambda_\Sigma, i_\Sigma)$ by $\mathbb{V}(\widetilde{b})$ in the definition of α_p. The action of δ on $\mathbb{V}(\widetilde{b})$ is defined via the $\mathbb{Q}_p$-isomorphism $J_b(\mathbb{Q}_p) \simeq \mathrm{Aut}^0(\mathbb{V}(\widetilde{b}))$, which is canonical up to $J_b(\mathbb{Q}_p)$-conjugacy.) Consider the following commutative diagram coming from the functoriality of the map $\kappa_{(\cdot)}$ (see §4). The element $\beta_p \in X^*(Z(\widehat{H_0})^{\Gamma(p)})$ is defined to be the image of α_p. Since b_δ maps to b in $B(G_{\mathbb{Q}_p}, -\mu)$, both α_p and β_p map to $-\mu_1 \in X^*(Z(\widehat{G})^{\Gamma(p)})$ by the bottom arrows. Recall that $\kappa_{G_0}(b) = -\mu_1$.

$$\begin{array}{ccccc} B((I_0)_{\mathbb{Q}_p}) & \longrightarrow & B((H_0)_{\mathbb{Q}_p}) & \longrightarrow & B(G_{\mathbb{Q}_p}) \\ \downarrow{\scriptstyle \kappa_{I_0}} & & \downarrow{\scriptstyle \kappa_{H_0}} & & \downarrow{\scriptstyle \kappa_{G_0}} \\ X^*(Z(\widehat{I_0})^{\Gamma(p)}) & \longrightarrow & X^*(Z(\widehat{H_0})^{\Gamma(p)}) & \longrightarrow & X^*(Z(\widehat{G})^{\Gamma(p)}) \end{array}$$

Finally we describe α_∞, β_∞. We can choose an elliptic maximal real torus T of G containing γ_0. We also choose a $G(\mathbb{R})$-conjugate of $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \to G$ factoring through T and use it to define μ_h (see §5). Then we see that μ_h belongs to $X_*(T) = X^*(\widehat{T})$ and that the image of μ_h in $X^*(\widehat{T}^{\Gamma(\infty)})$ is independent of choices (See [Kot90, p.167]). By restricting this image via the canonical embedding $Z(\widehat{I_0}) \hookrightarrow Z(\widehat{H_0}) \hookrightarrow \widehat{T}$, we get elements

$$\alpha_\infty \in X^*(Z(\widehat{I_0})^{\Gamma(\infty)}) \quad \text{and} \quad \beta_\infty \in X^*(Z(\widehat{H_0})^{\Gamma(\infty)}).$$

We are ready to define the elements $\alpha(\gamma_0,\gamma,\delta) \in \mathfrak{K}(I_0/\mathbb{Q})^D$ and $\beta(\gamma_0,\gamma,\delta) \in \mathfrak{K}(H_0/\mathbb{Q})^D$. As we assumed that γ_0 is elliptic, it follows that $\mathfrak{K}(I_0/\mathbb{Q}) = (\bigcap_v Z(\widehat{I_0})^{\Gamma(v)} Z(\widehat{G}))/Z(\widehat{G})$ ([Kot90, p.166]). We extend α_v to an element α'_v of $X^*(Z(\widehat{I_0})^{\Gamma(v)} Z(\widehat{G}))$ so that on $Z(\widehat{G})$, α'_v is $-\mu_1$ if $v = p$, and μ_1 if $v = \infty$, and trivial if $v \neq p, \infty$. Similarly, we define $\beta'_v \in X^*(Z(\widehat{H_0})^{\Gamma(v)} Z(\widehat{G}))$. The elements

$\alpha(\gamma_0;\gamma,\delta)$ in $\mathfrak{K}(I_0/\mathbb{Q})^D$ and $\beta(\gamma_0;\gamma,\delta)$ in $\mathfrak{K}(H_0/\mathbb{Q})^D$ are defined by

$$\begin{aligned}\alpha(\gamma_0;\gamma,\delta) &= \left(\prod_{v\neq p,\infty} \alpha'_v|_{\mathfrak{K}(I_0/\mathbb{Q})}\right)\cdot(\alpha'_p\alpha'_\infty)|_{\mathfrak{K}(I_0/\mathbb{Q})},\\ \beta(\gamma_0;\gamma,\delta) &= \left(\prod_{v\neq p,\infty} \beta'_v|_{\mathfrak{K}(H_0/\mathbb{Q})}\right)\cdot(\beta'_p\beta'_\infty)|_{\mathfrak{K}(H_0/\mathbb{Q})}.\end{aligned}$$

In terms of the vanishing of $\alpha(\gamma_0;\gamma,\delta)$, we want to single out those Kottwitz triples which are expected to come from the moduli data of Igusa varieties. This motivates the following definition.

Definition 10.7. A Kottwitz triple $(\gamma_0;\gamma,\delta)$ is called *effective* if $\alpha(\gamma_0;\gamma,\delta)$ is trivial. We define the set KT_b^{eff} to be the subset of KT_b consisting of the equivalence classes of effective Kottwitz triples.

For later use, we record two lemmas regarding a Kottwitz triple $(\gamma_0;\gamma,\delta)$. Define $I_\delta := Z_{J_b}(\delta)$.

Lemma 10.8. *The group I_δ is an inner form of I_0 over $\mathbb{Q}_p$.*

Proof. Let s be the positive integer in the decency equation (6) for $\widetilde{b}$. Denote by σ the Frobenius element in $\mathrm{Gal}(L/\mathbb{Q}_p)$ or $\mathrm{Gal}(L_s/\mathbb{Q}_p)$. To avoid confusion, for any $\tau\in\Gamma(p)$ we will sometimes write τ_M (resp. τ_J) for the τ-action on the points of M_b (resp. J_b). Recall from Lemma 4.2 that there is an isomorphism $\psi: M_b \xrightarrow{\sim} J_b$ over L_s such that $\psi^{-1}\psi^\sigma = \mathrm{Int}(\widetilde{b})$. Viewing ψ as a $\overline{\mathbb{Q}}_p$-morphism by base change, define $c_\tau\in M_b(\overline{\mathbb{Q}}_p)$ for each $\tau\in\Gamma(p)$ so that $\mathrm{Int}(c_\tau)=\psi^{-1}\psi^\tau$. (Here ψ^τ is $\tau_J\psi\tau_M^{-1}$ by definition.) The condition $\delta\sim_{st}\gamma_0$ means that there exists $x\in G(\overline{\mathbb{Q}}_p)$ such that

$$\psi^{-1}(\delta) = x\gamma_0x^{-1}. \tag{29}$$

Define a $\overline{\mathbb{Q}}_p$-morphism $\theta:=\psi\circ\mathrm{Int}(x)$. This induces a $\overline{\mathbb{Q}}_p$-isomorphism $I_0\xrightarrow{\sim} I_\delta$. Indeed, $\theta(\gamma_0)=\delta$ implies that θ induces a $\overline{\mathbb{Q}}_p$-isomorphism $Z_{M_b}(\gamma_0)\xrightarrow{\sim} I_\delta$, but $I_0 = Z_{M_b}(\gamma_0)$ by the acceptability of δ. For any $\tau\in\Gamma(p)$,

$$\theta^{-1}\theta^\tau = \mathrm{Int}(x^{-1})\psi^{-1}\psi^\tau\mathrm{Int}(x^{\tau_M}) = \mathrm{Int}(x^{-1}c_\tau x^{\tau_M}).$$

On the other hand,

$$x^{\tau_M}\gamma_0x^{-\tau_M} = \psi^{-1}(\delta)^{\tau_M} = \mathrm{Int}(c_\tau)\psi^{-1}(\delta^{\tau_J}) = \mathrm{Int}(c_\tau)\psi^{-1}(\delta). \tag{30}$$

By (29) and (30), we conclude that $x^{-1}c_\tau x^{\tau_M}\in I_0(\overline{\mathbb{Q}}_p)$. Therefore I_δ is a $\mathbb{Q}_p$-inner form of I_0 given by $\tau\mapsto\mathrm{Int}(x^{-1}c_\tau x^\tau)$ in $H^1(\mathbb{Q}_p,\mathrm{Int}(I_0))$.

