HONDA-TATE THEORY FOR SHIMURA VARIETIES

MARK KISIN, KEERTHI MADAPUSI PERA, AND SUG WOO SHIN

Abstract. A Shimura variety of Hodge type is a moduli space for abelian varieties equipped with a certain collection of Hodge cycles. We show that the Newton strata on such varieties are non-empty provided the corresponding group G is quasi-split at p, confirming a conjecture of Rapoport in this case. Under the same condition, we conjecture that every mod p isogeny class on such a variety contains the reduction of a special point. This is a refinement of Honda-Tate theory. We prove a large part of this conjecture for Shimura varieties of PEL type. Our results make no assumption on the availability of a good integral model for the Shimura variety. In particular, the group G may be ramified at p.

Contents

Introduction

A Shimura variety, $\mathrm{Sh}(G, X)$, of Hodge type may be thought of as a moduli space for abelian varieties equipped with a particular family of Hodge cycles. This interpretation gives rise to a natural integral model $\mathscr{S} = \mathscr{S}(G, X)$. For a mod p point, $x \in \mathscr{S}(\bar{\mathbb{F}}_p)$, one has the attached abelian variety $\mathcal{A}_x$ and its p-divisible group $\mathcal{G}_x = \mathcal{A}_x[p^\infty]$. In this paper, we study the two related questions of classifying the isogeny classes of $\mathcal{G}_x$ and $\mathcal{A}_x$. We are able to do this for quite general groups G, as our methods do not require any particular information about $\mathscr{S}$; for example we do not assume that $\mathscr{S}$ has good reduction.

The isogeny class of $\mathcal{G}_x$ is determined by its rational Dieudonné module $\mathbb{D}_x$, which is an $L = W(\bar{\mathbb{F}}_p)[1/p]$-vector space equipped with a Frobenius semi-linear operator $b_x\sigma$, where $b_x \in G(L)$ is an element which is well defined up to σ-conjugacy, $b_x \mapsto g^{-1}b_x\sigma(g)$, and σ denotes the Frobenius automorphism of L. The element b_x is subject to a group theoretic analogue of Mazur's inequality [RR96, Thm. 4.2], and the set consisting of σ-conjugacy classes which satisfy this condition is denoted $B(G, \mu)$, where $\mu : \mathbb{G}_m \to G$ is the inverse of the cocharacter μ_X (up to conjugacy) attached to X. (See 1.1.5 and 1.2.3 below for precise definitions.) Let $\mathbb{D}$ denote the pro-torus whose character group is $\mathbb{Q}$. Each $b \in G(L)$ gives rise to the so-called

Newton cocharacter $\nu_b : \mathbb{D} \to G$, defined over L, whose conjugacy class is defined over $\mathbb{Q}_p$ and depends only on the σ-conjugacy class $[b]$. The slope decomposition of $\mathbb{D}_x$ is given by ν_{b_x}. For $[b] \in B(G,\mu)$, the corresponding subset $S_{[b]} \subset \mathscr{S}(\bar{\mathbb{F}}_p)$ is called the *Newton stratum* corresponding to $[b]$ so that a point $x \in \mathscr{S}(\bar{\mathbb{F}}_p)$ belongs to $S_{[b]}$ if and only if $[b_x] = [b]$. Our first result is on the non-emptiness of Newton strata. (The converse is known, i.e. if $S_{[b]}$ is non-empty then $b \in B(G,\mu)$. See Lemma 1.3.8.)

Theorem 1. *Suppose that $b \in B(G,\mu)$ and that the $G(L)$-conjugacy class of ν_b has a representative which is defined over $\mathbb{Q}_p$. Then $S_{[b]}$ is non-empty. In particular, $S_{[b]}$ is always non-empty when $G_{\mathbb{Q}_p}$ is quasi-split.*

Rapoport has conjectured [Rap05, Conj. 7.1] that $S_{[b]}$ is non-empty for every $b \in B(G,\mu)$; see also the paper of He-Rapoport [HR17]. Previous results on the non-emptiness of $S_{[b]}$ have been obtained by a number of authors - see the papers of Wedhorn [Wed99] and Wortmann [Wor13] for the μ-ordinary case (of hyperspecial level), that of Viehmann-Wedhorn [VW13] for the PEL case of type A and C (of hyperspecial level), and the recent work of Zhou [Zho17] for many cases of parahoric level. These all rely on an understanding of the fine structure of a suitable integral model of $\mathrm{Sh}(G,X)$.

Our method involves constructing a special point whose reduction lies in $S_{[b]}$. This is essentially a group theoretic problem, as the Newton stratum of a special point can be computed in terms of the torus and cocharacter attached to a special point. When $G_{\mathbb{Q}_p}$ is unramified, this problem was already solved by Langlands-Rapoport [LR87, Lem. 5.2]. This was independently observed by Lee [Lee18], who also used it to show non-emptiness of Newton strata in this case. If $S_{[b]}$ contains the reduction of a special point, then it is easy to see that the $G(L)$-conjugacy class of ν_b has a representative which is defined over $\mathbb{Q}_p$. Thus the result of Theorem 1 is the best possible using this method.

Along the way we confirm an expectation of Rapoport–Viehmann [RV14, Rem. 8.3] on cocharacters and isocrystals. (See Remark 1.1.14 below.) We also show the Newton stratification has some of the expected properties:

Theorem 2. *For every $b \in B(G,\mu)$, $S_{[b]} \subset \mathscr{S}(\bar{\mathbb{F}}_p)$ is locally closed for the Zariski topology. One has the following closure relations, where $\preceq$ is the partial order on the set of conjugacy classes of Newton cocharacters (see 1.1.1):*

$$\overline{S}_{[b]} \subset \bigcup_{\nu_{b'} \preceq \nu_b} S_{[b']}.$$

This theorem is proved by showing the existence of isocrystals with G-structure on $\mathscr{S}$. This may be of independent interest, but is rather technical so is left to the appendix. (Recently Hamacher and Kim [HK17] proved similar results for the case of Kisin-Pappas models by a different argument.) As a corollary, we obtain generalizations of the theorems of Wedhorn and Wortmann on the density of the μ-ordinary locus.

Theorem 3. *If the special fibre of $\mathscr{S}$ is locally integral then the μ-ordinary locus is dense in the special fibre.*

We now discuss the problem of classifying $\mathcal{A}_x$ up to isogeny. For the moduli space of polarized abelian varieties, this is closely related to Honda-Tate theory, which asserts that the isogeny class of an abelian variety A over $\mathbb{F}_q$ is determined by the characteristic polynomial of the q-Frobenius on the ℓ-adic cohomology $H^1(A,\mathbb{Q}_\ell)$, with $\ell \nmid q$, and that the isogeny class of A contains the reduction of a special point. Using this fact one can describe precisely which characteristic polynomials can occur. For $x \in \mathscr{S}(G,X)(\mathbb{F}_q)$ one expects that the q-Frobenius arises from a $\gamma \in G(\mathbb{Q})$ whose $G(\bar{\mathbb{Q}})$-conjugacy class is independent of ℓ, although it is in general not a complete invariant for the isogeny class of A. We make the following conjecture:

Conjecture 1. *If $G_{\mathbb{Q}_p}$ is quasi-split then the isogeny class of any $x \in \mathscr{S}(\bar{\mathbb{F}}_p)$ contains the reduction of a special point.*

Here if $x, x' \in \mathscr{S}(\bar{\mathbb{F}}_p)$, then $\mathcal{A}_x, \mathcal{A}_{x'}$ are defined to be in the same isogeny class if there is an isogeny $i : \mathcal{A}_x \to \mathcal{A}_{x'}$ such that for each of the Hodge cycles $s_{\alpha,x}$ carried by $\mathcal{A}_x$, i takes $s_{\alpha,x}$ to $s_{\alpha,x'}$. More precisely, the Hodge cycles $s_{\alpha,x}$ can be viewed via either ℓ-adic cohomology for $\ell \neq p$, or crystalline cohomology. We require that i takes $s_{\alpha,x}$ to $s_{\alpha,x'}$ in each of these cohomology theories.

When G is unramified this conjecture was proved by one of us [Kis17]; see also [Zho17] for some cases of parahoric Shimura varieties. The methods of *loc. cit* require rather fine information about the special fibre of $\mathscr{S}$, and are rather different from the ones employed in this paper which require almost no information about integral models.

Even for the moduli space of polarized abelian varieties the conjecture is a more refined statement than Honda-Tate theory, since the definition of isogeny class involves isogenies which respect polarizations. As we shall explain, it can nevertheless be deduced from Honda-Tate theory with some extra arguments, but remarkably these do not seem to be in the literature; the closest is perhaps [Kot92, §17].

To state our main result in the direction of the conjecture, we recall that the group of automorphisms of $\mathcal{A}_x$ in the isogeny category is naturally the $\mathbb{Q}$-points of an algebraic group $I'_x = \underline{\mathrm{Aut}}_{\mathbb{Q}}\mathcal{A}_x$ over $\mathbb{Q}$. Similarly one can define the subgroup $I = I_x \subset I'_x$ consisting of isogenies which respect Hodge cycles in ℓ-adic and crystalline cohomology. The set of isogenies (respecting Hodge cycles) between $\mathcal{A}_x$ and $\mathcal{A}_{x'}$ is likewise the $\mathbb{Q}$-points of a scheme $\mathcal{P}(x, x')$ which is either empty or a torsor under I_x. We say that $\mathcal{A}_x$ and $\mathcal{A}_{x'}$ are $\bar{\mathbb{Q}}$-isogenous if $\mathcal{P}(x, x')$ is nonempty. This is equivalent to asking that there is a finite extension $F/\mathbb{Q}$ and an isomorphism $\mathcal{A}_x \otimes F \to \mathcal{A}_{x'} \otimes F$ (for example as fppf sheaves) respecting Hodge cycles. We say that $\mathcal{A}_x$ and $\mathcal{A}_{x'}$ are $\mathbb{Q}$-isogenous if $\mathcal{P}(x, x')$ is a trivial torsor.

Theorem 4. *Suppose that G is quasi-split at p, and that (G, X) is a PEL Shimura datum of type A or C, then for any $x \in \mathscr{S}(\bar{\mathbb{F}}_p)$ the abelian variety $\mathcal{A}_x$ is $\bar{\mathbb{Q}}$-isogenous to $\mathcal{A}_{x'}$, with x' the reduction of a special point.*

Our main result is actually more precise, as we show that one can construct special points associated to any maximal torus $T \subset I$. There is also a slightly weaker version of the first part of the theorem in the case of PEL type D; see 2.3.16. In fact we prove an analogous theorem for (G, X) of Hodge type, conditional on a version of Tate's theorem for abelian varieties equipped with Hodge cycles - see below.

When G is unramified the first part of the above theorem was proved by Zink [Zin83]. Note that in *loc. cit.* Zink's theorem says that $\mathcal{A}_x$ is isogenous (not just $\bar{\mathbb{Q}}$-isogenous) to the reduction of a special point, however his definition does not require that isogenies respect polarizations, and it is not hard to see that one can then produce a $\mathbb{Q}$-isogeny from a $\bar{\mathbb{Q}}$-isogeny (the corresponding torsor turns out to be trivial).

When G^{ab} satisfies the Hasse principle one can replace $\bar{\mathbb{Q}}$-isogenies by $\mathbb{Q}$-isogenies in Theorem 4. For example one has

Theorem 5. *Suppose that G is quasi-split at p, and that (G, X) is a PEL Shimura datum of type C or of type A_n with n odd. Then for any $x \in \mathscr{S}(\bar{\mathbb{F}}_p)$, $\mathcal{A}_x$ is $\mathbb{Q}$-isogenous to $\mathcal{A}_{x'}$, with x' the reduction of a special point, so that Conjecture 1 holds in this case.*

One of the key ingredients in Honda-Tate theory is Tate's theorem on the Tate conjecture for morphisms between abelian varieties over finite fields [Tat66]. We prove an analogue of this result for (G, X) of Hodge type, and for *automorphisms* of abelian varieties equipped with the corresponding collection of Hodge cycles. To explain this, for each $\ell \neq p$, let $I_\ell \subset$

$\mathrm{Aut}(H^1(\mathcal{A}_x, \mathbb{Q}_\ell))$ be the subgroup which fixes the Hodge cycles $s_{\alpha,x}$ and commutes with the q-Frobenius for $q = p^r$ and r sufficiently large (ordered multiplicatively). We define a similar group I_p using crystalline cohomology.

Theorem 6. *For every ℓ (including $\ell = p$) the natural map*

$$I \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to I_\ell$$

is an isomorphism. In particular the (absolute) rank of I is equal to the rank of G.

The proof uses the finiteness of $\mathscr{S}(\mathbb{F}_q)$ (when level is fixed) as in [Kis17], as well as a result of Noot on the independence of ℓ of the conjugacy class of Frobenius as an element of $G(\mathbb{Q}_\ell)$. Note that a similar finiteness condition plays a crucial role in [Tat66].

Using this result, one knows that any maximal torus $T \subset I$ has the same rank as G. We show that, when $G_{\mathbb{Q}_p}$ is quasi-split, any such T can be viewed as (transferred to) a subgroup of G. Our results on non-emptiness of Newton strata then imply that there is a special point $\tilde{x}' \in \mathrm{Sh}(G, X)$ with associated torus T. If x' is the reduction of $\tilde{x}'$, then $\mathcal{A}_x$ and $\mathcal{A}_{x'}$ should be $\bar{\mathbb{Q}}$-isogenous. Indeed this follows from a version of Tate's theorem with Hodge cycles. When $x = x'$ this is Theorem 6 above, but we do not know how to prove such a theorem when $x \neq x'$, except in the PEL case, when one can use Tate's original result to deduce the first part of Theorem 4. Finally the second part is proved via an analysis of the local behavior of the torsor $\mathcal{P}(x, x')$.

Acknowledgements

M.K was partially supported by NSF grant DMS-1601054. K.M.P. was partially supported by NSF grant DMS-1502142/1802169. S.W.S. was partially supported by NSF grant DMS-1501882/1802039.

Notational conventions

Given a connected reductive group G over a field F, we write $G^{\mathrm{der}} \subset G$ for its derived subgroup and $G^{\mathrm{sc}} \to G^{\mathrm{der}}$ for the simply connected cover of its derived group.

Fix an algebraic closure $\bar{F}$ for F. For any torus T over F, we set

$$X_*(T) = \mathrm{Hom}(\mathbb{G}_{m,\bar{F}}, T_{\bar{F}})\ ;\ X^*(T) = \mathrm{Hom}(T_{\bar{F}}, \mathbb{G}_{m,\bar{F}})$$

for the cocharacter and character groups of T, respectively. Write $\mathbb{D}$ for the multiplicative pro-group scheme over $\mathbb{Q}_p$ with character group $\mathbb{Q}$. A homomorphism $\mathbb{D}_{\bar{F}} \to T_{\bar{F}}$ gives an element of $X_*(T)_{\mathbb{Q}} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, and vice versa. We often refer to a homomorphism $\mathbb{D} \to G$ (defined over an extension of F) as a cocharacter of G by standard abuse of terminology.

For a maximal torus T in the reductive group G, we write $W(G, T)$ for the absolute Weyl group of G relative to T, and we denote by $\pi_1(G)$ the algebraic fundamental group of G [Bor98]: It is a $\mathrm{Gal}(\bar{F}/F)$-module, functorial in G, and canonically isomorphic to $X_*(T)/X_*(T^{\mathrm{sc}})$, where T^{sc} is the preimage of T in G^{sc}.

For v a place of $\mathbb{Q}$, we fix an algebraic closure $\bar{\mathbb{Q}}_v$ for $\mathbb{Q}_v$ (here, $\mathbb{Q}_\infty = \mathbb{R}$ and $\bar{\mathbb{Q}}_\infty = \mathbb{C}$). We also fix an algebraic closure $\bar{\mathbb{Q}}$, along with embeddings $\iota_v : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_v$, for every place v. Set $\Gamma_v = \mathrm{Gal}(\bar{\mathbb{Q}}_v/\mathbb{Q}_v)$ and $\Gamma = \Gamma_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. We will use our chosen embeddings to view Γ_v as a subgroup of Γ.

When E is a number field, the ring of integers of E is denoted by $\mathscr{O}_E$.

1. Non-emptiness of Newton strata

1.1. **Local results.** Fix a rational prime p. Let G be a connected reductive group over $\mathbb{Q}_p$. Fix a maximal torus $T \subset G$ defined over $\mathbb{Q}_p$ and a Borel subgroup $B \subset G_{\bar{\mathbb{Q}}_p}$ containing $T_{\bar{\mathbb{Q}}_p}$. Positive roots and coroots of T in G will be determined by B.

1.1.1. Set

$$\mathcal{N}(G) = (X_*(T)_{\mathbb{Q}}/W(G,T))^{\Gamma_p}.$$

This space has a more canonical description that $\mathcal{N}(G)$ is the space of $G(\bar{\mathbb{Q}}_p)$-conjugacy classes of homomorphisms $\mathbb{D}_{\bar{\mathbb{Q}}_p} \to G_{\bar{\mathbb{Q}}_p}$ that are defined over $\mathbb{Q}_p$.

Let $\overline{\mathcal{C}} \subset X_*(T)_{\mathbb{R}}$ be the closed dominant Weyl chamber determined by B. Each class $\bar{\nu} \in \mathcal{N}(G)$ has a unique representative $\nu \in X_*(T)_{\mathbb{Q}} \cap \overline{\mathcal{C}}$. There is a natural partial order $\preceq_G$ on $X_*(T)_{\mathbb{R}}$ and $\mathcal{N}(G)$, also denoted by $\preceq$ if there is no danger of confusion, determined as follows; cf. [RR96, 2.2, 2.3]: Given $\bar{\nu}_1, \bar{\nu}_2 \in \mathcal{N}(G)$ with representatives $\nu_1, \nu_2 \in X_*(T)_{\mathbb{Q}} \cap \overline{\mathcal{C}}$, we have $\bar{\nu}_1 \preceq \bar{\nu}_2$ if and only if $\nu_2 - \nu_1$ is a nonnegative linear combination of positive coroots.

There is a unique map $\mathcal{N}(G) \to \pi_1(G)^{\Gamma_p} \otimes \mathbb{Q}$ which is functorial in G and induces the identity map when G is a torus [RR96, Thm. 1.15].

1.1.2. Let $W = W(\bar{\mathbb{F}}_p)$ be the ring of Witt vectors for an algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$, and write L for its fraction field. We fix an algebraic closure $\bar{L}$ for L along with an embedding $\bar{\mathbb{Q}}_p \hookrightarrow \bar{L}$. Let $\sigma : W \to W$ be the unique automorphism lifting the p-power Frobenius on $\bar{\mathbb{F}}_p$. As in [Kot85], we will denote by $B(G)$ the set of σ-conjugacy classes in $G(L)$, so that two elements $b_1, b_2 \in G(L)$ are in the same class in $B(G)$ if and only if there exists $c \in G(L)$ with $b_1 = cb_2\sigma(c)^{-1}$.

Recall the following maps from [RR96, Thm. 1.15], which are functorial in G:

$$\kappa_G : B(G) \to \pi_1(G)_{\Gamma_p} \; ; \; \bar{\nu}_G : B(G) \to \mathcal{N}(G).$$

A class $[b] \in B(G)$ is *basic* if $\bar{\nu}_G([b])$ is the class of a central cocharacter of G. We write $B(G)_b \subset B(G)$ for the subset of basic classes.

The maps $\kappa_G, \bar{\nu}_G$ have the following properties:

(1.1.2.1) The diagram

$$\begin{tikzcd} B(G) \arrow[r, "\kappa_G"] \arrow[d, "\bar{\nu}_G"'] & \pi_1(G)_{\Gamma_p} \arrow[d] \\ \mathcal{N}(G) \arrow[r] & (\pi_1(G) \otimes \mathbb{Q})^{\Gamma_p} \end{tikzcd}$$

commutes. Here, the vertical map on the right-hand side is induced by the usual isomorphism averaging over each Γ_p-orbit, cf. [RR96, p.162]:

$$(\pi_1(G) \otimes \mathbb{Q})_{\Gamma_p} \xrightarrow{\simeq} (\pi_1(G) \otimes \mathbb{Q})^{\Gamma_p}.$$

The bottom horizontal map is uniquely characterized as a functorial map in G that is the natural identification when G is a torus. See [RR96, Thm. 1.15] for details.

(1.1.2.2) [Kot85, 4.3, 4.4]: Given $b \in G(L)$ representing a class $[b] \in B(G)$, the conjugacy class $\bar{\nu}_G([b])$ is represented by a cocharacter $\nu_b : \mathbb{D}_L \to G_L$ that is characterized uniquely by the following property: There exists $c \in G(L)$ and an integer $r \in \mathbb{Z}_{>0}$ such that $r\nu_b$ factors through a cocharacter $\mathcal{G}_{m,L} \to G_L$, that $c(r\nu_b)c^{-1}$ is defined over the fixed field of σ^n on L, and that

$$cb\sigma(b)\sigma^2(b)\cdots\sigma^r(b)\sigma^r(c)^{-1} = c(r\nu_b)(p)c^{-1}.$$

This implies that $\nu_{\sigma(b)} = \sigma(\nu_b)$ and that, for every $g \in G(L)$,

$$\nu_{gb\sigma(g)^{-1}} = g\nu_b g^{-1}.$$

(1.1.2.3) [Kot97, 4.13]: The map

$$(\kappa_G, \bar{\nu}_G) : B(G) \to \pi_1(G)_{\Gamma_p} \times \mathcal{N}(G)$$

is injective. Furthermore, the restriction of κ_G to $B(G)_b$ induces a bijection:

$$B(G)_b \xrightarrow{\simeq} \pi_1(G)_{\Gamma_p}.$$

(1.1.2.4) [Kot85, 2.5]: When $G = T$ is a torus, κ_T is an isomorphism, and can be described explicitly: Let E/L be a finite extension over which T is split, and let $N_{E/L} : T(E) \to T(L)$ be the associated norm map. Fix a uniformizer $\pi \in E$. Then we have a commutative diagram:

$$\begin{tikzcd} X_*(T) \arrow[rr, "\nu \mapsto {[N_{E/L}(\nu(\pi))]}"] \arrow[dr] & & B(T) \arrow[dl, "\kappa_T"', "\simeq"] \\ & X_*(T)_{\Gamma_p}. & \end{tikzcd}$$

1.1.3. Later we will often make the following hypothesis on G and $[b]$:

(1.1.3.1) The class $[b]$ contains a representative $b \in G(L)$ such that the cocharacter ν_b is defined over $\mathbb{Q}_p$.

Given $[b]$ satisfying the above condition, we fix such a representative and denote the corresponding cocharacter by $\nu_G([b])$. Let $M_{[b]} \subset G$ be the centralizer of $\nu_G([b])$: This is a $\mathbb{Q}_p$-rational Levi subgroup of G.

Note that (1.1.3.1) is always satisfied if G is quasi-split over $\mathbb{Q}_p$ as one can see from (1.1.2.2); cf. [Kot85, p.219]. If $[b]$ is basic (but G is possibly not quasi-split), (1.1.3.1) is still satisfied as (1.1.2.2) shows that ν_b is σ-invariant central cocharacter of G for any representative b.

1.1.4. Suppose that $b \in G(L)$. Consider the group scheme J_b over $\mathbb{Q}_p$ that attaches to every $\mathbb{Q}_p$-algebra R the group:

$$J_b(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} L) : \; gb = b\sigma(g)\}.$$

By construction, there is a natural map of group schemes over L: $J_{b,L} \to G_L$.

If $b' = gb\sigma(g)^{-1}$ is another representative of $[b] \in B(G)$, then conjugation by g induces an isomorphism of $\mathbb{Q}_p$-groups:

$$\mathrm{int}(g) : J_b \xrightarrow{\simeq} J_{b'}.$$

As shown in [RR96, 1.11], J_b is a reductive group over $\mathbb{Q}_p$. A more precise statement holds: Let $M_{\nu_b} \subset G_L$ be the centralizer of ν_b. By replacing b by a σ-conjugate if necessary, we can arrange to have (1.1.2.2):

$$b\sigma(b)\sigma^2(b)\cdots\sigma^{r-1}(b) = (r\nu_b)(p), \tag{1.1.4.1}$$

with ν_b defined over $\mathbb{Q}_{p^r}$ and $r \in \mathbb{Z}_{\geq 1}$. Then M_{ν_b} is also defined over $\mathbb{Q}_{p^r}$, and b belongs to $G(\mathbb{Q}_{p^r})$. Moreover, the natural map $J_{b,L} \to G_L$ is defined over $\mathbb{Q}_{p^r}$ and identifies $J_{b,\mathbb{Q}_{p^r}}$ with M_{ν_b}.

Under hypothesis (1.1.3.1), the discussion in (1.1.2.2) and (1.1.3) tells us that M_{ν_b} is a pure inner twist of $M_{[b]}$ by the $M_{[b]}$-torsor (which is trivial by Steinberg's theorem) of elements of $G_{\mathbb{Q}_{p^r}}$ conjugating ν_b to $\nu_G([b])$.

Combining the previous two paragraphs, we find that J_b is equipped with an inner twisting $J_b \xrightarrow{\simeq} M_{[b]}$ over $\mathbb{Q}_p$ (cf. also [Kot85, 5.2]).

1.1.5. We return to the general setup, disregarding (1.1.3.1) up to (1.1.13) below. Let $\xi : G \xrightarrow{\simeq} G^*$ be an inner twisting for a quasi-split group G^* over $\mathbb{Q}_p$. Let $B^* \subset G^*$ be a Borel subgroup over $\mathbb{Q}_p$ and $T^* \subset B^*$ a maximal torus over $\mathbb{Q}_p$. Write $\overline{\mathcal{C}^*} \subset X_*(T^*)_{\mathbb{R}}$ for the B^*-dominant chamber.

If the $G(\bar{\mathbb{Q}}_p)$-conjugacy class of a cocharacter $\nu : \mathbb{D}_{\bar{\mathbb{Q}}_p} \to G_{\bar{\mathbb{Q}}_p}$ is defined over $\mathbb{Q}_p$ then so is the $G^*(\bar{\mathbb{Q}}_p)$-conjugacy class of $\xi \circ \nu$. Thus ξ induces a map $\mathcal{N}_\xi : \mathcal{N}(G) \to \mathcal{N}(G^*)$, depending only on the $G^*(\bar{\mathbb{Q}}_p)$-conjugacy class of ξ.

Let $\{\mu\}$ be a conjugacy class of cocharacters $\mathbb{G}_{m,\bar{\mathbb{Q}}_p} \to G_{\bar{\mathbb{Q}}_p}$, and let $\mu^* \in X_*(T^*) \cap \overline{\mathcal{C}^*}$ be the dominant representative for $\xi \circ \{\mu\}$. Let $\Gamma_{\mu^*} \subset \Gamma_p$ be the stabilizer of μ^*, and set

$$N\mu^* = \frac{1}{[\Gamma_p : \Gamma_{\mu^*}]} \sum_{\sigma \in \Gamma_p/\Gamma_{\mu^*}} \sigma\mu^* \in X_*(T^*)_{\mathbb{Q}}^{\Gamma_p}.$$

We will write $\bar{\mu}^*$ for the image of $N\mu^*$ in $\mathcal{N}(G^*)$.