□

Lemma 10.9.

$$\begin{aligned}\alpha_p(\gamma_0;\delta') &= \alpha_p(\gamma_0;\delta)+\mathrm{inv}_p(\delta,\delta') \quad (31)\\ \alpha_p(\gamma'_0;\delta) &= \alpha_p(\gamma_0;\delta)+\mathrm{inv}_p(\gamma_0,\gamma'_0) \quad (32)\end{aligned}$$

hold in $X^*(Z(\widehat{I}_0)^{\Gamma(p)})$, where $\mathrm{inv}_p(\delta,\delta')$ is viewed as an element of $X^*(Z(\widehat{I}_0)^{\Gamma(p)})$ via the canonical $\Gamma(p)$-equivariant isomorphism $Z(\widehat{I}_\delta)\simeq Z(\widehat{I}_0)$ (by Lemma 10.8).

Proof. Before the proof, we record a general fact in the first paragraph. Let r be a positive integer. Let γ_1 and γ_2 be semisimple elements of $G(L_r)$ which are conjugate in $G(\overline{\mathbb{Q}}_p)$. Then γ_1 and γ_2 are conjugate in $G(L)$. This follows from an easy application of Steinberg's vanishing theorem. Now suppose that $\gamma_1, \gamma_2 \in G(\mathbb{Q}_p)$. Let $x_0 \in G(\overline{\mathbb{Q}}_p)$ and $x \in G(L)$ be such that $\gamma_2 = x_0\gamma_1x_0^{-1}$ and $\gamma_2 = x\gamma_1x^{-1}$. Let $I_1 := Z_G(\gamma_1)$. Recall ([Kot85, 1.7-1.8]) that the map $H^1(\mathbb{Q}_p, I_1) \to B(I_1)$ of (5) is given by

$$H^1(\mathbb{Q}_p, I_1) \hookrightarrow H^1(\overline{L}/\mathbb{Q}_p, I_1) \simeq H^1(L/\mathbb{Q}_p, I_1) \simeq B(I_1) \tag{33}$$

where the inverse of the second map is given by the inflation map. (It is an isomorphism by Steinberg's vanishing theorem.) The last isomorphism sends a cocycle $\sigma \mapsto \gamma$ to the image of γ in $B(I_1)$. It is easy to see that the cocycle $\tau \mapsto x_0^{-1}x_0^\tau$ (up to coboundary) in $H^1(\mathbb{Q}_p, I_1)$ maps to the image of $x^{-1}x^\sigma$ in $B(I_1)$ under the composite map of (33).

Now we begin to prove (31). Recall from the proof of Lemma 10.8 that we have an L_s-isomorphism $\psi : M_b \xrightarrow{\sim} J_b$. Let $y \in G(L)$ be an element such that $\psi^{-1}(\delta) = y\gamma_0y^{-1}$. Such a y exists by the discussion of the last paragraph since $\psi^{-1}(\delta)$ and γ_0 are conjugate in $G(\overline{\mathbb{Q}}_p)$. We may view y as an equivalence (in the sense of §2) from $V \otimes_{\mathbb{Q}} L$ ($= V_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L$) with the natural Hermitian $B \otimes_F F_{\mathbb{Q}_p}(\gamma_0) \otimes_{\mathbb{Q}_p} L$-module structure onto $V \otimes_{\mathbb{Q}} L$ with the natural Hermitian $B \otimes_F F_{\mathbb{Q}_p}(\delta) \otimes_{\mathbb{Q}_p} L$-module structure, where the latter is the underlying Hermitian structure of the G-isocrystal $\mathbb{V}(\widetilde{b})$ (defined in Remark 4.15). Recall that $\mathbb{V}(\Sigma, \lambda_\Sigma, i_\Sigma)$ may be replaced by $\mathbb{V}(\widetilde{b})$ in the definition of the α_p-invariant. We let y play the role of f in the definition of $\alpha_p(\gamma_0; \delta)$. Then

$$\widetilde{b}_\delta = y^{-1}\widetilde{b}y^\sigma$$

and $\alpha_p(\gamma_0; \delta) = \kappa_{I_0}(b_\delta)$. On the other hand, it is easy to check that $\psi \circ \mathrm{Int}(y)$ gives an L-isomorphism $I_0 \xrightarrow{\sim} I_\delta$, which will be called θ.

Consider the following commutative diagram where the left rectangle comes from (5) and the right rectangle from [Kot97, 4.13].

$$\begin{array}{ccccc}
H^1(\mathbb{Q}_p, I_\delta) & \longrightarrow & B(I_\delta) & \xrightarrow{\cdot b_\delta} & B(I_0) \\
\downarrow{\scriptstyle \alpha_{I_\delta,p}} & & \downarrow{\scriptstyle \kappa_{I_\delta}} & & \downarrow{\scriptstyle \kappa_{I_0}} \\
A_p(I_\delta) & \longrightarrow & X^*(Z(\widehat{I_\delta})^{\Gamma(p)}) & \xrightarrow{\kappa_{I_0}(b_\delta)} & X^*(Z(\widehat{I_0})^{\Gamma(p)})
\end{array} \tag{34}$$

The right top horizontal arrow is induced by $i \mapsto \theta^{-1}(i)\widetilde{b}_\delta$. The right bottom arrow is simply the addition by $\kappa_{I_0}(b_\delta)$ via the canonical identification $Z(\widehat{I_\delta}) = Z(\widehat{I_0})$. On the other hand, there exist $j_0 \in J_b(\overline{\mathbb{Q}}_p)$ and $j \in J_b(L)$ such that $\delta' = j_0\delta j_0^{-1}$ and $\delta' = j\delta j^{-1}$. Let $c(\delta, \delta') \in H^1(\mathbb{Q}_p, I_\delta)$ be given by the cocycle $\tau \mapsto j_0^{-1}j_0^\tau$. By the earlier discussion $c(\delta, \delta')$ maps to $j^{-1}j^\sigma$ under $H^1(\mathbb{Q}_p, I_\delta) \to B(I_\delta)$. Observe that $\widetilde{b}_{\delta'}$ may be defined using the equivalence $\psi(j)y$ in the same way as $\widetilde{b}_\delta$ was defined using y. Thus

$$\widetilde{b}_{\delta'} = (jy)^{-1}\widetilde{b}(jy)^\sigma = y^{-1}\psi(j)^{-1}\widetilde{b}\psi(j)^\sigma y^\sigma.$$

The image of $c(\delta, \delta')$ in $B(I_0)$ is given by $\theta^{-1}(j^{-1}j^\sigma)\widetilde{b}_\delta \in I_0(L)$, which is equal to

$$y^{-1}\psi^{-1}(j^{-1}j^\sigma)yy^{-1}\widetilde{b}y^\sigma = y^{-1}\psi^{-1}(j^{-1})\mathrm{Int}(\widetilde{b})(\psi^{-1}(j)^\sigma)\widetilde{b}y^\sigma = y^{-1}\psi^{-1}(j^{-1})\widetilde{b}\psi^{-1}(j)^\sigma y^\sigma = \widetilde{b}_{\delta'}.$$

Since $\alpha_{I_\delta,p}(c(\delta, \delta')) = \mathrm{inv}_p(\delta, \delta')$, the commutativity of (34) shows that

$$\kappa_{I_0}(b_{\delta'}) = \kappa_{I_0}(b_\delta) + \mathrm{inv}_p(\delta, \delta')$$

which is nothing but (31).

We prove (32) in a similar way using an analogue of the diagram (34), with I_δ (resp. I_0) replaced by I_0 (resp. I'_0) where $I'_0 := Z_G(\gamma'_0)$.

□

11 Auxiliary invariants and vanishing of invariants

Throughout this section, let (A, i) be an object of AV_B^0 such that there is a quasi-isogeny $j : \Sigma \to A[p^\infty]$ compatible with $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$-action. Let $M := Z(\mathrm{End}_B^0(A))$. Let λ be a $(B \otimes_F M, * \otimes c)$-polarization of A. We do not suppose that (A, λ, i) represents an equivalence class of PIC_b. Let $(\gamma_0; \gamma, \delta) \in KT_b$ be such that it corresponds to the p-adic type for (A, i) via Lemma 10.4. Recall that an F-embedding $M \hookrightarrow F(\gamma_0)$ is given. Let $i' : B \otimes_F F(\gamma_0) \to \mathrm{End}^0(A)$ be any map extending i.