Let $\mu^\sharp$ be the image of $\{\mu\}$ in $\pi_1(G)_{\Gamma_p}$. (The image of μ^* in $\pi_1(G^*)_{\Gamma_p}$ is equal to $\mu^\sharp$ via the canonical isomorphism $\pi_1(G)_{\Gamma_p} = \pi_1(G^*)_{\Gamma_p}$.) Given $[b] \in B(G)$, we will say that the pair $([b], \{\mu\})$ is *G-admissible* or simply *admissible*, if two conditions hold:

(1.1.5.1) $\kappa_G([b]) = \mu^\sharp$.

(1.1.5.2) $\mathcal{N}_\xi(\bar{\nu}_G([b])) \preceq \bar{\mu}^*$.

If G is quasi-split then we may and will take $G = G^*$ and ξ to be the identity map so that $\mathcal{N}_\xi$ is also the identity map.

Lemma 1.1.6. *Given a conjugacy class $\{\mu\}$ as above, let $[b_{\mathrm{bas}}(\mu)] \in B(G)_b$ denote the unique basic class such that $\kappa_G([b_{\mathrm{bas}}(\mu)]) = \mu^\sharp$. Then $([b_{\mathrm{bas}}(\mu)], \{\mu\})$ is admissible.*

Proof. The condition (1.1.5.1) is tautological, and (1.1.5.2) follows from [RR96] Prop. 2.4(ii) and the commutativity of (1.1.2.1). □

Definition 1.1.7. Let $T' \subset G$ be a maximal torus over $\mathbb{Q}_p$. We will call an admissible pair $([b], \{\mu\})$ *T'-special* if there exists a representatives $b' \in T'(L)$ (resp. $\mu' \in X_*(T')$) of $[b]$ (resp. $[\mu']$) such that the pair $([b']_{T'}, \mu')$ is an admissible pair for T'. Here, we write $[b']_{T'}$ for the σ-conjugacy class of b' in $T'(L)$. We say that $([b], \{\mu\})$ is *special* if it is *T'-special* for some maximal torus $T' \subset G$.

Lemma 1.1.8. *Suppose that $([b], \{\mu\})$ is an admissible pair for G with $[b]$ basic. Then $([b], \{\mu\})$ is T'-special for any elliptic maximal torus $T' \subset G$. More precisely, for any $\mu' \in X_*(T')$ in $\{\mu\}$, $[b_{\mathrm{bas}}(\mu')] \in B(T')$ maps to $[b] \in B(G)$.*

Proof. Let $T' \subset G$ be an elliptic maximal torus, and let $\mu' \in X_*(T')$ be a representative for $\{\mu\}$. As T' is elliptic, $[b_{\mathrm{bas}}(\mu')] \in B(T')$ maps to a basic class $[b'] \in B(G)$ [Kot85, 5.3]. Moreover, $\kappa_G([b'])$ is the image in $\pi_1(G)_{\Gamma_p}$ of $\mu'^{,\sharp} = \kappa_{T'}([b_{\mathrm{bas}}(\mu')])$, and so must be equal to $\mu^\sharp$. Hence, $[b'] = [b_{\mathrm{bas}}(\mu)] = [b]$. □

1.1.9. From here until (1.1.13) we are concerned with quasi-split groups. Let H_0 be an absolutely simple quasi-split adjoint group over a finite extension $F/\mathbb{Q}_p$. Fix a Borel subgroup $B_0 \subset H_0$ and a maximal torus $T_0 \subset B_0$ over F.

Set $H = \mathrm{Res}_{F/\mathbb{Q}_p} H_0$, $B = \mathrm{Res}_{F/\mathbb{Q}_p} B_0$, $T = \mathrm{Res}_{F/\mathbb{Q}_p} T_0$ and $X = X_*(T)$. The last is a free $\mathbb{Z}$-module with an action of Γ_p, and the choice of B_0 equips it with a Γ_p-invariant positive chamber $\mathcal{C} \subset X_{\mathbb{Q}}$. As above, we have a Galois averaging map $N : \mathcal{C} \to \mathcal{C}$ with image in $\mathcal{C}^{\Gamma_p}$.

Lemma 1.1.10. *Let $F'/\mathbb{Q}_p$ be the unramified extension with $[F' : \mathbb{Q}_p] = [F : \mathbb{Q}_p]$. Then there is a quasi-split absolutely simple adjoint group H_0' over F' equipped with a Borel subgroup B_0' and a maximal torus $T_0' \subset B_0'$ with the following properties:*

(1.1.10.1) Let $(H', B', T') = \mathrm{Res}_{F'/\mathbb{Q}_p}(H'_0, B'_0, T'_0)$*. Then there is an isomorphism of triples:*

$$(H, B, T) \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p \xrightarrow{\simeq} (H', B', T') \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p.$$

(1.1.10.2) Let $\mathcal{C}' \subset X'_{\mathbb{Q}}$ *be the positive chamber of* $X' = X_*(T')$ *determined by* B'*, and let* $N' : \mathcal{C}' \to \mathcal{C}'$ *be the Galois averaging map. Then the isomorphism in (1.1.10.1) can be chosen such that the induced isomorphism* $\mathcal{C}' \xrightarrow{\simeq} \mathcal{C}$ *carries the endomorphism* N' *to* N*.*

Proof. We begin by explicating the averaging map N. Let D be the Dynkin diagram of H: It is a disjoint union

$$\bigsqcup_{\sigma : F \to \bar{\mathbb{Q}}_p} D_0,$$

where D_0 is the Dynkin diagram for H_0. The action of Γ_p permutes the connected components of this diagram in the usual way, and for each $\sigma : F \to \bar{\mathbb{Q}}_p$, the stabilizer $\Gamma_\sigma \subset \Gamma_p$ of σ (that is, the pointwise stabilizer of $\sigma(F)$) acts on D_0 via a homomorphism

$$\rho_\sigma : \Gamma_\sigma \to \mathrm{Aut}(D_0).$$

Fix an embedding $\sigma_0 : F \to \bar{\mathbb{Q}}_p$, and let $\tau \in \Gamma_p$ be such that $\tau \circ \sigma_0 = \sigma$. Then ρ_σ is equal to the composition

$$\Gamma_\sigma \xrightarrow{\gamma \mapsto \tau^{-1}\gamma\tau} \Gamma_{\sigma_0} \xrightarrow{\rho_{\sigma_0}} \mathrm{Aut}(D_0).$$

The simple coroots in X are in canonical bijection with pairs (σ, d_0), where $\sigma : F \to \bar{\mathbb{Q}}_p$ and $d_0 \in D_0$ is a vertex. Write $\alpha^\vee(\sigma, d_0)$ for the simple coroot associated with such a pair.

The Γ_p-orbit of $\alpha^\vee(\sigma, d_0)$ consists of simple coroots $\alpha^\vee(\sigma', d'_0)$ where $d'_0 \in D_0$ is in the Γ_σ-orbit of $\alpha^\vee$, and $\sigma' : F \to \bar{\mathbb{Q}}_p$ is arbitrary. Therefore, if $d_{0,1}, \ldots, d_{0,r} \in D_0$ comprise the Γ_σ-orbit of d_0, we have

$$N\alpha^\vee(\sigma, d_0) = \frac{1}{r[F : \mathbb{Q}_p]} \sum_{\substack{\sigma' : F \to \bar{\mathbb{Q}}_p \\ 1 \le i \le r}} \alpha^\vee(\sigma', d_{0,i}).$$

Fix an embedding $\sigma'_0 : F' \to \bar{\mathbb{Q}}_p$. We now claim that we can find a quasi-split group H'_0 over F' with a Borel subgroup $B'_0 \subset H'_0$ and a maximal torus $T'_0 \subset B'_0$ with the following properties:

- There is an isomorphism
$$(H'_0, B'_0, T'_0) \otimes_{F', \sigma'_0} \bar{\mathbb{Q}}_p \xrightarrow{\simeq} (H_0, B_0, T_0) \otimes_{F, \sigma_0} \bar{\mathbb{Q}}_p.$$
- If D'_0 is the Dynkin diagram of H'_0, identified with D_0 via the above isomorphism, then the induced action of $\Gamma_{\sigma'_0}$ on D_0 has the same orbits as those of the action of Γ_{σ_0}.

Observe that the claim implies the lemma by choosing a bijection between $\mathrm{Hom}(F, \bar{\mathbb{Q}}_p)$ and $\mathrm{Hom}(F', \bar{\mathbb{Q}}_p)$ carrying σ_0 to σ'_0. Indeed, ((1.1.10.1)) follows from the first part of the claim, and ((1.1.10.2)) from the second; since N' and N are linear, it suffices to compare them on the set of simple coroots.

Let us prove the claim. Suppose first that the image of Γ_{σ_0} in $\mathrm{Aut}(D_0)$ is cyclic. Consider a map $\Gamma_{\sigma'_0} \to \mathrm{Aut}(D_0)$ which has the same image as Γ_{σ_0} and factors through the Galois group of an unramified extension of F'. Then we can take H'_0 to be the quasi-split outer form of H_0 over F' associated to this map.

The only remaining case is when D_0 is of type D_4, and Γ_{σ_0} surjects onto $\mathrm{Aut}(D_0)$. In this case, the subgroup of index 2 still acts transitively on each orbit of $\mathrm{Aut}(D_0)$ in D_0, and we choose $\Gamma_{\sigma'_0} \to \mathrm{Aut}(D_0)$ with image this index two subgroup, and factoring through the Galois group of an unramified extension of F', and H'_0 the corresponding quasi-split outer form of H_0. The proof of the claim is complete.

□

1.1.11. Assume that G is quasi-split over $\mathbb{Q}_p$. Let B be a Borel subgroup of G over $\mathbb{Q}_p$ and $T \subset B$ a maximal torus over $\mathbb{Q}_p$. Let $M \subset G$ be a standard Levi subgroup. Recall that this means that M is the centralizer of a split torus $T_1 \subset T$. Note that we may regard $X_*(Z_M)_{\mathbb{Q}}^{\Gamma_p}$ as a subset of $\mathcal{N}(M)$.

Lemma 1.1.12. *Let $\mu, \mu_M \in X_*(T)$ be cocharacters having the same image in $\pi_1(G)$ and let $[b_M] \in B(M)_b$ be the unique basic class with $\kappa_M([b_M]) = \mu_M^\sharp$. Then*

(1.1.12.1) $\bar{\nu}_M([b_M])$ is equal to the image of $\mu_M^\sharp$ in

$$(\pi_1(M) \otimes \mathbb{Q})^{\Gamma_p} \simeq (X_*(Z_M) \otimes \mathbb{Q})^{\Gamma_p}.$$

(1.1.12.2) $([b_M], \{\mu\})$ is G-admissible if and only if $\bar{\nu}_M([b_M]) \preceq_G \bar{\mu}$.

Proof. The first claim follows from the commutativity of (1.1.2.1). By definition the G-admissibility of $([b_M], \{\mu\})$ is equivalent to asking that $\bar{\nu}_M([b_M]) \preceq_G \bar{\mu}$, and that $\mu_M^\sharp$ maps to $\mu^\sharp$ in $\pi_1(G)_{\Gamma_p}$. However, since μ_M and μ have the same image in $\pi_1(G)$, the second condition is automatic. □

Proposition 1.1.13. *Suppose that G is quasi-split over $\mathbb{Q}_p$. Let $\mu \in X_*(T)$ be minuscule, and $[b_M] \in B(M)_b$ such that $([b_M], \{\mu\})$ is G-admissible. Then there exists $w \in W(G,T)$ such that $([b_M], \{w \cdot \mu\})$ is M-admissible.*

Proof. First, suppose that G is unramified. We fix a reductive model of G over W, again denote by G, such that T extends to a maximal torus $T \subset G$ over W. Then M extends to a Levi subgroup $M \subset G$ over W.

By a theorem of Wintenberger [Win05], the admissibility of $([b_M], \{\mu\})$ implies that there exists $g \in G(L)$ such that $g^{-1} b_M \sigma(g)$ belongs to $G(W)\mu(p)G(W)$. By the Iwasawa decomposition, after modifying g by an element of $G(W)$, we can assume that $g = nm$, where $m \in M(L)$ and $n \in N(L)$, where $N \subset G$ is the unipotent radical of the (positive) parabolic subgroup of G with Levi subgroup M. Then an argument with the Satake transform [LR87, Lem. 5.2] (cf. also the proof of [Kis17, (2.2.2)]) shows that $m^{-1} b_M \sigma(m)$ belongs to $M(W)\mu'(p)M(W)$, where $\mu' \in X_*(T)$ is a cocharacter of M which is $G(L)$-conjugate of μ. More precisely, the Satake transform is used to show that $\mu' \preceq \mu$, and the minuscule nature of μ allows us to conclude that μ' is conjugate to μ. Write $\mu' = w \cdot \mu$ with $w \in W(G,T)$. By a result of Rapoport-Richartz [RR96, Theorem 4.2], $([b_M], \{w \cdot \mu\})$ is M-admissible.