Let us define $\alpha(\gamma_0; (A, \lambda, i'))$ and $\beta(\gamma_0; (A, \lambda, i))$. First define α_v and β_v for $v \neq p, \infty$. Let $(c_v)_{v \neq p,\infty}$ be the element in $H^1(\mathbb{Q}, I_0(\overline{\mathbb{A}}^{\infty,p}))$ measuring the difference of $V \otimes \mathbb{A}^{\infty,p}$ and $(V^p A)_\lambda$ as Hermitian $B \otimes_F F(\gamma_0) \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules, using Lemma 3.3. (The Hermitian $B \otimes_F F(\gamma_0) \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-module structure on $V \otimes \mathbb{A}^{\infty,p}$ is induced by i'.) Let $(d_v)_{v \neq p,\infty}$ be the element in $H^1(\mathbb{Q}, H_0(\overline{\mathbb{A}}^{\infty,p}))$ measuring the difference of $V \otimes \mathbb{A}^{\infty,p}$ and $V^p A$ as Hermitian $B \otimes_F M \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules. Using the maps in the diagram (28), we get elements $\alpha_v \in A_v(I_0)$ and $\beta_v \in A_v(H_0)$ corresponding to c_v and d_v. We will also view α_v and β_v as elements of $X^*(Z(\widehat{I_0})^{\Gamma(v)})$ and $X^*(Z(\widehat{H_0})^{\Gamma(v)})$, respectively. To define α_p for $(\gamma_0; (A, \lambda, i'))$ and β_p for $(\gamma_0; (A, \lambda, i))$, we only need to replace $\mathbb{V}(\Sigma, \lambda_\Sigma, i_\Sigma)$ equipped with $B \otimes_F F_{\mathbb{Q}_p}(\delta)$-action by $\mathbb{V}(A[p^\infty], \lambda, i')$ with $B \otimes_F F_{\mathbb{Q}_p}(\gamma_0)$-action in the definition of α_p and β_p for $(\gamma_0; \gamma, \delta)$. The resulting elements $\alpha_p \in X^*(Z(\widehat{I_0})^{\Gamma(p)})$ and $\beta_p \in X^*(Z(\widehat{H_0})^{\Gamma(p)})$ map to $-\mu_1 \in X^*(Z(\widehat{G})^{\Gamma(p)})$ as before. The local components α_∞ and β_∞ are defined to be the same as for $(\gamma_0; \gamma, \delta)$. Finally the element $\alpha(\gamma_0; (A, \lambda, i'))$ in $X^*(Z(\widehat{I_0})^\Gamma)$ is defined to be $\prod_v \alpha_v|_{Z(\widehat{I_0})^\Gamma}$. Likewise, $\beta(\gamma_0, (A, \lambda, i))$ in $X^*(Z(\widehat{H_0})^\Gamma)$ is defined to be $\prod_v \beta_v|_{Z(\widehat{H_0})^\Gamma}$.

Now we introduce a variant of the above construction. Here we assume that $((A, \lambda, i), [a]) \in FP_b^{AV}$. Recall that there exists an embedding $\iota_{(A,\lambda,i)} : H_{(A,\lambda,i)}(\mathbb{A}^\infty) \hookrightarrow G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$. Let $\gamma_0 \in G(\mathbb{Q})$ be as before. (Namely, there exist γ and δ such that $(\gamma_0; \gamma, \delta)$ lies in KT_b and corresponds to (A, i).) Suppose that $\iota_{(A,\lambda,i)}(a)$ is conjugate to γ_0 in $G(\overline{\mathbb{A}}^\infty)$, via any $J_b(\overline{\mathbb{Q}}_p) \hookrightarrow G(\overline{\mathbb{Q}}_p)$ as in Definition 10.1. Then we can define $\alpha(\gamma_0; (A, \lambda, i), [a])$ in $X^*(Z(\widehat{I_0})^\Gamma)$ as $\prod_v \alpha_v|_{Z(\widehat{I_0})^\Gamma}$ where α_v are as follows. In case $v \neq p, \infty$, we reuse the previous definition of α_v for $\alpha(\gamma_0; (A, \lambda, i'))$ where $(V^p A)_\lambda$ is viewed as a Hermitian $B \otimes_F F(\gamma_0) \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ via an isomorphism $F(a) \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \simeq F(\gamma_0) \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ such that $a \mapsto \gamma_0$. For $v = p$ and $v = \infty$, the definition of α_v is the same as in the case of $\alpha(\gamma_0; (A, \lambda, i'))$.

We will need yet another auxiliary invariant where no reference to $(\gamma_0; \gamma, \delta)$ is made. Let (A, i) be as in the beginning of this section. (Drop the assumption $((A, \lambda, i), [a]) \in FP_b^{AV}$.) Suppose that N is a product of fields which are totally real or CM, and that N embeds into $\mathrm{End}_B^0(A)$ as a maximal commutative semisimple F-subalgebra. Then i naturally extends to $i' : B \otimes_F N \to \mathrm{End}^0(A)$. Let λ be a $(B \otimes_F N, * \otimes c)$-polarization of A with respect to i'. Assume that W is a $B \otimes_F N$-module with a $* \otimes c$-Hermitian pairing $\langle \cdot, \cdot \rangle_W$ such that $W \simeq V$ as B-modules. Define a $\mathbb{Q}$-torus T by $T(R) = \{g \in N \otimes_{\mathbb{Q}} R \mid gg^c \in R^\times\}$ for any $\mathbb{Q}$-algebra R. Then we construct

$$\alpha(N, W; (A, \lambda, i)) = \prod_v \alpha_v|_{\widehat{T}^\Gamma} \in X^*(\widehat{T}^\Gamma)$$

where the local components $\alpha_v \in X^*(\widehat{T}^{\Gamma(v)})$ are given as follows. The elements α_v for $v \neq \infty$ are defined in the same way as for $\alpha(\gamma_0; (A, \lambda, i'))$ except that we replace V, I_0, $F(\gamma_0)$ respectively with W, T, N in the previous definition. To describe α_∞, choose an $\mathbb{R}$-algebra homomorphism $h' : \mathbb{C} \to \mathrm{End}_{B_\mathbb{R}}(W_\mathbb{R})$ such that the pairing $(v,w) \mapsto \langle v, h'(\sqrt{-1})w\rangle_W$ is positive definite on $W_\mathbb{R}$. Viewing h' as an $\mathbb{R}$-morphism $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_\mathbb{R}$, choose a $G(\mathbb{R})$-conjugate h'' of h' that factors through T. We get $\mu_{h''} \in X_*(T) = X^*(\widehat{T})$ from h'' (as in §5). Via restriction this gives the element $\alpha_\infty \in X^*(\widehat{T}^{\Gamma(\infty)})$, which is independent of the choice of h' and h''.

We conclude this section with the following lemma which is an important step in proving the vanishing of Kottwitz invariants in certain cases.