Now, let G be an arbitrary quasi-split group. We can assume that G is adjoint. Indeed, let $\tilde{M} \subset G^{\mathrm{ad}}$ denote the image of M, and $[b_M^{\mathrm{ad}}] \in B(\tilde{M})_b$ the image of $[b_M]$. If $w \in W(G,T)$ is such that $([b_M^{\mathrm{ad}}], \{w \cdot \mu^{\mathrm{ad}}\})$ is $\tilde{M}$-admissible, then we claim that $([b_M], \{w \cdot \mu\})$ is M-admissible. To see this, note that the difference $\kappa_M([b_M]) - (w \cdot \mu)^\sharp$ is contained in the intersection of the kernels of the maps

$$\pi_1(M)_{\Gamma_p} \to \pi_1(\tilde{M})_{\Gamma_p} \text{ and } \pi_1(M)_{\Gamma_p} \to \pi_1(G)_{\Gamma_p}.$$

The kernel of the first map is the image of $X_*(Z_G)_{\Gamma_p} \to \pi_1(M)_{\Gamma_p}$. The composite $X_*(Z_G)_{\Gamma_p} \to \pi_1(G)_{\Gamma_p} \to X_*(G^{\mathrm{ab}})_{\Gamma_p}$ has torsion kernel, so the intersection must be a torsion group. However, by [CKV15, 2.5.12(2)], the kernel of the second map is torsion free. Hence the intersection is trivial.

Next, by considering the simple factors of G separately, we can assume that G is also simple. Therefore, $G = \mathrm{Res}_{F/\mathbb{Q}_p} G_0$, where $F/\mathbb{Q}_p$ is a finite extension, and G_0 is an absolutely simple, quasi-split adjoint group over F. We may also assume that

$$T = \mathrm{Res}_{F/\mathbb{Q}_p} T_0 \; ; \; B = \mathrm{Res}_{F/\mathbb{Q}_p} B_0,$$

where $T_0 \subset G_0$ (resp. $B_0 \subset G_0$) is a maximal torus (resp. Borel subgroup).

By (1.1.10), we can find an *unramified* group G', a Borel subgroup $B' \subset G'$ and a maximal torus $T' \subset B'$, as well as an isomorphism

$$\xi : (G, B, T) \otimes \bar{\mathbb{Q}}_p \xrightarrow{\simeq} (G', B', T') \otimes \bar{\mathbb{Q}}_p$$

such that the induced isomorphism of positive chambers $\eta : \mathcal{C} \xrightarrow{\simeq} \mathcal{C}'$ commutes with Galois averaging maps.

Recall that M is the centralizer of T_1, which is a split torus in T. Set $\mu' = \eta(\mu)$ and $T_1' = \xi(T_1)$. Since η commutes with Galois averaging maps, the elements in $X_*(T_1')$ are equal to their own Galois averages, and hence are Γ_p-invariant. Hence the subtorus $T_1' \subset G'$ is defined over $\mathbb{Q}_p$ and is again split. Let $M' \subset G'$ be the centralizer of T_1'. Then ξ carries M onto M'.

Let $\mu_M \in X_*(T)$ be a cocharacter such that $\mu_M^\sharp = \kappa_M([b_M])$, and such that μ_M and μ have the same image in $\pi_1(G)$, and set $\mu'_M = \eta(\mu_M)$. Let $[b'_M] \in B(M')_b$ be the unique basic class with $\mu'^\sharp_M = \kappa_{M'}([b'_M])$. Then using Lemma 1.1.12 one sees that $([b'_M], \{\mu'\})$ is G'-admissible. Hence, by what we saw in the unramified case, there exists $w \in W(G', T') = W(G, T)$ such that $([b'_M], \{w \cdot \mu'\})$ is M'-admissible. By Lemma 1.1.6 this is equivalent to $\mu'^\sharp_M = (w \cdot \mu')^\sharp$ in $\pi_1(M')_{\Gamma_p}$. This implies that $\mu^\sharp_M - (w \cdot \mu)^\sharp$ in $\pi_1(M)_{\Gamma_p}$ is torsion, since its image under the averaging map in (1.1.2.1) is 0. Since this difference maps to 0 in $\pi_1(G)_{\Gamma_p}$, it follows, as above, that $\mu^\sharp_M = (w \cdot \mu)^\sharp$, and hence, applying Lemma 1.1.6 again, that $([b_M], \{w \cdot \mu\})$ is M-admissible. □

Remark 1.1.14. The previous proposition confirms that part (ii) of [RV14, Lemma 8.2] holds generally for quasi-split groups as expected. (See their Remark 8.3. In fact they do not assume that $[b_M]$ is basic in $B(M)$ but one can reduce to the basic case by [Kot85, Prop 6.2].) Further we extend the proposition to non-quasi-split groups below.

Corollary 1.1.15. *Let G be an arbitrary connected reductive group over $\mathbb{Q}_p$ with a $\mathbb{Q}_p$-rational Levi subgroup M. Let $\mu : \mathbb{G}_m \to M$ be a minuscule cocharacter and $[b_M] \in B(M)_b$ such that $([b_M], \{\mu\})$ is G-admissible. Then there exists $w \in W(G, M) := N_G(M)/M$ such that $([b_M], \{w \cdot \mu\})$ is M-admissible.*

The assumptions of the corollary imply hypothesis (1.1.3.1) for $[b_M]$ (as an element of $B(M)$ or $B(G)$) by (1.1.3). In other words, the corollary is vacuous unless (1.1.3.1) is satisfied.

Proof. We reduce the proof to the quasi-split case. We will freely use the notation from (1.1.5). So let $\xi : G \xrightarrow{\simeq} G^*$ denote an inner twisting. Let P be a $\mathbb{Q}_p$-rational parabolic subgroup with M as a Levi factor. Then the $G^*(\bar{\mathbb{Q}}_p)$-conjugacy class of $\xi(P)$ is defined over $\mathbb{Q}_p$. Since G^* is quasi-split, there exists $g \in G^*(\bar{\mathbb{Q}}_p)$ such that $P^* := g\xi(P)g^{-1}$ is $\mathbb{Q}_p$-rational. We replace ξ by $g\xi g^{-1}$ so that $\xi(P) = P^*$. Put $M^* := \xi(M)$ so that $\xi|_M : M \xrightarrow{\simeq} M^*$ is an inner twisting. We use ξ to identify $W(G, M) \simeq W(G^*, M^*) := N_{G^*}(M^*)/M^*$. We may assume that $B^* \subset P^*$ and $T^* \subset M^*$.

We have a chain of isomorphisms

$$B(M)_b \overset{\kappa_M}{\simeq} \pi_1(M)_{\Gamma_p} = \pi_1(M^*)_{\Gamma_p} \overset{\kappa_{M^*}^{-1}}{\simeq} B(M^*)_b,$$

where the second map is a canonical isomorphism; cf. [RR96, 1.13]. Write $[b_{M^*}] \in B(M^*)_b$ for the image of $[b_M]$. Let μ^* be the $B^* \cap M^*$-dominant representative in $X_*(T^*)$ of the $M^*(\bar{\mathbb{Q}}_p)$-conjugacy class of $\xi|_M \circ \mu$. We claim that $([b_{M^*}], \{\mu^*\})$ is G^*-admissible. Once this is shown, (1.1.13) implies that there exists $w^* \in W(G^*, M^*)$ such that $([b_{M^*}], \{w^* \cdot \mu^*\})$ is M^*-admissible. Writing $w \in W(G, M)$ for the image of w^*, the M-admissibility of $([b_M], \{w \cdot \mu\})$ follows from this.

It remains to prove the claim, i.e. to verify that $\kappa_{G^*}([b_{M^*}]) = (\mu^*)^\#$ and that $\bar{\nu}_{G^*}([b_{M^*}]) \preceq_{G^*} \bar{\mu}^*$. We will deduce this from the assumption that $([b_M], \{\mu\})$ is G-admissible via compatibility of various maps. The former condition follows from the construction of $[b_{M^*}]$ and μ^*, using the functoriality of the Kottwitz map and the fact that the canonical isomorphisms $\pi_1(M) = \pi_1(M^*)$ and $\pi_1(G) = \pi_1(G^*)$ are compatible with the Levi embeddings $M \subset G$ and $M^* \subset G^*$. For the latter condition, since we know $\mathcal{N}_\xi(\bar{\nu}_G([b_M])) \preceq_{G^*} \bar{\mu}^*$, it suffices to check that

$$\mathcal{N}_\xi(\bar{\nu}_G([b_M])) = \bar{\nu}_{G^*}([b_{M^*}]).$$

By [Kot97, 4.4] the Newton maps $\mathcal{N}_{\xi|_M} \circ \bar{\nu}_M : B(M)_b \to \mathcal{N}(M^*)$ and $\bar{\nu}_{M^*} : B(M^*)_b \to \mathcal{N}(M^*)$ factor through the natural inclusion $X_*(A_{M^*})_{\mathbb{Q}} \subset \mathcal{N}(M^*)$, where A_{M^*} is the maximal split torus in the center of M^*. Also the images $\mathcal{N}_{\xi|_M}(\bar{\nu}_M([b_M]))$ and $\bar{\nu}_{M^*}([b_{M^*}])$ in $X_*(A_{M^*})_{\mathbb{Q}}$ are determined by $\kappa_M([b_M])$ and $\kappa_{M^*}([b_{M^*}])$ as elements of $\pi_1(M)_{\Gamma_p} = \pi_1(M^*)_{\Gamma_p}$ (via the canonical isomorphism $X_*(A_{M^*})_{\mathbb{Q}} \simeq \pi_1(M^*)_{\Gamma_p} \otimes \mathbb{Q}$). Since $\kappa_M([b_M]) = \kappa_{M^*}([b_{M^*}])$ by construction, we obtain that $\mathcal{N}_{\xi|_M}(\bar{\nu}_M([b_M])) = \bar{\nu}_{M^*}([b_{M^*}])$. This implies $\mathcal{N}_\xi(\bar{\nu}_G([b_M])) = \bar{\nu}_{G^*}([b_{M^*}])$ since the maps $\mathcal{N}(M) \to \mathcal{N}(G)$ and $\mathcal{N}(M^*) \to \mathcal{N}(G^*)$ induced by Levi embeddings are compatible with $\mathcal{N}_{\xi|_M}$, $\mathcal{N}_\xi$, and likewise for the maps $B(M) \to B(G)$ and $B(M^*) \to B(G^*)$. The proof is complete.

□

1.1.16. Let $b \in G(L)$. We continue to allow G to be non-quasi-split but assume hypothesis (1.1.3.1) on G and $[b]$. Recall that the group J_b defined in (1.1.4) is equipped with an inner twisting $J_b \xrightarrow{\simeq} M_{[b]}$. In particular, $\nu_G([b])$ induces a central cocharacter $\nu_{b,J} : \mathbb{D} \to J_b$ defined over $\mathbb{Q}_p$.

If $T' \subset J_b$ is a maximal torus over $\mathbb{Q}_p$, then a *transfer* of T' to $M_{[b]}$ is an embedding $T' \hookrightarrow M_{[b]}$ over $\mathbb{Q}_p$ which is $M_{[b]}(\bar{\mathbb{Q}}_p)$-conjugate to the composite

$$T' \hookrightarrow J_b \xrightarrow{\simeq} M_{[b]}.$$

A transfer of T' to $M_{[b]}$ always exists either if G is quasi-split ([Lan89, Lemma 2.1]) or if T' is elliptic ([Kot86, Section 10]).

Corollary 1.1.17. *Assume hypothesis (1.1.3.1). Let $([b], \{\mu\})$ be an admissible pair for G with $\{\mu\}$ minuscule. Let $T' \subset J_b$ be a maximal torus. Assume that its transfer $j : T' \hookrightarrow M_{[b]}$ exists. Then $([b], \{\mu\})$ is $j(T')$-special.*

In particular, there exists $\mu_{T'} \in X_*(T')$ such that $j \circ \mu_{T'}$ lies in the G-conjugacy class $\{\mu\}$, and such that we have:

$$\nu_{b,J} = N\mu_{T'} \in X_*(T')_{\mathbb{Q}}^{\Gamma_p}.$$

Proof. Note that J_b and $M_{[b]}$ are both subgroups of G over L. After replacing b by a σ-conjugate satisfying (1.1.4.1), we may assume that $J_{b,L}$ is identified with M_{ν_b}, and that the inner twisting $J_b \xrightarrow{\simeq} M_{[b]}$ is given by composing this identification with conjugation by an element $h \in G(L)$ that carries ν_b to $\nu_G([b])$. In particular, then $\nu_G([b]) = \mathrm{int}(h)(\nu_{b,J})$ as G-valued cocharacters.

Now, view T' as a subtorus of G, via j, let $T_1 \subset T'$ be the maximal split subtorus, and let $M \subset G$ be the centralizer of T_1, so that $T' \subset M$ is an elliptic maximal torus. Let $T_2 \supset T_1$ be a maximal split torus in G containing T_1. After conjugating our fixed torus $T \subset G$, we may assume that T is the centralizer of T_2, so that $M \supset T$ is a standard Levi subgroup.

The scheme of elements of $M_{[b],L}$ which conjugate the inclusion $j_0 : T' \hookrightarrow J_b \xrightarrow{\simeq} M_{[b]}$ into j is a T'-torsor over L. By Steinberg's theorem this torsor is trivial. Hence, there exists $m \in M_{[b]}(L)$ such that $mj_0m^{-1} = j$. Now, a simple computation, using the definition of J_b, shows that $b_M = mh \cdot b \cdot \sigma(mh)^{-1}$ commutes with $j(T'(\mathbb{Q}_p))$. Since $T_1(\mathbb{Q}_p)$ is Zariski dense in T_1, this shows that b_M belongs to $M(L)$. Moreover, since $\nu_{b_M} = \nu_G([b])$ is central in M, b_M is in fact basic in M.