Lemma 11.1. *Suppose that (A,i) is an object of AV_B^0 such that there is an isomorphism $j : \Sigma \xrightarrow{\sim} A[p^\infty]$ compatible with $\mathcal{O}_B \otimes_\mathbb{Z} \mathbb{Z}_p$-action. Put $M := Z(\mathrm{End}^0(A,i))$. Suppose that N is a product of fields which are totally real or CM, and that N embeds into $\mathrm{End}^0(A,i)$ as a maximal commutative semisimple F-subalgebra. Then we can find a $(B \otimes_F N, * \otimes c)$-polarization $\lambda_0 : A \to A^\vee$ and a $B \otimes_F N$-module W_0 with a $* \otimes c$-Hermitian pairing $\langle\cdot,\cdot\rangle_0 : W_0 \times W_0 \to \mathbb{Q}$ such that*

- *(i) $W_0 \otimes_\mathbb{Q} \mathbb{A}^{\infty,p}$ and $(V^pA)_{\lambda_0}$ are equivalent as $B \otimes_\mathbb{Q} \mathbb{A}^{\infty,p}$-Hermitian modules where $(V^pA)_{\lambda_0}$ is the $B \otimes_F N \otimes_\mathbb{Q} \mathbb{A}^{\infty,p}$-module V^pA with the λ_0-Weil pairing, and*
- *(ii) $W_0 \otimes_\mathbb{Q} \mathbb{R}$ and $V \otimes_\mathbb{Q} \mathbb{R}$ are equivalent as $B \otimes_\mathbb{Q} \mathbb{R}$-modules with Hermitian pairings $\langle\cdot,\cdot\rangle_0$ and $\langle\cdot,\cdot\rangle$ (the latter from the PEL datum), respectively.*

Remark 11.2. Given any object (A,i) of AV_B^0 such that there exists a quasi-isogeny $j : \Sigma \to A[p^\infty]$ compatible with $\mathcal{O}_B \otimes_\mathbb{Z} \mathbb{Z}_p$-action, we can always find an object (A',i') which is isomorphic in AV_B^0 to (A,i) such that there is an isomorphism $j : \Sigma \xrightarrow{\sim} A[p^\infty]$ compatible with $\mathcal{O}_B \otimes_\mathbb{Z} \mathbb{Z}_p$-action. This is possible using the argument in the proof of Lemma 7.1.

Proof. Note that i naturally extends to an F-algebra map $B \otimes_F N \hookrightarrow \mathrm{End}^0(A)$, which we call i'. Write $N = \prod_t N_t$ as a product of fields, and decompose $(A,i') = \oplus_{t\in I}(A_t, i_t)$ accordingly where i_t is a $\mathbb{Q}$-algebra map $B \otimes_F N_t \hookrightarrow \mathrm{End}^0(A_t)$. Then N_t is a maximal commutative subalgebra of $\mathrm{End}^0(A_t,i_t)$ for each t. By maximality of N_t, there is an isomorphism $B \otimes_F N_t \simeq M_n(N_t)$. By Lemma 9.2, we may choose a $(B \otimes_F N_t, * \otimes c)$-polarization $\lambda_t : A_t \to A_t^\vee$ for each t. By putting them together, we have a $(B \otimes_F N, * \otimes c)$-polarization $\lambda_0 : A \to A^\vee$.

Arguing as in [HT01, p.170-171], we find a lifting $(\widetilde{A}_t, \widetilde{\lambda}_t, \widetilde{i}_t)$ of (A_t, λ_t, i_t) with respect to the fixed reduction map $\overline{\iota}_p : \mathcal{O}_{\mathbb{Q}_p^{ur}} \to \overline{\mathbb{F}}_p$, where $\widetilde{A}_t$ is an abelian scheme over $\mathcal{O}_{\mathbb{Q}_p^{ur}}$, $\widetilde{\lambda}_t$ is a polarization of $\widetilde{A}_t$, and $\widetilde{i}_t : B \otimes_F N_t \to \mathrm{End}^0(\widetilde{A}_t)$ such that $\ddagger_{\widetilde{\lambda}_t}$ restricts to $* \otimes c$ via $\widetilde{i}_t$.

Set $W_t := H_1((\widetilde{A}_t \times_{\mathcal{O}_{\mathbb{Q}_p^{ur}},\iota_p} \mathbb{C})(\mathbb{C}), \mathbb{Q})$. This is a $B \otimes_F N_t$-module with a $* \otimes c$-Hermitian pairing coming from $\widetilde{\lambda}_t$. So $W_0 := \oplus_{t\in I} W_t$ is equipped with a $B \otimes_F N$-module structure with a $* \otimes c$-Hermitian pairing $\langle\cdot,\cdot\rangle_0 : W_0 \times W_0 \to \mathbb{Q}$. By construction we see that $W_0 \otimes \mathbb{A}^{\infty,p}$ is equivalent to V^pA as Hermitian $B \otimes_F N \otimes_\mathbb{Q} \mathbb{A}^{\infty,p}$-modules.

It remains to prove that $W_0 \otimes \mathbb{R} \simeq V \otimes \mathbb{R}$ as Hermitian $B \otimes \mathbb{R}$-modules. Put $\widetilde{A} = \prod_t \widetilde{A}_t$. Observe that $\mathrm{Lie}\,\widetilde{A}$ is a module over $\mathcal{O}_F \otimes_\mathbb{Z} \mathcal{O}_{\mathbb{Q}_p^{ur}} \simeq \prod_\xi \mathcal{O}_{\mathbb{Q}_p^{ur}}$ where ξ runs over the set $\mathrm{Hom}_\mathbb{Z}(\mathcal{O}_F, \mathcal{O}_{\mathbb{Q}_p^{ur}})$. Accordingly we decompose $\mathrm{Lie}\,\widetilde{A} = \oplus_\xi(\mathrm{Lie}\,\widetilde{A})_\xi$. Similarly, we have decompositions of $\mathcal{O}_F \otimes_\mathbb{Z} \overline{\mathbb{F}}_p$-modules $\mathrm{Lie}\,A = \oplus_\zeta(\mathrm{Lie}\,A)_\zeta$ and $\mathrm{Lie}\,\Sigma = \oplus_\zeta(\mathrm{Lie}\,\Sigma)_\zeta$ where ζ runs over the set $\mathrm{Hom}_\mathbb{Z}(\mathcal{O}_F, \overline{\mathbb{F}}_p)$. We have

$$\mathrm{rank}_{\mathcal{O}_{\mathbb{Q}_p^{ur}}}(\mathrm{Lie}\,(\widetilde{A}))_\xi = \dim_{\overline{\mathbb{F}}_p}(\mathrm{Lie}\,(A))_{\overline{\iota}_p\circ\xi} \overset{\text{via } j}{=} \dim_{\overline{\mathbb{F}}_p}(\mathrm{Lie}\,(\Sigma))_{\overline{\iota}_p\circ\xi} = \dim_\mathbb{C}(V^1_{\iota_p\circ\xi}) \tag{35}$$

in which the last equality follows from the determinant condition imposed on Σ (see §5).

Now observe that there is an isomorphism of $\mathcal{O}_F \otimes_\mathbb{Z} \mathbb{C}$-modules

$$W_0 \otimes_\mathbb{Q} \mathbb{R} \simeq (\mathrm{Lie}\,\widetilde{A}) \otimes_{\mathcal{O}_{\mathbb{Q}_p^{ur}},\iota_p} \mathbb{C}.$$

According to $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{C} \simeq \prod_{\tau \in \mathrm{Hom}(\mathcal{O}_F,\mathbb{C})} \mathbb{C}$, we decompose the left hand side as $W_0 \otimes_{\mathbb{Q}} \mathbb{R} \simeq \oplus_\tau (W_0 \otimes_{\mathbb{Q}} \mathbb{R})_\tau$. We deduce from (35) that

$$\dim_{\mathbb{C}}(W_0 \otimes_{\mathbb{Q}} \mathbb{R})_\tau = \dim_{\mathbb{C}} V^1_\tau.$$

Hence $W_0 \otimes \mathbb{R}$ and $V \otimes \mathbb{R}$ are isomorphic $B \otimes \mathbb{R}$-modules. In the case of a PEL datum of type (C) (i.e. symplectic case), we are done since there is a unique Hermitian $B \otimes \mathbb{R}$-module structure on $V \otimes \mathbb{R}$ up to equivalence. In the case of type (A), the argument of [HT01, p.172-173] shows that $W_0 \otimes \mathbb{R}$ with $\langle \cdot, \cdot \rangle_0$ is equivalent to $V \otimes \mathbb{R}$ with $\langle \cdot, \cdot \rangle$ as Hermitian $B \otimes \mathbb{R}$-modules. (The argument of Harris and Taylor shows that the classifying invariants ([HT01, p.49]) for the two Hermitian pairings are respectively given by the numbers $\dim_{\mathbb{C}}(W_0 \otimes_{\mathbb{Q}} \mathbb{R})_\tau$ and $\dim_{\mathbb{C}} V^1_\tau$, which are the same by the above identity.)

□

Corollary 11.3. *Suppose that N, W_0 and (A, λ_0, i) are as in Lemma 11.1. Then $\alpha(N, W_0; (A, \lambda_0, i))$ is trivial.*

Proof. By Lemma 11.1, $\alpha_v(N, W_0; (A, \lambda_0, i))$ vanishes when $v \neq p, \infty$. The key fact that $\alpha_p \alpha_\infty$ is trivial is proved using the same argument as in the last paragraph of §13 in [Kot92]. □

12 Main lemmas

First we will describe how we associate a Kottwitz triple $(\gamma_0, \gamma, \delta)$ to an element $((A, \lambda, i), [a]) \in FP^{AV}_b$. This will lead to a natural map from the set FP^{AV}_b to the set KT_b.

We begin with a fixed element $((A, \lambda, i), [a]) \in FP^{AV}_b$ and put $z = [(A, \lambda, i)]$. Recall that we defined an embedding $\iota_z : H_z(\mathbb{A}^\infty) \hookrightarrow G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ which is well-defined up to $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-conjugacy. We simply let (γ, δ) be the image of $a \in H_z(\mathbb{A}^\infty)$ under ι_z.

It requires more effort to determine the element γ_0. The point is that there is an F-algebra embedding $i' : F(a) \hookrightarrow \mathrm{End}_B(V)$ compatible with involutions c and $\#$, respectively on the source and the target, by Lemma 14.1 of [Kot92]. In fact, we need to check two implicit assumptions underlying Lemma 14.1 of Kottwitz since it is those assumptions that make his proof work. Firstly, we verify that there exists an F-algebra embedding of $F(a)$ into $\mathrm{End}_B(V)$ (with no condition on involutions). This can be checked locally at every place v. For $v \neq p, \infty$, ι_z gives such an embedding. For $v = p$, it is enough to remark that $B_{\mathbb{Q}_p}$ splits. For $v = \infty$, one can easily check case by case for types (A) and (C). Secondly, we verify that the $\mathbb{Q}_p$-group G_0 given by $G_0(\mathbb{Q}_p) = \{g \in \mathrm{End}_B(V)_{\mathbb{Q}_p} | gg^\# = 1\}$ is quasi-split over $\mathbb{Q}_p$. This follows from our original assumption on the PEL datum.

Now we are ready to describe γ_0. Using the embedding i' in the last paragraph, we set γ_0 to be the element $i'(a)$ in $\mathrm{End}_B(V)$. As i' is compatible with involutions, we see that γ_0 lies in $G(\mathbb{Q})$. Note that by construction there is an isomorphism of F-algebras $F(\gamma_0) \simeq F(a)$ such that $\gamma_0 \mapsto a$.

It remains to show that the triple $(\gamma_0; \gamma, \delta)$ we just constructed is a Kottwitz triple and well-defined up to equivalence. The well-definedness is immediate from the construction: the triple changes only within its equivalence class as we vary the choice of i and ι_z and the representative $((A, \lambda, i), [a])$ in its equivalence class in FP^{AV}_b. The element γ_0 is semisimple and elliptic over $\mathbb{R}$ since a is so. The acceptability of δ is inherited from a. By construction γ_0 and γ are conjugate in $G(\mathbb{A}^{\infty,p})$ and γ_0 and δ are conjugate in $G(\overline{\mathbb{Q}}_p)$.

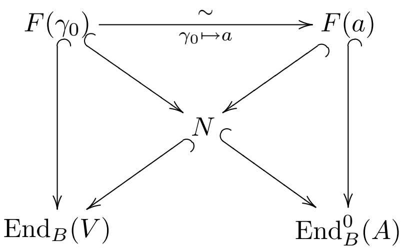
Lemma 12.1. *The image of the natural map from FP^{AV}_b to KT_b defined above is contained in KT^{eff}_b.*

Proof. Let $(\gamma_0; \gamma, \delta)$ be the image of $((A, \lambda, i), [a])$. The long exact sequence arising from

$$1 \to Z(\widehat{G}) \to Z(\widehat{I_0}) \to Z(\widehat{I_0})/Z(\widehat{G}) \to 1$$

yields a natural map $Z(\widehat{I_0})^\Gamma \to \mathfrak{K}(I_0/\mathbb{Q})$ which is surjective ([Kot92, p.425]). Thus we get an injection of groups $\mathfrak{K}(I_0/\mathbb{Q})^D \hookrightarrow X^*(Z(\widehat{I_0})^\Gamma)$. The Kottwitz invariants are defined so that $\alpha(\gamma_0;\gamma,\delta)$ maps to $\alpha(\gamma_0;(A,\lambda,i),[a])$ under this map. Therefore the proof of the lemma boils down to showing that $\alpha(\gamma_0;(A,\lambda,i),[a])$ is trivial.

Before further reduction steps, we need some preparation. Set $I = Z_{H_z}(a)$. Arguing as in the top of [Kot92, p.424], we see that the $\mathbb{Q}$-groups I_0 and I are inner forms of each other. Note that there are natural inclusions $I_0(\mathbb{Q}) \subset \mathrm{End}_{B\otimes_F F(\gamma_0)}(V)$ and $I(\mathbb{Q}) \subset \mathrm{End}^0_{B\otimes_F F(a)}(A)$. We choose a maximal torus T of I so that T is elliptic at ∞ and all the finite places where I_0 is not quasi-split. Then T transfers to I_0 locally at every place. As T is elliptic over $\mathbb{R}$, the argument at the end of proof of Lemma 14.1 in [Kot92] shows that T transfers globally to I_0. Let N be the centralizer of $T(\mathbb{Q})$ in $\mathrm{End}^0_{B\otimes_F F(a)}(A)$. Then N is a product of fields which are CM or totally real. Moreover N is a maximal commutative semisimple F-subalgebra of $\mathrm{End}^0_B(A)$ equipped with an F-algebra embedding $F(a) \hookrightarrow N$. By the second paragraph of [Kot92, p.426], the transfer of T into I_0 provides an inclusion $N \hookrightarrow \mathrm{End}_{B\otimes_F F(\gamma_0)}(V)$ which maps $a \in N$ to γ_0. To summarize we have the following commutative diagram where all maps are compatible with involutions (complex conjugation c on $F(\gamma_0)$, $F(a)$, N; $\#$ on $\mathrm{End}_B(V)$; $\ddagger_\lambda$ on $\mathrm{End}^0_B(A)$).



(36)

These maps induce (1) a $B\otimes_F N$-polarization λ' which extends the $B\otimes_F F(a)$-polarization structure of λ (2) an F-algebra map $i' : B\otimes_F N \hookrightarrow \mathrm{End}^0(A)$ extending $B\otimes_F F(a) \hookrightarrow \mathrm{End}^0(A)$ given by i and a, and (3) a Hermitian $B\otimes_F N$-module structure on V with the pairing $\langle\cdot,\cdot\rangle$ which extends the Hermitian $B\otimes_F F(\gamma_0)$-module structure given by γ_0. So the invariant $\alpha(N,V;(A,\lambda',i')) \in X^*(\widehat{T}^\Gamma)$ makes sense.

We claim that $\alpha(N,V;(A,\lambda',i'))$ maps to $\alpha(\gamma_0;(A,\lambda,i),[a]) \in X^*(Z(\widehat{I_0})^\Gamma))$ via the inclusion $Z(\widehat{I_0}) \hookrightarrow \widehat{T}$. Keeping the compatibility (36) in mind, let us verify that $\alpha_v(N,V;(A,\lambda',i'))$ is sent to $\alpha_v(\gamma_0;(A,\lambda,i),[a])$ for every v. This is clear from the definition when $v=\infty$. For $v \neq p,\infty$, this follows from the functoriality of the map $\alpha_{(\cdot),v}$ in Lemma 2.3. For $v=p$, we use the functoriality of the map $\kappa_{(\cdot)}$ in §4. In other words, we appeal to the following commutative diagrams for $v\neq\infty,p$ and $v=p$, respectively.

$$\begin{array}{ccc} H^1(\mathbb{Q}_v,T) & \longrightarrow & H^1(\mathbb{Q}_v,I_0) \\ \downarrow{\scriptstyle \alpha_{T,v}} & & \downarrow{\scriptstyle \alpha_{I_0,v}} \\ X^*(\widehat{T}^{\Gamma(v)}) & \longrightarrow & X^*(Z(\widehat{I_0})^{\Gamma(v)}) \end{array} \qquad \begin{array}{ccc} B(T) & \longrightarrow & B(I_0) \\ \downarrow{\scriptstyle \kappa_T} & & \downarrow{\scriptstyle \kappa_{I_0}} \\ X^*(\widehat{T}^{\Gamma(p)}) & \longrightarrow & X^*(Z(\widehat{I_0})^{\Gamma(p)}) \end{array}$$

As a result of the above claim, it suffices to prove that $\alpha(N,V;(A,\lambda',i'))$ is trivial. By Lemma 11.1, we can find a $B\otimes_F N$-polarization λ_0 and a Hermitian $B\otimes_F N$-module $W=W_0$ satisfying the conditions in that lemma. Thus the invariants $\alpha(N,W;(A,\lambda',i'))$ and $\alpha(N,W;(A,\lambda_0,i'))$ may be considered. As further reduction steps, we make two claims, which are very similar to the ones found in the proof of [Kot92, Lem 13.2].