By Lemma 1.1.13, there exists $w \in W(G,T)$, such that $([b_M], \{w \cdot \mu\})$ is M-admissible. (Here we may take $\mu \in X_*(T)$ the dominant representative of $\{\mu\}$.) It follows by Lemma 1.1.8

that $([b_M], \{w \cdot \mu\})$ is T'-special. In particular, there exists $\mu_{T'} \in X_*(T')$ in $\{\mu\}$ such that $\nu_{b_M} = N\mu_{T'}$. Hence, if we think of $N\mu_{T'}$ as a J_b-valued cocharacter via the natural inclusion $T' \subset J_b$, then $\nu_{b,J} = N\mu_{T'}$.

□

1.2. **Global results.**

Lemma 1.2.1. *Let T be a torus over $\mathbb{Q}$. For any prime p, the restriction map*

$$\ker\big(H^1(\mathbb{Q}, T) \to H^1(\mathbb{Q}_p, T)\big) \to H^1(\mathbb{R}, T)$$

is surjective.

Proof. For each place v of $\mathbb{Q}$, there is a canonical isomorphism [Kot86, (1.1.1)]:

$$j_v : H^1(\mathbb{Q}_v, T) \xrightarrow{\simeq} X_*(T)^{\mathrm{tors}}_{\Gamma_v}.$$

Write $\bar{j}_v$ for the composition of this map with the natural projection $X_*(T)^{\mathrm{tors}}_{\Gamma_v} \to X_*(T)^{\mathrm{tors}}_{\Gamma}$. We then have an exact sequence [Kot86, Prop. 2.6]:

$$H^1(\mathbb{Q}, T) \to \oplus_v H^1(\mathbb{Q}_v, T) \xrightarrow{\oplus \bar{j}_v} X_*(T)^{\mathrm{tors}}_{\Gamma}.$$

So, given a class $\alpha_\infty \in H^1(\mathbb{R}, T)$, it suffices to find $\ell \neq p$ and a class $\alpha_\ell \in H^1(\mathbb{Q}_\ell, T)$ such that $\bar{j}_\ell(\alpha_\ell) = -\bar{j}_\infty(\alpha_\infty)$. Indeed, once we have done this, we can take the element $(\alpha_v) \in \oplus_v H^1(\mathbb{Q}_v, T)$, with $\alpha_v = 0$ for $v \neq \infty, \ell$: This will be the image of an element $\alpha \in H^1(\mathbb{Q}, T)$ mapping to $\alpha_\infty \in H^1(\mathbb{R}, T)$ and to the trivial element in $H^1(\mathbb{Q}_p, T)$.

The remainder of the proof now proceeds as in [Lan83, 7.16]. We choose a finite Galois extension $E \subset \bar{\mathbb{Q}}$ over which T splits. Then complex conjugation on $\mathbb{C}$ induces an automorphism σ_∞ of E. We now choose $\ell \neq p$ such that E is unramified over ℓ and such that, for some place $v|\ell$ of E, the Frobenius σ_v at v is conjugate to σ_∞. We can further assume that v is induced from the embedding $E \hookrightarrow \bar{\mathbb{Q}}_\ell$. If $g \in \Gamma$ conjugates σ_v into σ_∞, then the automorphism of $X_*(T)$ given by g, induces an isomorphism $X_*(T)_{\Gamma_\infty} \xrightarrow{\simeq} X_*(T)_{\Gamma_\ell}$, which is compatible with projections onto $X_*(T)_\Gamma$. We use this isomorphism to identify $X_*(T)^{\mathrm{tors}}_{\Gamma_\infty}$ with $X_*(T)^{\mathrm{tors}}_{\Gamma_\ell}$. Now we may take $\alpha_\ell = -j_\ell^{-1}(j_\infty(\alpha_\infty))$. □

Lemma 1.2.2. *Let G be a connected reductive group over $\mathbb{Q}$. Suppose that we are given a finite set of places S of $\mathbb{Q}$ and, for each $v \in S$, a maximal torus $T_v \subset G_{\mathbb{Q}_v}$. Then there exists a maximal torus $T \subset G$ such that, for all $v \in S$, the inclusion $T_{\mathbb{Q}_v} \subset G_{\mathbb{Q}_v}$ is $G(\mathbb{Q}_v)$-conjugate to $T_v \subset G_{\mathbb{Q}_v}$.*

Proof. This is [Har66, Lemma 5.5.3], cf. [Bor98, 5.6.3]. □

1.2.3. Let (G, X) be a Shimura datum. Given $x \in X$, we have the associated homomorphism of $\mathbb{R}$-groups:

$$h_x : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}.$$

We also have the associated (minuscule) cocharacter:

$$\mu_x : \mathbb{G}_{m,\mathbb{C}} \xrightarrow{z \mapsto (z,1)} \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \xrightarrow{\simeq} \mathbb{S}_{\mathbb{C}} \xrightarrow{h_x} G_{\mathbb{C}}.$$

The $G(\mathbb{R})$-conjugacy class of h_x, and hence the $G(\mathbb{C})$-conjugacy class $\{\mu_X\}_\infty$ of μ_x, is independent of the choice of x. Let $E \subset \mathbb{C}$ be the reflex field for (G, X): This is the field of definition of $\{\mu_X\}_\infty$, and is a finite extension of $\mathbb{Q}$.

The embedding $\iota_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ allows us to view $E \subset \mathbb{C}$ as a subfield of $\bar{\mathbb{Q}}$, so that we may regard $\{\mu_X\}_\infty$ as a conjugacy class $\{\mu_X\}$ of cocharacters of $G_{\bar{\mathbb{Q}}}$.

1.2.4. We will use the embedding ι_p to view $\{\mu_X\}$ as a conjugacy class $\{\mu_X\}_p$ of cocharacters of $G_{\bar{\mathbb{Q}}_p}$.

Proposition 1.2.5. *Let $[b] \in B(G_{\mathbb{Q}_p})$ be a class such that $([b], \{\mu_X^{-1}\}_p)$ is admissible. Assume hypothesis (1.1.3.1) holds for $[b]$. Then there exist a maximal torus $T \subset G$ and an element $x \in X$ with h_x factoring through $T_{\mathbb{R}}$ (in which case $\mu_x^{-1} \in X_*(T)$) such that $[b_{\mathrm{bas}}(\mu_x)] \in B(T_{\mathbb{Q}_p})$ maps to $[b] \in B(G_{\mathbb{Q}_p})$.*

Proof. This proof is directly inspired by that of [LR87, 5.12].

By (1.1.17), there exist a maximal torus $T_p \subset G_{\mathbb{Q}_p}$ (chosen to be elliptic if $G_{\mathbb{Q}_p}$ is not quasi-split so that the transfer to $M_{[b]}$ exists) and a representative $\mu_p \in X_*(T_p)$ of $\{\mu_X\}_p$ such that $[b_{\mathrm{bas}}(\mu_p^{-1})] \in B(T_p)$ maps to $[b] \in B(G_{\mathbb{Q}_p})$.

Choose $y \in X$, and let $T_\infty \subset G_{\mathbb{R}}$ be a maximal torus such that h_y factors through T_∞. By (1.2.2), we can find a maximal torus $T \subset G$ such that $T_{\mathbb{Q}_p}$ (resp. $T_{\mathbb{R}}$) is $G(\mathbb{Q}_p)$-conjugate to T_p (resp. $G(\mathbb{R})$-conjugate to T_∞).

Choose $g_p \in G(\mathbb{Q}_p)$ such that $g_pT_pg_p^{-1} = T_{\mathbb{Q}_p}$, and let $\mu_T : \mathbb{G}_{m,\bar{\mathbb{Q}}} \to T_{\bar{\mathbb{Q}}}$ be the unique cocharacter, which, after base-change along ι_p, is identified with $\mathrm{int}(g_p)(\mu_p)$. Then $[b_{\mathrm{bas}}(\mu_T^{-1})]$ maps to $[b]$.

Choose $g_\infty \in G(\mathbb{R})$ such that $g_\infty T_\infty g_\infty^{-1} = T_{\mathbb{R}}$. After base-change along ι_∞, the cocharacter μ_T is $G(\mathbb{C})$-conjugate to $\mu_\infty = \mathrm{int}(g_\infty)(\mu_y)$. Therefore, there exists an element $\omega \in W(G,T)(\mathbb{C})$ such that $\omega(\mu_\infty) = \mu_T$.

We can identify $W(G,T)$ with $N_{G^{\mathrm{sc}}}(T^{\mathrm{sc}})/T^{\mathrm{sc}}$. Let $n \in N_{G^{\mathrm{sc}}}(T^{\mathrm{sc}})(\mathbb{C})$ be any element mapping to ω. Since T^{sc} is anisotropic over $\mathbb{R}$, the element ω acts on T^{sc} by an $\mathbb{R}$-automorphism. Hence $n\bar{n}^{-1} \in T^{\mathrm{sc}}(\mathbb{C})$. The cocycle carrying complex conjugation to $n\bar{n}^{-1}$ determines a class $\alpha_\infty \in H^1(\mathbb{R}, T^{\mathrm{sc}})$ depending only on ω (not on the choice of n). By (1.2.1), we can find a class $\alpha \in H^1(\mathbb{Q}, T^{\mathrm{sc}})$ mapping to $\alpha_\infty \in H^1(\mathbb{R}, T^{\mathrm{sc}})$, as well as to the trivial class in $H^1(\mathbb{Q}_p, T^{\mathrm{sc}})$.

By construction, the image of α_∞ in $H^1(\mathbb{R}, G^{\mathrm{sc}})$ is trivial. Therefore, by the Hasse principle and the Kneser vanishing theorem for simply connected groups, the image of α in $H^1(\mathbb{Q}, G^{\mathrm{sc}})$ is trivial. This means that we can find $g \in G^{\mathrm{sc}}(\bar{\mathbb{Q}})$ such that, for any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, $g\sigma(g)^{-1} \in T^{\mathrm{sc}}(\bar{\mathbb{Q}})$, and such that α is represented by the $T^{\mathrm{sc}}(\bar{\mathbb{Q}})$-valued cocycle $\sigma \mapsto g\sigma(g)^{-1}$.

In particular, if we view g as an element of $G^{\mathrm{sc}}(\mathbb{C})$ via ι_∞, there exists $t \in T^{\mathrm{sc}}(\mathbb{C})$ such that $g\bar{g}^{-1} = tn\bar{n}^{-1}\bar{t}^{-1}$.

Now, μ_∞ and $\mathrm{int}(g^{-1})(\mu_T)$ are conjugate under $h = g^{-1}tn \in G(\mathbb{R})$, and the maximal torus $\mathrm{int}(g^{-1})(T_{\bar{\mathbb{Q}}}) \subset G_{\bar{\mathbb{Q}}}$ is defined over $\mathbb{Q}$. Replacing T with this torus, and μ_T with $\mathrm{int}(g^{-1})(\mu_T)$, we see that μ_T is of the form μ_x for $x \in X$, and that the pair (T, μ_x) satisfies the conclusions of the proposition. □

1.3. **Shimura varieties of Hodge type.** One may view (1.2.5) as showing the non-emptiness of Newton strata in the special fiber of the Shimura variety associated with (G,X). We will now make this assertion precise in the case where (G,X) is of Hodge type, where the moduli spaces of abelian varieties give us a natural way to construct integral models.

1.3.1. Recall that, given a symplectic space (V,ψ) over $\mathbb{Q}$, we can attach to it the Siegel Shimura datum $(\mathcal{G}_V, \mathcal{H}_V)$, where $\mathcal{G}_V = \mathrm{GSp}(V,\psi)$ is the group of symplectic similitudes and $\mathcal{H}_V$ is the union of the Siegel half-spaces associated with (V,ψ).

Let (G,X) be a Shimura datum of *Hodge type*. This means that there exists a faithful symplectic representation (V,ψ) of G such that the associated map $G \hookrightarrow \mathcal{G}_V$ extends to an embedding of Shimura data $(G,X) \hookrightarrow (\mathcal{G}_V, \mathcal{H}_V)$. We denote by $E = E(G,X)$ the reflex field of (G,X).

1.3.2. Fix a $\mathbb{Z}_{(p)}$-lattice $V_{(p)} \subset V$ on which ψ is $\mathbb{Z}_{(p)}$-valued. Set $V_p = \mathbb{Z}_p \otimes V_{(p)}$, and let $\mathcal{K}_p \subset \mathcal{G}_V(\mathbb{Q}_p)$ (resp. $K_p \subset G(\mathbb{Q}_p)$) be the stabilizer of $V_p \subset V_{\mathbb{Q}_p}$.

Given a sufficiently small compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$, we can find a neat compact open subgroup $\mathcal{K}^p \subset \mathcal{G}_V(\mathbb{A}_f^p)$ such that, with $K = K_pK^p$ and $\mathcal{K} = \mathcal{K}_p\mathcal{K}^p$, the map of Shimura varieties

$$\mathrm{Sh}_K \coloneqq \mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{\mathcal{K}} \coloneqq \mathrm{Sh}_{\mathcal{K}}(\mathcal{G}_V, \mathcal{H}_V) \otimes E$$

is a closed immersion [Kis10, 2.1.2].

The variety $\mathrm{Sh}_{\mathcal{K}}$ admits an integral model $\mathcal{S}_{\mathcal{K}}$ over $\mathbb{Z}_{(p)}$, which is an open and closed subscheme of the moduli scheme parameterizing polarized abelian schemes (A, λ) up to prime-to-p isogeny, and equipped with additional level structures away from p. Let $\mathcal{A}$ denote the universal abelian scheme over $\mathcal{S}_{\mathcal{K}}$ up to prime-to-p isogeny.