The first claim is that $\alpha(N,V;(A,\lambda',i')) = \alpha(N,W;(A,\lambda',i'))$. To prove this, denote by t the element of $H^1(\mathbb{Q},T)$ measuring the difference of V and W as Hermitian $B\otimes_F N$-modules. For each place v, write t_v for the image of t under $H^1(\mathbb{Q},T)\to H^1(\mathbb{Q}_v,T)\to X^*(\widehat{T}^{\Gamma(v)})$. We see that $\alpha_v(N,V;(A,\lambda',i')) = \alpha_v(N,W;(A,\lambda',i')) + t_v$ in $X^*(\widehat{T}^{\Gamma(v)})$. Taking product over all v, we get $\alpha(N,V;(A,\lambda',i')) = \alpha(N,W;(A,\lambda',i'))$ since $\prod_v t_v = 1$ by the exact sequence of Lemma 2.3.

The second claim is that $\alpha(N,W;(A,\lambda',i')) = \alpha(N,W;(A,\lambda_0,i'))$. Observe that T is isomorphic to $H_{\lambda'}$ defined by $H_{\lambda'}(\mathbb{Q}) = \{g\in \mathrm{End}_{B\otimes_F N}(A)|gg^{\ddagger_{\lambda'}}\in\mathbb{Q}^\times\}$. We write t for the element of $\ker(H^1(\mathbb{Q},H_{\lambda'})\to H^1(\mathbb{R},H_{\lambda'}))$ measuring the difference between λ' and λ_0 (Lemma 9.2). Let t_v be the image of t under $H^1(\mathbb{Q},T)\to H^1(\mathbb{Q}_v,T)\to X^*(\widehat{T}^{\Gamma(v)})$. Then $\alpha_v(N,W;(A,\lambda_0,i')) - \alpha_v(N,W;(A,\lambda',i')) = t_v$ in $X^*(\widehat{T}^{\Gamma(v)})$ for every place v (including the case $v=\infty$ where both sides are trivial). Therefore an application of Lemma 2.3 as before shows that $\alpha(N,W;(A,\lambda,i')) = \alpha(N,W;(A,\lambda_0,i'))$.

As a result of the previous claims, we only need to prove that $\alpha(N,W;(A,\lambda_0,i'))$ is trivial. This is exactly Corollary 11.3.

□

Corollary 12.2. *Suppose that $(\gamma_0;\gamma,\delta)\in KT_b$ corresponds to (A,i) via Lemma 10.4 and Corollary 8.5. Let $\widetilde{i}: B\otimes_F F(\gamma_0)\hookrightarrow \mathrm{End}^0(A)$ be an extension of i and λ be a $B\otimes_F F(\gamma_0)$-polarization of A with respect to $\widetilde{i}$. Then $\alpha(\gamma_0;(A,\lambda,\widetilde{i}))$ and $\beta(\gamma_0;(A,\lambda,i))$ are trivial.*

Proof. As in the second through the fourth paragraph of the proof of the last lemma, we can find $\alpha(N,V;(A,\lambda',i'))$ which maps to $\alpha(\gamma_0;(A,\lambda,\widetilde{i}))$. To do so, it is enough to replace a by γ_0 in the argument. (So $I = Z_{H_z}(\gamma_0)$, N is the centralizer of $T(\mathbb{Q})$ in $\mathrm{End}^0_{B\otimes_F F(\gamma_0)}(A)$, etc.) The triviality of $\alpha(N,V;(A,\lambda',i'))$ is proved in the same way as in the last lemma. Therefore $\alpha(\gamma_0;(A,\lambda,\widetilde{i}))$ is trivial.

Similarly we repeat the argument of the last lemma to find $\alpha(N,V;(A,\lambda',i'))$ which maps to $\beta(\gamma_0;(A,\lambda,i))$ under the natural map $X^*(\widehat{T}^\Gamma)\to X^*(Z(\widehat{H}_0)^\Gamma)$. To see this, replace I_0, I, $F(a)$, $F(\gamma_0)$ by H_0, $H_{(A,\lambda,i)}$, M, M in the proof of Lemma 12.1, respectively. Again, $\alpha(N,V;(A,\lambda',i'))$ is proved to be trivial in the same way. The proof is complete.

□

The next lemma will play a crucial role in rewriting the counting point formula in a group-theoretic way.

Lemma 12.3. *The map in Lemma 12.1 defines a set bijection from FP_b^{AV} onto KT_b^{eff}.*

Proof. We will construct the backward map from KT_b^{eff} to FP_b^{AV} which is inverse to the map in Lemma 12.1. So our starting point is a Kottwitz triple $(\gamma_0;\gamma,\delta)$ with $\alpha(\gamma_0;\gamma,\delta)$ being trivial.

Suppose that $(\gamma_0;\gamma,\delta)$ is mapped to a minimal p-adic type $(M,\vec{\eta},\vec{n})$ under the map in Lemma 10.4. This p-adic type over F naturally corresponds to a pair (A,i) as in (i) of Corollary 8.5. We choose an M-algebra embedding $F(\gamma_0)\hookrightarrow \mathrm{End}^0_B(A)$ whose existence is guaranteed since the embedding exists locally at every place (use (23)). This amounts to choosing an F-algebra embedding $i': B\otimes_F F(\gamma_0)\hookrightarrow \mathrm{End}^0(A)$. With respect to i', there exists a $B\otimes_F F(\gamma_0)$-polarization λ_1 by Lemma 9.2.

Our plan is to find a polarization λ such that $[(A,\lambda,i)]\in \mathrm{PIC}_b$ and an element $a\in H_{(A,\lambda,i)}(\mathbb{Q})$ such that $((A,\lambda,i),[a])$ belongs to FP_b^{AV} and maps to $(\gamma_0;\gamma,\delta)$ via the map of Lemma 12.1. First off, we will search for a $B\otimes_F M$-polarization λ. For this we will use the fact that $\beta(\gamma_0;\gamma,\delta)$ is trivial. The last fact is an immediate consequence of the fact that $\alpha(\gamma_0;\gamma,\delta)$ is trivial.

Let H_1 be the $\mathbb{Q}$-algebraic group with

$$H_1(\mathbb{Q}) = \{g\in \mathrm{End}^0_B(A)|g^{\ddagger_{\lambda_1}}g\in\mathbb{Q}^\times\}.$$

Note that H_1 is an inner form of H_0 as we may argue as in [Kot92, p.424]. (Recall the definition of H_0 from (27).) For each v, let $d_v = \beta_v(\gamma_0;\gamma,\delta) - \beta_v(\gamma_0;(A,\lambda_1,i))$ as an element of $X^*(Z(\widehat{H_0})^{\Gamma(v)}) = X^*(Z(\widehat{H_1})^{\Gamma(v)})$. In fact, each d_v for $v \neq \infty$ is trivial on the connected component of $Z(\widehat{H_0})^{\Gamma(v)}$ and may be viewed as an element of $A_v(H_0)$. Since d_∞ is trivial by the definition of $\beta_\infty(\gamma_0;(A,\lambda_1,i))$, we will view d_∞ as the trivial element of $A_\infty(H_0)$. We know that both $\beta(\gamma_0;\gamma,\delta)$ and $\beta(\gamma_0;(A,\lambda_1,i))$ are trivial in $X^*(Z(\widehat{H_0})^{\Gamma})$. Indeed, the former is trivial by assumption and the latter by Corollary 12.2. So their difference is trivial, namely the image of $(d_v)_v$ under $H^1(\mathbb{Q},H_1(\overline{\mathbb{A}})) \to \oplus_v A_v(H_1) \to A(H_1)$ is trivial. We deduce from Lemma 2.3 that there exists $d \in H^1(\mathbb{Q},H_1)$ mapping to $(d_v)_v \in H^1(\mathbb{Q},H_1(\overline{\mathbb{A}}))$. In particular, we have $d \in \ker(H^1(\mathbb{Q},H_1) \to H^1(\mathbb{R},H_1))$, which implies by the argument in the proof of [HT01, Lem V.4.3] that there exists a $B\otimes_F M$-polarization λ on A (with respect to $i'|_{B\otimes_F M}$) such that for all $v\neq\infty$, $\beta_v(\gamma_0;\gamma,\delta)-\beta_v(\gamma_0;(A,\lambda,i))$ is trivial. The last condition can be interpreted as the existence of the following isomorphisms.

$$V\otimes\mathbb{A}^{\infty,p} \simeq (V^pA)_\lambda \quad \text{as} \quad B\otimes_F M\otimes_{\mathbb{Q}}\mathbb{A}^{\infty,p}\text{-modules} \tag{37}$$
$$\mathbb{V}(\Sigma,\lambda_\Sigma,i_\Sigma) \simeq \mathbb{V}(A[p^\infty],\lambda,i) \quad \text{as} \quad \text{isocrystals with } B\otimes_F M\otimes_{\mathbb{Q}}\mathbb{Q}_p\text{-action} \tag{38}$$

which preserve Hermitian pairings up to $(\mathbb{A}^{\infty,p})^\times$ and $L^\times$, respectively. This proves that (A,λ,i) represents an element of PIC_b.

Next we construct an element $a\in H_{(A,\lambda,i)}(\mathbb{Q})$. Recall that we have a $B\otimes_F F(\gamma_0)$-polarization λ_1 with respect to i'. Let I_1 be the $\mathbb{Q}$-algebraic group with

$$I_1(\mathbb{Q}) = \{g\in \mathrm{End}^0_{B\otimes F(\gamma_0)}(A) | g^{\ddagger_{\lambda_1}}g\in\mathbb{Q}^\times\}.$$

As it was the case for H_1 and H_0, we see that I_1 is an inner form of I_0. We basically repeat the argument in the last paragraph, replacing $B\otimes_F M$ with $B\otimes_F F(\gamma_0)$ and H_1 with I_1. When $v\neq\infty$, let $e_v := \alpha_v(\gamma_0;\gamma,\delta)-\alpha_v(\gamma_0;(A,\lambda_1,i'))$, which may be viewed as elements of $A_v(I_0)=A_v(I_1)$. Let $e_\infty\in A_\infty(I_1)$ be the trivial element. We know that both $\alpha_v(\gamma_0;\gamma,\delta)$ and $\alpha_v(\gamma_0;(A,\lambda_1,i'))$ are trivial by the initial assumption and Corollary 12.2, respectively. As in the last paragraph, we can find $e\in\ker(H^1(\mathbb{Q},I_1)\to H^1(\mathbb{R},I_1))$ which maps to $(e_v)_v$ under $H^1(\mathbb{Q},I_1)\to H^1(\mathbb{Q},I_1(\overline{\mathbb{A}}))$. Moreover, e can be chosen so that e maps to d under $H^1(\mathbb{Q},I_1)\to H^1(\mathbb{Q},H_1)$. This can be seen from the following commutative diagram coming from Lemma 2.3. The left vertical map is surjective by [HT01, p.174].

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \ker^1(\mathbb{Q},I_1) & \longrightarrow & H^1(\mathbb{Q},I_1) & \longrightarrow & H^1(\mathbb{Q},I_1(\overline{\mathbb{A}})) & \longrightarrow & A(I_1) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \ker^1(\mathbb{Q},H_1) & \longrightarrow & H^1(\mathbb{Q},H_1) & \longrightarrow & H^1(\mathbb{Q},H_1(\overline{\mathbb{A}})) & \longrightarrow & A(H_1)
\end{array}$$

The cocycle e naturally corresponds to a $B\otimes_F F(\gamma_0)$-polarization λ' (with respect to i') by Lemma 9.3. The following properties of λ' result from the construction of e.

(i) $V\otimes\mathbb{A}^{\infty,p}$ and $(V^pA)_{\lambda'}$ are equivalent as $B\otimes_F F(\gamma_0)\otimes_{\mathbb{Q}}\mathbb{A}^{\infty,p}$-Hermitian modules

(ii) $\mathbb{V}(\Sigma,\lambda_\Sigma,i_\Sigma)\simeq\mathbb{V}(A[p^\infty],\lambda',i')$ as isocrystals with $B\otimes_F F(\gamma_0)\otimes_{\mathbb{Q}}\mathbb{Q}_p$-action, preserving Hermitian pairings up to $L^\times$

(iii) λ' is equivalent to λ as $B\otimes_F M$-polarizations (via $B\otimes_F M\hookrightarrow B\otimes_F F(\gamma_0)$)

The part (iii) implies that there exists $h\in\mathrm{End}^0_B(A)^\times$ such that $h^\vee\lambda' h=\gamma\lambda$ for some $\gamma\in\mathbb{Q}^\times$. Then the association $g\mapsto hgh^{-1}$ defines an M-algebra map $\mathrm{End}^0_B(A)\xrightarrow{\sim}\mathrm{End}^0_B(A)$ compatible with involutions $\ddagger_\lambda$ and $\ddagger_{\lambda'}$. The fact that λ' is a $B\otimes_F F(\gamma_0)$-polarization means that the map i' induces an M-algebra map $F(\gamma_0) \hookrightarrow \mathrm{End}^0_B(A)$ compatible with involutions c and $\ddagger_{\lambda'}$. Finally we define $a := h^{-1}i(\gamma_0)h$. Then $a^{\ddagger_\lambda}a = \gamma_0^c\gamma_0 \in \mathbb{Q}^\times$. Hence $a \in H_{(A,\lambda,i)}(\mathbb{Q})$, which is the desired element.

Now that we have explained how to associate $((A,\lambda,i),[a])$ to a Kottwitz triple $(\gamma_0;\gamma,\delta)$ whose Kottwitz invariant vanishes, we need to verify that $((A,\lambda,i),[a])$ is an element of FP_b^{AV}. For this, (37) and (38) tell us that (A,λ,i) represents an element of PIC_b, and the acceptability of $a \in H_z(\mathbb{Q})$ is inherited from δ. The well-definedness of $((A,\lambda,i),[a])$ is easy to check. By changing γ and δ in their conjugacy classes we change (A,λ,i) within its near equivalence class and a within its $\mathbb{A}$-conjugacy class. Although replacing γ_0 with a stably conjugate element may change H_0, it is easy to check that the resulting $((A,\lambda,i),[a])$ is unchanged.

Finally, we verify that the map constructed above is the inverse of the map in Lemma 12.1. Surjectivity follows from our construction. If $(\gamma_0;\gamma,\delta)$ maps to $((A,\lambda,i),[a])$ then it is readily checked that $\iota_{(A,\lambda,i)}(a)$ is conjugate to (γ,δ) in $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$. Since the stable conjugacy class of γ_0 is determined by the conjugacy class of (γ,δ), the image of a under the map in Lemma 12.1 is stably conjugate to γ_0 in $G(\mathbb{Q})$.