1.3.3. We will now use the notation from (1.1.2). Given a point $s_0 \in \mathcal{S}_{\mathcal{K}}(\bar{\mathbb{F}}_p)$, we obtain the associated Dieudonné F-crystal $\mathbb{D}(\mathcal{A}_{s_0})$ over W. Set $D_{s_0} = \mathbb{D}(\mathcal{A}_{s_0})_{\mathbb{Q}}$: This is an F-isocrystal over $L = W[p^{-1}]$, so that it is equipped with a σ-semi-linear bijection $\varphi : D_{s_0} \to D_{s_0}$.

Given a finite extension $L' \subset \bar{L}$ of L and a point $s \in \mathcal{S}_{\mathcal{K}}(L')$ specializing to s_0, we obtain two canonical comparison isomorphisms:

(1.3.3.1) The Berthelot-Ogus isomorphism:

$$H^1_{\mathrm{dR}}(\mathcal{A}_s/L') \xrightarrow{\simeq} L' \otimes_L D_{s_0}.$$

(1.3.3.2) The p-adic comparison isomorphism:

$$B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} H^1_{\text{ét}}(\mathcal{A}_{s,\bar{L}}, \mathbb{Q}_p) \xrightarrow{\simeq} B_{\mathrm{cris}} \otimes_L D_{s_0}.$$

The two isomorphisms are compatible with the de Rham comparison isomorphism:

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^1_{\text{ét}}(\mathcal{A}_{s,\bar{L}}, \mathbb{Q}_p) \xrightarrow{\simeq} B_{\mathrm{dR}} \otimes_{L'} H^1_{\mathrm{dR}}(\mathcal{A}_s/L'). \tag{1.3.3.3}$$

1.3.4. Let $\boldsymbol{V}_{\mathrm{dR}}$ be the (cohomological) de Rham realization of $\mathcal{A}$: It is a vector bundle over $\mathrm{Sh}_{\mathcal{K}}$ with integrable connection, and its fiber at each point $s \in \mathrm{Sh}_{\mathcal{K}}(\kappa)$ (κ a field of characteristic 0) is the de Rham cohomology $H^1_{\mathrm{dR}}(\mathcal{A}_s/\kappa)$.

Let $\widehat{V}^p(\mathcal{A})$ be the prime-to-p Tate module of $\mathcal{A}$: This is a smooth $\mathbb{A}_f^p$-sheaf over $\mathrm{Sh}_{\mathcal{K}}$. Write $\boldsymbol{V}^p$ for its dual; then the fiber of $\boldsymbol{V}^p$ at any point $s \in \mathrm{Sh}_{\mathcal{K}}(\kappa)$, with κ algebraically closed, is identified with the étale cohomology group $H^1_{\text{ét}}(\mathcal{A}_s, \mathbb{A}_f^p)$.

Finally, write $T_p(\mathcal{A})$ for the p-adic Tate module of $\mathcal{A}$, and set $V_p(\mathcal{A}) = \mathbb{Q}_p \otimes T_p(\mathcal{A})$. Write $\boldsymbol{V}_p$ for the dual $(V_p(\mathcal{A}))^\vee$. We will set

$$\widehat{V}(\mathcal{A}) = \widehat{V}^p(\mathcal{A}) \times V_p(\mathcal{A}) \quad \text{and} \quad \boldsymbol{V}_{\text{ét}} = \boldsymbol{V}^p \times \boldsymbol{V}_p.$$

Fix tensors $\{s_\alpha\} \subset V^\otimes$ such that G is their pointwise stabilizer in $\mathrm{GL}(V)$. Here and below, the superscript $\otimes$ means the direct sum of $V^{\otimes n} \otimes V^{*\otimes m}$ for all $m, n \geq 0$. Then there exist global sections:

$$\{s_{\alpha,\mathrm{dR}}\} \subset H^0(\mathrm{Sh}_K, \boldsymbol{V}^\otimes_{\mathrm{dR}}) \; ; \; \{s_{\alpha,\text{ét}}\} \subset H^0(\mathrm{Sh}_K, \boldsymbol{V}^\otimes_{\text{ét}})$$

with the following properties:

(1.3.4.1) Given an algebraically closed field κ of characteristic 0 and a point $s \in \mathrm{Sh}_K(\kappa)$, there exists an isomorphism

$$V_{\mathbb{A}_f} \xrightarrow{\simeq} H^1_{\text{ét}}(\mathcal{A}_s, \mathbb{A}_f) = \boldsymbol{V}_{\text{ét},s},$$

determined up to conjugacy by $G(\mathbb{A}_f)$, carrying $\{s_\alpha\}$ to $\{s_{\alpha,\text{ét},s}\}$.

(1.3.4.2) For each α, let $s_{\alpha,p}$ be the projection of $s_{\alpha,\text{ét}}$ onto $\boldsymbol{V}_p$. Then, given a finite extension L'/L and a point $s \in \mathrm{Sh}_K(L')$, the isomorphism (1.3.3.3) carries $\{1 \otimes s_{\alpha,p,s}\}$ to $\{1 \otimes s_{\alpha,\mathrm{dR},s}\}$.

The construction of these tensors is described in [Kis10, (2.2)]: The key point is a theorem of Deligne showing that all Hodge cycles on abelian varieties over $\mathbb{C}$ are absolutely Hodge. Property (1.3.4.1) now holds by construction. Property (1.3.4.2) is a theorem of Blasius-Wintenberger [Bla94].

1.3.5. Fix a place $v|p$ of E, and an embedding $k(v) \hookrightarrow \bar{\mathbb{F}}_p$. We denote by

$$\mathscr{S}_K \hookrightarrow \mathscr{O}_{E,(v)} \otimes_{\mathbb{Z}_{(p)}} \mathcal{S}_{\mathcal{K}}.$$

the normalization of the Zariski closure of Sh_K in $\mathscr{O}_{E,(v)} \otimes \mathcal{S}_{\mathcal{K}}$.

Proposition 1.3.6. *For every point $s_0 \in \mathscr{S}_{K,k(v)}(\bar{\mathbb{F}}_p)$, there exists a canonical collection of φ-invariant tensors $\{s_{\alpha,\mathrm{cris},s_0}\} \subset D_{s_0}^{\otimes}$ characterized by the following property: For any lift $s \in \mathscr{S}_K(\bar{L})$ of s_0, the isomorphism (1.3.3.2) carries $\{s_{\alpha,p,s}\}$ to $\{s_{\alpha,\mathrm{cris},s_0}\}$.*

Proof. The proof of this can essentially be found in [Kis10, (2.3.5)]; however, since it is not given there in the generality we require, we review the key steps here. Write $L' = E_v L \subset \bar{L}$; here, we are embedding $E_v \hookrightarrow \bar{L}$ via the fixed embedding $\bar{\mathbb{Q}}_p \hookrightarrow \bar{L}$. Let $\widehat{\mathcal{U}}$ be the formal scheme over W pro-representing the deformation functor for the p-divisible group $\mathcal{A}_{s_0}[p^\infty]$: this is formally smooth over W. Let $\widehat{U}$ be the formal scheme obtained by completing $\mathscr{S}_K \otimes_{\mathscr{O}_{E,(v)}} \mathscr{O}_{L'}$ along s_0..

We have a finite map of normal formal schemes over $\mathscr{O}_{L'}$, $\widehat{U} \to \widehat{\mathcal{U}}_{L'}$. Taking their rigid analytic fibers (in the sense of Berthelot; cf. [Jon95, 7.3]), we obtain a map $\widehat{U}^{\mathrm{an}} \to \widehat{\mathcal{U}}^{\mathrm{an}}_{L'}$ of smooth, irreducible rigid analytic spaces over L'. This map is a closed immersion, since the map $\mathrm{Sh}_K \to \mathrm{Sh}_{\mathcal{K}}$ is.

Since $\widehat{\mathcal{U}}_{L'}$ is formally smooth, $\widehat{\mathcal{U}}^{\mathrm{an}}_{L'}$ is a rigid analytic open ball over L', and, for any two points $s, s' \in \widehat{\mathcal{U}}^{\mathrm{an}}(\bar{L})$, p-adic parallel transport using the Gauss-Manin connection on $\boldsymbol{V}_{\mathrm{dR}}$ gives us a canonical isomorphism:

$$H^1_{\mathrm{dR}}(\mathcal{A}_s/\bar{L}) \xrightarrow{\simeq} H^1_{\mathrm{dR}}(\mathcal{A}_{s'}/\bar{L}). \tag{1.3.6.1}$$

Suppose now that s, s' lie in $\widehat{U}^{\mathrm{an}}(\bar{L})$. Since the sections $s_{\alpha,\mathrm{dR}}$ over Sh_K are horizontal for the connection, and since $\widehat{U}^{\mathrm{an}}$ is smooth and irreducible over L', for each α this isomorphism carries $s_{\alpha,\mathrm{dR},s}$ to $s_{\alpha,\mathrm{dR},s'}$.

Given $s \in \widehat{U}^{\mathrm{an}}(\bar{L})$, the isomorphism (1.3.3.2) carries the tensors $\{s_{\alpha,p,s}\}$ to φ-invariant tensors $\{s_{\alpha,\mathrm{cris},s}\} \subset D_{s_0}^{\otimes}$. To prove the proposition, it is now enough to show: If s' is a different lift, giving rise to φ-invariant tensors $\{s_{\alpha,\mathrm{cris},s'}\} \subset D_{s_0}^{\otimes}$, then, for each α, we have $s_{\alpha,\mathrm{cris},s'} = s_{\alpha,\mathrm{cris},s}$.

By the compatibility of (1.3.3.2) with (1.3.3.1), and by (1.3.4.2), the pre-image of $1 \otimes s_{\alpha,\mathrm{cris},s}$ (resp. $1 \otimes s_{\alpha,\mathrm{cris},s'}$) in $H^1_{\mathrm{dR}}(\mathcal{A}_s/\bar{L})^{\otimes}$ (resp. in $H^1_{\mathrm{dR}}(\mathcal{A}_{s'}/\bar{L})^{\otimes}$) under (1.3.3.1) is exactly $s_{\alpha,\mathrm{dR},s}$ (resp. $s_{\alpha,\mathrm{dR},s'}$). Therefore, we only need to show that the composition:

$$H^1_{\mathrm{dR}}(\mathcal{A}_s/\bar{L}) \xrightarrow{\simeq} \bar{L} \otimes D_{s_0} \xrightarrow{\simeq} H^1_{\mathrm{dR}}(\mathcal{A}'_s/\bar{L})$$

is the parallel transport isomorphism (1.3.6.1). This follows from [BO83, 2.9]. □

1.3.7. It follows from (1.3.6) and (1.3.4.1) that there exists an isomorphism $L \otimes_{\mathbb{Q}} V \xrightarrow{\simeq} D_{s_0}$ carrying $\{1 \otimes s_\alpha\}$ to $\{s_{\alpha,\mathrm{cris},s_0}\}$. Indeed the scheme of such isomorphisms is a G-torsor by (1.3.4.1), and a G-torsor over L is trivial by Steinberg's theorem. Under this isomorphism, the map $\varphi : D_{s_0} \to D_{s_0}$ pulls back to an automorphism of $L \otimes V$ of the form $\sigma \otimes b_{s_0}$, with $b_{s_0} \in G(L)$ well-determined up to σ-conjugacy. Therefore, s_0 determines a canonical class $[b_{s_0}] \in B(G_{\mathbb{Q}_p})$.

Assume that $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ has been chosen such that the associated embedding $E \hookrightarrow \bar{\mathbb{Q}}_p$ induces the place v.

Lemma 1.3.8. *The pair $([b_{s_0}], \{\mu_X^{-1}\}_p)$ is admissible.*

Proof. This is a consequence of a result of Wintenberger; cf. corollary to [Win97, 4.5.3]. □

Proposition 1.3.9. *Assume hypothesis (1.1.3.1) for $G_{\mathbb{Q}_p}$ and $[b]$. Then the pair $([b], \{\mu_X^{-1}\}_p)$ is admissible if and only if there exists $s_0 \in \mathscr{S}_K(\bar{\mathbb{F}}_p)$ such that $[b] = [b_{s_0}]$.*

Proof. The 'if' part is (1.3.8)

Suppose that we are given $[b] \in B(G_{\mathbb{Q}_p})$ with $([b], \{\mu_X^{-1}\}_p)$ admissible. Then (1.2.5) gives us a maximal torus $T \subset G$ and an $x \in X$ such that h_x factors through $T_{\mathbb{R}}$, and such that $[b_{\mathrm{bas}}(\mu_x^{-1})] \in B(T_{\mathbb{Q}_p})$ maps to $[b] \in B(G_{\mathbb{Q}_p})$.

Now, consider the 0-dimensional Shimura variety $\mathrm{Sh}_0 = \mathrm{Sh}_{K\cap T(\mathbb{A}_f)}(T, h_x)$: This is a finite étale scheme over the reflex field $E_T = E(T, h_x)$. Fix a place $v'|p$ of E_T lying above v. The normalization of $\operatorname{Spec} \mathscr{O}_{E_T,(v')}$ in Sh_0 gives us a canonical normal integral model $\mathscr{S}_0$ for Sh_0 over $\mathscr{O}_{E_T,(v')}$.

Since all CM abelian varieties over number fields have everywhere potentially good reduction, the map $\mathrm{Sh}_0 \to E_T \otimes_E \mathrm{Sh}_K$ extends to a map of $\mathscr{O}_{E_T,(v')}$-schemes $\mathscr{S}_0 \to \mathscr{O}_{E_T,(v')} \otimes_{\mathscr{O}_{E,(v)}} \mathscr{S}_K$.

Therefore, to prove the theorem, we can replace $(G, [b], \{\mu_X^{-1}\})$ with $(T, [b_{\mathrm{bas}}(\mu_x^{-1})], \mu_x^{-1})$, and reduce to the case where $G = T$ is a torus. Choose any point $s_0 \in \mathscr{S}_0(\bar{\mathbb{F}}_p)$. By (1.3.8), the pair $([b_{s_0}], \mu_x^{-1})$ is admissible for $T_{\mathbb{Q}_p}$. But then we must have $[b_{s_0}] = [b]$. □

1.3.10. Given a scheme S in characteristic p, let F-Isoc(S) be the category of F-isocrystals over S (cf. [RR96, §3]): This is the isogeny category obtained by localizing the category of F-crystals over S. It is a $\mathbb{Q}_p$-linear (non-neutral) Tannakian category, whose identity object $\mathbf{1}$ corresponds to the structure sheaf on the crystalline site of S over $\mathbb{Z}_p$.

Recall that for G a reductive group over $\mathbb{Q}_p$, an F-isocrystal with G-structure over S [RR96, 3.3] is an exact faithful tensor functor

$$\mathrm{Rep}_{\mathbb{Q}_p} G \to \text{F-Isoc}(S).$$

Here $\mathrm{Rep}_{\mathbb{Q}_p} G$ denotes the category of finite dimensional $\mathbb{Q}_p$-representations of G.

The crystalline realization of the universal abelian scheme $\mathcal{A}$ over $\mathscr{S}_K$ gives us a canonical object $\mathcal{D}$ in F-Isoc$(\mathscr{S}_K \otimes_{\mathscr{O}_{E,(v)}} \bar{\mathbb{F}}_p)$. For each point $s_0 \in \mathscr{S}_K(\bar{\mathbb{F}}_p)$, the restriction of $\mathcal{D}$ over s_0 is realized by the F-isocrystal D_{s_0}.

The proof of the following proposition is rather technical. Since it is used only in (1.3.13) and (1.3.15) below, and the rest of the paper does not depend on it, we relegate it to an appendix, where we prove a stronger statement; see Corollary A.7 below.

Proposition 1.3.11. *For each α, there exists a morphism*

$$\boldsymbol{s}_\alpha : \mathbf{1} \to \mathcal{D}^{\otimes}$$

whose restriction to any point $s_0 \in \mathscr{S}_K(\bar{\mathbb{F}}_p)$ is $s_{\alpha,\mathrm{cris},s_0}$.

Corollary 1.3.12. *The association $V \mapsto \mathcal{D}$ extends to an F-isocrystal with G-structure over $\mathscr{S}_K \otimes \bar{\mathbb{F}}_p$.*

Proof. Let S be a connected component of $\mathscr{S}_K$. We shall again write $\mathcal{D}$ for $\mathcal{D}|_S$. Let $C_{\mathcal{D}}$ be the smallest full Tannakian subcategory of F-Isoc(S) containing $\mathcal{D}$. It suffices to construct, for each S, an exact faithful tensor functor $\omega : \mathrm{Rep}_{\mathbb{Q}_p} G \to C_{\mathcal{D}}$ which sends V to $\mathcal{D}$.

First consider the associated L-linear category $C_{\mathcal{D},L} = C_{\mathcal{D}} \otimes L$, which is obtained from $C_{\mathcal{D}}$ by tensoring the Hom sets by L, and adjoining the direct summands corresponding to idempotents in the endomorphism algebra of each object [Del79, 2.1]. Choose $s_0 \in S(\bar{\mathbb{F}}_p)$. Pulling isocrystals back to s_0 induces an L-fibre functor $\omega_{s_0} : C_{\mathcal{D},L} \to \text{F-Isoc}(s_0)$ which takes $\mathcal{D}$ to D_{s_0}, and $C_{\mathcal{D},L}$ is equivalent to the category $\mathrm{Rep}_L G_{s_0}$ where $G_{s_0} = \mathrm{Aut}_{\{s_{\alpha,\mathrm{cris},s_0}\}} D_{s_0}$, the group of automorphisms of D_{s_0} respecting the tensors $s_{\alpha,\mathrm{cris},s_0}$.

Let $P(s_0) = \underline{\mathrm{Isom}}_{s_\alpha}(V_L, D_{s_0})$, the scheme of L-linear maps from V_L to D_{s_0} taking s_α to $s_{\alpha,\mathrm{cris},0}$. Then $P(s_0)$ is a G-torsor. If W is in $\mathrm{Rep}_{\mathbb{Q}_p} G$, then $W^{\mathcal{D}} = G\backslash(W \times P(s_0))$ is an L-representation of G_{s_0}. We consider the composite functor

$$\omega_L : \mathrm{Rep}_{\mathbb{Q}_p} G \stackrel{W \mapsto W^{\mathcal{D}}}{\to} \mathrm{Rep}_L G_{s_0} \simeq C_{\mathcal{D},L}.$$

It remains to show that the above functor factors through $C_{\mathcal{D}}$. For this, note that any object of $\mathrm{Rep}_{\mathbb{Q}_p} G$ is the kernel of a map $e : W \to W$ where W is a direct sum of objects of the form $V_{m,n} := V^{\otimes n} \otimes V^{*\otimes m}$. Now $\omega_L(V_{m,n}) = \mathcal{D}^{\otimes m} \otimes \mathcal{D}^{*\otimes n}$ lies in $C_{\mathcal{D}}$. Since e can be considered as a morphism $1 \to W^* \otimes W$, we see that by Proposition 1.3.11, $\omega_L(e)$ lies in $C_{\mathcal{D}}$, and so does its kernel. Similarly if $e : W_1 \to W_2$ is any map in $\mathrm{Rep}_{\mathbb{Q}_p} G$, then e may be regarded as a map $1 \to W_1^* \otimes W_2$ so $\omega_L(e)$ is in $C_{\mathcal{D}}$ by Proposition 1.3.11. □

Theorem 1.3.13.

(1.3.13.1) If $s_0 \in \mathscr{S}_K(\bar{\mathbb{F}}_p)$, then

$$\{s_0' \in \mathscr{S}_K(\bar{\mathbb{F}}_p) : \ \bar{\nu}_G([b_{s_0'}]) \preceq \bar{\nu}_G([b_{s_0}])\} \subset \mathscr{S}_K(\bar{\mathbb{F}}_p)$$

is a Zariski closed subset.

(1.3.13.2) Let $B(G_{\mathbb{Q}_p}, \{\mu_X^{-1}\}_p) \subset B(G_{\mathbb{Q}_p})$ be the subset consisting of those classes $[b]$ such that $([b], \{\mu_X^{-1}\}_p)$ is admissible. Then, for every $[b] \in B(G_{\mathbb{Q}_p}, \{\mu_X^{-1}\}_p)$ satisfying hypothesis (1.1.3.1), the subset:

$$S_{[b]} = \{s_0 \in \mathscr{S}_K(\bar{\mathbb{F}}_p) : \ [b_{s_0}] = [b]\}$$

is non-empty and locally closed in $\mathscr{S}_K(\bar{\mathbb{F}}_p)$ for the Zariski topology.

(1.3.13.3) Let $\overline{S}_{[b]}$ be the closure of $S_{[b]}$ in $\mathscr{S}_K(\bar{\mathbb{F}}_p)$; then we have an inclusion of Zariski closed subsets:

$$\overline{S}_{[b]} \subset \bigsqcup_{\bar{\nu}_G([b']) \preceq \bar{\nu}_G([b])} S_{[b']}.$$

Proof. Assertions (1.3.13.1) and (1.3.13.3) follow immediately from (1.3.11) and Grothendieck's semicontinuity theorem for Newton polygons of F-isocrystals [Kat79, Theorem 2.3.1]; cf. the proof of [RR96, Theorem 3.6]. Assertion (1.3.13.2) follows from (1.3.13.1) and (1.3.9). □

As we noted in (1.1.3), the second part of the theorem implies that the stratum $S_{[b]}$ is non-empty if either $[b]$ is basic or $G_{\mathbb{Q}_p}$ is quasi-split.

1.3.14. As in (1.1.5) we fix an inner twisting $\xi : G_{\mathbb{Q}_p} \to G^*$ over $\bar{\mathbb{Q}}_p$, a Borel $B^* \subset G^*$, and a maximal torus $T^* \subset B^*$ over $\mathbb{Q}_p$. Let μ be the B^*-dominant representative of $\xi \circ \{\mu_X\}_p$. There is a unique $[b_\mu] \in B(G_{\mathbb{Q}_p}, \{\mu^{-1}\})$ with $\mathcal{N}_\xi(\bar{\nu}_G([b_\mu])) = \bar{\mu}^{-1}$ (which, of course, does not depend on the choice of B^* or T^*). The corresponding subset $S_{[b_\mu]} \subset \mathscr{S}_K(\bar{\mathbb{F}}_p)$ is the *μ-ordinary stratum.* By (1.3.9) and (1.3.13), this stratum is a non-empty Zariski open subspace.

Corollary 1.3.15. *Suppose that the special fiber $\mathscr{S}_{K,k(v)}$ is locally integral. Then $S_{[b_\mu]}$ is dense in $\mathscr{S}_{K,k(v)}$.*

Proof. If the special fiber is locally integral, it follows from [MP15, Cor. 4.1.11] that every connected component of $\mathscr{S}_K$ has irreducible special fiber. This implies that $G(\mathbb{A}_f^p)$ acts transitively on the set of irreducible components of $\mathscr{S}_{K,k(v)}$. So the result follows from the non-emptiness of $S_{[b_\mu]}$. □

Remark 1.3.16. In the situation where $p > 2$ and K_p is hyperspecial, so that it is of the form $G_{(p)}(\mathbb{Z}_p)$ for a reductive model $G_{(p)}$ of G over $\mathbb{Z}_{(p)}$, the main theorem of [Kis10] shows that $\mathscr{S}_{K,k(v)}$ is smooth. So the corollary applies to give the density of the μ-ordinary locus in this situation. This special case is already known due to D. Wortmann [Wor13].

Using the results of one of us and Pappas, we can prove the following:

Corollary 1.3.17. *Suppose that $p > 2$, and that G splits over a tamely ramified extension. Then the embedding $G \hookrightarrow \mathcal{G}_V$ can be chosen such that $S_{[b_\mu]}$ is dense in $\mathscr{S}_{K,k(v)}$.*

Proof. For $K_p \subset K'_p \subset G(\mathbb{Q}_p)$ a compact open subgroup the map $\mathscr{S}_{K_pK^p} \to \mathscr{S}_{K'_pK^p}$ is finite and surjective, and $S_{[b_\mu]}$ is the preimage of the corresponding subset of $\mathscr{S}_{K'_pK^p}$. Hence it suffices to show the statement for some compact open K_p. When K_p is a special parahoric, this follows from [KP15, Cor. 0.3]. $\square$

2. CM lifts and independence of ℓ

2.1. Tate's theorem with additional structures.

2.1.1. We keep the notation introduced in §1.3, so that (G, X) is a Shimura datum of Hodge type, equipped with an embedding of Shimura data $(G, X) \hookrightarrow (\mathcal{G}_V, \mathcal{H}_V)$, and we have a finite map $\mathscr{S}_K \to \mathscr{O}_{E,(v)} \otimes \mathcal{S}_{\mathcal{K}}$, which is an embedding on generic fibres.

We set

$$\mathcal{S}_{\mathcal{K}_p} = \varprojlim_{\mathcal{K}^p \subset \mathcal{G}_V(\mathbb{A}_f^p)} \mathcal{S}_{\mathcal{K}^p\mathcal{K}_p} \; ; \; \mathscr{S}_{K_p} = \varprojlim_{K^p \subset G(\mathbb{A}_f^p)} \mathscr{S}_{K^pK_p}.$$

The transition maps in the inverse systems are finite étale, and so the limits are schemes over $\mathbb{Z}_{(p)}$ (resp. $\mathscr{O}_{E,(v)}$). By construction, we have a map

$$\iota_p : \mathscr{S}_{K_p} \to \mathscr{O}_{E,(v)} \otimes \mathcal{S}_{\mathcal{K}_p}.$$

The scheme $\mathcal{S}_{\mathcal{K}_p}$ is open and closed in the moduli space of triples $(A, \lambda, \varepsilon)$, where (A, λ) is a polarized abelian scheme up to prime-to-p isogeny and

$$\varepsilon : \underline{\mathbb{A}}_f^p \otimes V \xrightarrow{\simeq} \widehat{V}^p(A)$$

is an isomorphism of smooth $\mathbb{A}_f^p$-sheaves carrying the symplectic form ψ to an $\underline{\mathbb{A}}_f^{p,\times}$-multiple of the Weil pairing $\widehat{T}^p(\lambda)$ on the prime-to-p Tate module

$$\widehat{V}^p(A) = (\varprojlim_{p \nmid n} A[n]) \otimes \mathbb{Q}.$$

2.1.2. Let $\mathrm{Sh}_{K_p} = E \otimes_{\mathscr{O}_{E,(v)}} \mathscr{S}_{K_p}$ be the generic fiber. For each α, let $s_{\alpha,\mathbb{A}_f^p}$ be the projection of $s_{\alpha,\text{ét}}$ onto $H^0\big(\mathrm{Sh}_{K_p}, (\widehat{V}^p(\mathcal{A}))^{\otimes}\big)$. Since $\mathscr{S}_{K_p}$ is normal, $s_{\alpha,\mathbb{A}_f^p}$ extends to a section over $\mathscr{S}_{K_p}$.