To see injectivity, suppose that both $((A,\lambda,i),[a])$ and $((A',\lambda',i'),[a'])$ map to $(\gamma_0;\gamma,\delta)$. Let $z := [(A,\lambda,i)]$ and $z' = [(A',\lambda',i')]$. Consider the minimal p-adic type over F for (A,i), defined on the F-algebra M_z (see Proposition 8.4). Using the F-algebra embedding $\zeta : M_z \hookrightarrow F(a)$ in Lemma 8.6, get an equivalent p-adic type $(F(a),(\eta_t),(\vec{n}_t))$. We claim that the last p-adic type is equivalent to the p-adic type constructed from $(\gamma_0;\gamma,\delta)$ in Lemma 10.4 under an isomorphism $F(a) \simeq F(\gamma_0)$ taking a to γ_0. This is so because (η_t) and $(\vec{n}_t)$ are determined by the valuation of a_t at each place over p of the fields $F(a_t)$, if we write $F(a) = \prod_t F(a_t)$ as a product of fields and denote by (a_t) the image of a. The point is that since a is acceptable, the p-adic valuation of a_t recovers the slope of the part of $A[p^\infty]$ on which a_t acts. So (η_t) and $(\vec{n}_t)$ can be recovered in view of (ii) of Corollary 8.5. In fact, we constructed a p-adic type from $(\gamma_0;\gamma,\delta)$ using the p-adic valuations of γ_0 in the fields F_t where $F(\gamma_0) = \prod_t F_t$ is a decomposition into fields. This proves our claim. As a consequence, the p-adic type for (A,i) is equivalent to the one for (A',i') since both are equivalent to the one constructed from $(\gamma_0;\gamma,\delta)$. It is easy to verify that (A,λ,i) and (A',λ',i') are nearly equivalent using the earlier part of the current proof involving β_v-invariant. Therefore there is an isomorphism $H_z \simeq H_{z'}$ by Lemma 9.6, which is well-defined up to $\mathbb{Q}$-conjugacy. It remains to see that a and a' are $H_z(\mathbb{A})$-conjugate via this isomorphism. Without loss of generality we may assume that $(A',\lambda',i') = (A,\lambda,i)$. Since $\mathbb{C}$-conjugacy and $\mathbb{R}$-conjugacy coincide in $H_z(\mathbb{R})$, it suffices to check that a and a' are $H_z(\mathbb{A}^\infty)$-conjugate. By Lemma 3.3, a and a' are $H_z(\mathbb{A}^{\infty,p})$-conjugate if and only if the Hermitian $B \otimes_F F(a) \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-module $(V^pA)_\lambda$ and the Hermitian $B \otimes_F F(a') \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-module $(V^pA)_\lambda$ are equivalent via $F(a) \simeq F(a')$ with $a \mapsto a'$. Similarly, a and a' are $H_z(\mathbb{Q}_p)$-conjugate if and only if the isocrystal $\mathbb{V}(A[p^\infty],\lambda,i)$ with the Hermitian $B \otimes_F F_{\mathbb{Q}_p}(a)$-pairing is isomorphic to the isocrystal $\mathbb{V}(A[p^\infty],\lambda,i)$ with the Hermitian $B \otimes_F F_{\mathbb{Q}_p}(a')$-pairing such that the two pairings match up to $L^\times$. (The last fact, an analogue of (part of) Lemma 3.3, can be proved analogously as that lemma.) But we know that $\iota_z(a)$ and $\iota_z(a')$ are conjugate in $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ since they are conjugate to (γ,δ) therein. This implies that the two Hermitian modules above are equivalent and the two isocrystals above are isomorphic with additional structure.(Apply Lemma 3.3 and its analogue for isocrystals again.) Therefore a and a' are $H_z(\mathbb{A}^\infty)$-conjugate.

□

13 Final form of the counting point formula

We go back to the analysis of cohomology of Igusa varieties. Assuming that $\varphi \in C_c^\infty(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$ is an acceptable function, we combine Lemma 7.4 and Lemma 8.6 to obtain the following expression for $\mathrm{tr}\,(\varphi|H_c(\mathrm{Ig}_b,\mathscr{L}_\xi))$.

$$\sum_{z\in\mathrm{PIC}_b}\sum_{[a]\in H_z(\mathbb{Q})/\sim}\mathrm{vol}(\iota_z(Z_{H_z}(a)(\mathbb{Q}))\backslash\iota_z(Z_{H_z}(a)(\mathbb{A})))\cdot\mathrm{tr}\,\xi(\iota_z(a))\cdot O^{G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p)}_{\iota_z(a)}(\varphi).$$

In view of Lemma 9.6, the summand in the above sum depends only on $H_z(\mathbb{A})$-conjugacy class of a and near equivalence class of (A,λ,i). Thus we merge terms to rewrite the sum over the set FP^{AV}_b. Proceeding exactly as in the proof of [HT01, Lem V.3.3] (but keeping the expression $|A(Z_{H_z}(a))|$ and not changing it into κ_B in their notation), we arrive at the expression (39). In the equality $Z_{H_z}(a)(\mathbb{R})^1$ denotes the kernel of the map $Z_{H_z}(a)(\mathbb{R})\to\mathbb{R}^\times_{>0}$ given by $x\mapsto|x^{\ddagger_z}x|_\mathbb{R}$. For the choice of appropriate Haar measures on $Z_{H_z}(a)$, one may read Theorem 13.1 below, replacing I_0 with $Z_{H_z}(a)$.

$$\mathrm{tr}\,(\varphi|H_c(\mathrm{Ig}_b,\mathscr{L}_\xi))=\sum_{(z,[a])\in FP^{AV}_b}\mathrm{vol}(Z_{H_z}(a)(\mathbb{R})^1)^{-1}|A(Z_{H_z}(a))|\cdot\mathrm{tr}\,\xi(\iota_z(a))\cdot O^{G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p)}_{\iota_z(a)}(\varphi)\quad(39)$$

Recall that $I_0=Z_G(\gamma_0)$ as usual. The $\mathbb{R}$-group I_∞ denotes the inner form of I_0 over $\mathbb{R}$ which is compact modulo center. (In fact, $I_\infty(\mathbb{R})\simeq Z_{H_z}(a)(\mathbb{R})$ since both are compact modulo center inner forms of I_0 over $\mathbb{R}$.) We define $I_0(\mathbb{A})^1$ to be the kernel of $I_0(\mathbb{A})\to\mathbb{R}^\times_{>0}$ given by $x\mapsto|x^\# x|_{\mathbb{A}^\times}$. Applying Lemma 12.3, we rewrite (39) in terms of Kottwitz triples to obtain the final result.

Theorem 13.1. *If $\varphi\in C^\infty_c(G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p))$ is acceptable, then*

$$\mathrm{tr}\,(\varphi|H_c(\mathrm{Ig}_b,\mathscr{L}_\xi))=\sum_{(\gamma_0;\gamma,\delta)\in KT^{\mathrm{eff}}_b}\mathrm{vol}(I_\infty(\mathbb{R})^1)^{-1}|A(I_0)|\cdot\mathrm{tr}\,\xi(\gamma_0)\cdot O^{G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p)}_{(\gamma,\delta)}(\varphi)$$

with the following choice of Haar measures. Choose the Tamagawa measure on $I_0(\mathbb{A})^1$. Choose Haar measures on $I_0(\mathbb{A}^\infty)$ and $I_0(\mathbb{R})^1$ compatibly with the measure on $I_0(\mathbb{A})^1$ via the exact sequence

$$1\to I_0(\mathbb{R})^1\to I_0(\mathbb{A})^1\to I_0(\mathbb{A}^\infty)\to 1.$$

We define Haar measures on $Z_G(\gamma)(\mathbb{Q}_v)$ ($v\neq p,\infty$), $I_\delta(\mathbb{Q}_p)$ and $I_\infty(\mathbb{R})^1$ compatibly with those on $I_0(\mathbb{Q}_v)$, $I_0(\mathbb{Q}_p)$ and $I_0(\mathbb{R})^1$, respectively (i.e. compatible choice of measures on inner forms in the sense of [Kot88, p.631]).

Remark 13.2. The only implicit assumption for the above theorem is that the Igusa variety Ig_b should arise from an unramified integral PEL datum (Definition 5.2). In particular G should be unramified over $\mathbb{Q}_p$.

Remark 13.3. It is worth noting that our formula is very similar to Kottwitz's formula [Kot92, (19.5)] and indeed inspired by it. However there is no naive explicit relation between the two formulas. Observe that our orbital integrals at p are quite different from those of Kottwitz and that the triples $(\gamma_0;\gamma,\delta)$ have a somewhat different meaning.

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