Over $\mathscr{S}_{K_p}$, ι_p induces an isomorphism

$$\eta : \mathbb{A}_f^p \otimes V \xrightarrow{\simeq} \widehat{V}^p(\mathcal{A}) \tag{2.1.2.1}$$

carrying s_α to $s_{\alpha,\mathbb{A}_f^p,s_0}$ for every α. In particular, for any $s_0 \in \mathscr{S}_{K_p}(\bar{\mathbb{F}}_p)$, the stabilizer in $\mathrm{GL}\big(\widehat{V}^p(\mathcal{A}_{s_0})\big)$ of the collection $\{s_{\alpha,\mathbb{A}_f^p,s_0}\}$ is canonically identified with $G_{\mathbb{A}_f^p}$.

2.1.3. Let $\underline{\mathrm{Aut}}_{\mathbb{Q}}(\mathcal{A}_{s_0})$ be the algebraic group over $\mathbb{Q}$ attached to the group of units in the endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(\mathcal{A}_{s_0}) \coloneqq \mathbb{Q} \otimes \mathrm{End}(\mathcal{A}_{s_0})$: Over $\mathbb{R}$, it is compact modulo the scalars. Therefore, it and any of its closed subgroups are reductive groups over $\mathbb{Q}$.

Now, $\underline{\mathrm{Aut}}_{\mathbb{Q}}(\mathcal{A}_{s_0})$ acts naturally on $\widehat{V}^p(\mathcal{A}_{s_0})$ and D_{s_0}. Let $I^p_{s_0} \subset \underline{\mathrm{Aut}}_{\mathbb{Q}}(\mathcal{A}_{s_0})$ be the closed subgroup that fixes the tensors $\{s_{\alpha,\mathbb{A}_f^p,s_0}\} \subset \widehat{V}^p(\mathcal{A}_{s_0})^{\otimes}$, and let $I_{s_0} \subset I^p_{s_0}$ be the largest closed subgroup that also fixes the tensors $\{s_{\alpha,\mathrm{cris},s_0}\} \subset D_{s_0}^{\otimes}$.

Since $G(\mathbb{A}_f^p)$ acts naturally on the right on $\mathcal{S}_{\mathcal{K}_p}$ and Sh_{K_p}, compatibly with the map ι_p, this action extends to $\mathscr{S}_{K_p}$.

Recall that $\mathcal{A}_{s_0}$ is an abelian variety up to prime-to-p isogeny (so the notion of automorphism is understood accordingly). Set

$$I_{s_0}(\mathbb{Z}_{(p)}) = I_{s_0}(\mathbb{Q}_p) \cap \mathrm{Aut}(\mathcal{A}_{s_0}).$$

We can view this as a subgroup of $G(\mathbb{A}_f^p)$ via the embeddings:

$$I_{s_0}(\mathbb{Z}_{(p)}) \subset I_{s_0}(\mathbb{A}_f^p) \subset I_{s_0}^p(\mathbb{A}_f^p) \subset G(\mathbb{A}_f^p).$$

Lemma 2.1.4. *Suppose that $g_1, g_2 \in G(\mathbb{A}_f^p)$ are such that*

$$s_0 \cdot g_1 = s_0 \cdot g_2 \in \mathscr{S}_{K_p}(\bar{\mathbb{F}}_p).$$

Then g_1 and g_2 have the same image in $I_{s_0}(\mathbb{Z}_{(p)})\backslash G(\mathbb{A}_f^p)$.

Proof. The proof is essentially contained in [Kis17, 2.1.3].

The image of s_0 in $\mathcal{S}_{\mathcal{K}_p}(\bar{\mathbb{F}}_p)$ corresponds to the triple $(\mathcal{A}_{s_0}, \lambda_{s_0}, \varepsilon_{s_0})$ over $\bar{\mathbb{F}}_p$ under the moduli interpretation of $\mathcal{S}_{\mathcal{K}_p}$. For $g \in G(\mathbb{A}_f^p)$, the image of $s_0 \cdot g$ in $\mathcal{S}_{\mathcal{K}_p}(\bar{\mathbb{F}}_p)$ corresponds to the triple $(A_{s_0}, \lambda_{s_0}, \varepsilon_{s_0} \circ g)$. Therefore, if $s_0 \cdot g_1 = s_0 \cdot g_2$, then in particular, we have an isomorphism of triples:

$$(A_{s_0}, \lambda_{s_0}, \varepsilon_{s_0} \circ g_1) \xrightarrow{\simeq} (A_{s_0}, \lambda_{s_0}, \varepsilon_{s_0} \circ g_2).$$

This corresponds to an automorphism $\phi \in \mathrm{Aut}(\mathcal{A}_{s_0})$ (necessarily unique) such that

$$\widehat{V}^p(\phi) \circ \varepsilon_{s_0} \circ g_1 = \varepsilon_{s_0} \circ g_2$$

and ϕ carries $\{s_{\alpha,\mathrm{cris},s_0\cdot g_1}\} \subset D^{\otimes}_{s_0\cdot g_1}$ to $\{s_{\alpha,\mathrm{cris},s_0\cdot g_2}\} \subset D^{\otimes}_{s_0\cdot g_2}$. Note that here we are using (1.3.6).

The first condition implies that $\widehat{V}^p(\phi)$ fixes $\{s_{\alpha,\mathbb{A}_f^p,s_0}\}$. Since under the natural identifications $D_{s_0} = D_{s_0\cdot g_i}$ induced by the identifications $\mathcal{A}_{s_0} = \mathcal{A}_{s_0\cdot g_i}$, for $i = 1, 2$, the tensors $\{s_{\alpha,\mathrm{cris},s_0}\}$ are carried to $\{s_{\alpha,\mathrm{cris},s_0\cdot g_i}\}$, ϕ preserves the $\{s_{\alpha,\mathrm{cris},s_0}\}$. Hence ϕ must belong to

$$I_{s_0}(\mathbb{Z}_{(p)}) = I_{s_0}(\mathbb{Q}) \cap \mathrm{Aut}(\mathcal{A}_{s_0}).$$

□

2.1.5. Choose a neat compact open $K^p \subset G(\mathbb{A}_f^p)$. Set $K = K_pK^p$, and suppose that the image of s_0 in $\mathscr{S}_K(\bar{\mathbb{F}}_p)$ is defined over $\mathbb{F}_q$.

Then, for any $m \in \mathbb{Z}_{\geq 1}$, let γ_{m,s_0} denote the geometric, q^m-power Frobenius of $\mathcal{A}_{s_0}$. Then γ_{m,s_0} fixes the absolute Hodge cycle components $\{s_{\alpha,\mathbb{A}_f^p,s_0}\}$, and it fixes the crystalline components $\{s_{\alpha,\mathrm{cris},s_0}\}$ as these are φ-invariant. Hence $\gamma_{m,s_0} \in I_{s_0}(\mathbb{Q})$. In particular, γ_{m,s_0} induces a semi-simple automorphism γ^p_{m,s_0} of $\widehat{V}^p(\mathcal{A}_{s_0})$ which preserves $\{s_{\alpha,\mathbb{A}_f^p,s_0}\}$, and thus lies in $G(\mathbb{A}_f^p)$. Set

$$I_{\mathbb{A}_f^p,m,s_0} = \mathrm{Cent}_{G_{\mathbb{A}_f^p}}(\gamma^p_{m,s_0}).$$

If $m \mid m'$, then $\gamma^p_{m',s_0} = (\gamma^p_{m,s_0})^{m'/m}$, and so we have a natural inclusion:

$$I_{\mathbb{A}_f^p,m,s_0} \subset I_{\mathbb{A}_f^p,m',s_0}.$$

Set

$$I_{\mathbb{A}_f^p,s_0} = \varinjlim_m I_{\mathbb{A}_f^p,m,s_0}.$$

Then, for m sufficiently divisible, we have $I_{\mathbb{A}_f^p,s_0} = I_{\mathbb{A}_f^p,m,s_0}$, which is independent of choice of q.

For each $\ell \neq p$, write I_{ℓ,s_0} for the projection of $I_{\mathbb{A}_f^p,s_0}$ onto $G_{\mathbb{Q}_\ell}$: For m sufficiently divisible, this is the centralizer in $G_{\mathbb{Q}_\ell}$ of the projection γ_{m,ℓ,s_0} of γ^p_{m,s_0}.

For m sufficiently divisible, the Zariski closure in $G_{\mathbb{A}_f^p}$ of the subgroup generated by γ^p_{m,s_0} is a torus. Therefore, I_{ℓ,s_0} is a Levi subgroup of $G_{\mathbb{Q}_\ell}$ (over $\bar{\mathbb{Q}}_\ell$) and in particular connected, reductive.

The action of I_{s_0} on $\widehat{V}^p(\mathcal{A}_{s_0})$ gives us a canonical map of $\mathbb{A}_f^p$-groups:

$$\mathbb{A}_f^p \otimes I_{s_0} \to I_{\mathbb{A}_f^p,s_0}.$$

For each $\ell \neq p$, this gives us a map $i_\ell : \mathbb{Q}_\ell \otimes I_{s_0} \to I_{\ell,s_0}$, which is injective.

Proposition 2.1.6. *Let $\ell \neq p$ be a prime such that $G_{\mathbb{Q}_\ell}$ is split and such that the characteristic polynomial of γ_{m,ℓ,s_0} is split over $\mathbb{Q}_\ell$. Then i_ℓ is an isomorphism.*

Proof. By (2.1.4), we have surjective maps

$$G(\mathbb{A}_f^p) \to s_0 \cdot G(\mathbb{A}_f^p) \to I_{s_0}(\mathbb{Z}_{(p)})\backslash G(\mathbb{A}_f^p),$$

where the first map is the orbit map $g \mapsto s_0 \cdot g$, and the composite is the natural projection.

For any neat compact open $K^p \subset G(\mathbb{A}_f^p)$ with ℓ-primary factor $K_\ell \subset G(\mathbb{Q}_\ell)$, this implies that the image in $\mathscr{S}_K(\bar{\mathbb{F}}_p)$ of $s_0 \cdot I_{\ell,s_0}(\mathbb{Q}_\ell)$ surjects onto $I_{s_0}(\mathbb{Q}_\ell)\backslash I_{\ell,s_0}(\mathbb{Q}_\ell)/(K_\ell \cap I_{\ell,s_0}(\mathbb{Q}_\ell))$. Since $I_{\ell,s_0}(\mathbb{Q}_\ell)$ commutes with γ_{m,ℓ,s_0} for m sufficiently divisible, this image is in fact contained in $\mathscr{S}_K(\mathbb{F}_{q^m})$. In particular $I_{s_0}(\mathbb{Q}_\ell)\backslash I_{\ell,s_0}(\mathbb{Q}_\ell)/(K_\ell \cap I_{\ell,s_0}(\mathbb{Q}_\ell))$ is finite.

The proposition is now deduced just as in [Kis17, 2.1.7]. □

2.2. **Independence of ℓ and conjugacy classes.**

2.2.1. Let T be a torus over a field k. We say that $x \in T(k)$ is *neat* if the Zariski closure of $\langle x\rangle$, the group of points generated by x, is connected (that is a torus).

Lemma 2.2.2. *Let T be a torus over a number field k, $H \subset T(k)$ a Zariski dense subgroup, $W \subset \mathrm{Aut}_k T$ a finite group of automorphisms, and $\sigma \in \mathrm{Aut}_k T$. Suppose that for every $h \in H$ there exists $w \in W$ such that $w(h) = \sigma(h)$. Then $\sigma \in W$.*

Proof. After replacing k by an extension we may assume that T is a split torus. We may assume that H is finitely generated: otherwise there exists an infinite strictly increasing sequence of multiplicative subgroups $T_1 \subsetneq T_2 \subsetneq \cdots$ of T, with each T_i the Zariski closure of a subgroup of H. For i large enough the dimension of T_i must be independent of i, so for $j > i$, $T_j/T_i \subset T/T_i$ is a torsion subgroup generated by k-points. But the torsion subgroup of $T/T_i(k)$ is finite since k is a number field, which implies that our sequence is eventually constant.

Let $x_1, \dots, x_s \in H$ be a family of generators. Note that for any positive integer n, the subgroup generated by $x_1^n, \dots, x_s^n$ in $T(k)$ is also Zariski dense in T, as it has finite index in H and T is connected. Let $n = |\mu(k)|$, where $\mu(k)$ denotes the subgroup of roots of unity in k. Then for any $x \in T(k)$, x^n is neat. Indeed otherwise, there is a character $\chi \in X^*(T)$ with $\chi(x^n)$ a non-trivial root of unity, which implies $\chi(x)$ is a root of unity in $\mu(k)$ and hence $\chi(x^n) = 1$, a contradiction. Thus we may assume that $x_1, \dots, x_s$ are neat.

Next we show that for $i = 1, \dots, s$:

(2.2.2.1) There is a $w_i \in W$ such that $w_i(x_j) = \sigma(x_j)$ for $j = 1, \dots, i$.

(2.2.2.2) There is a neat element $s_i = x_1^{a_1} \dots x_i^{a_i}$ where $a_1 > a_2 > \cdots > a_i > 0$ such that if $w \in W$ satisfies $w(s_i) = s_i$ then $w(x_j) = x_j$ for $j = 1, \dots, i$.

We proceed by induction on i, the case $i = 1$ being evident. We may replace σ by $\sigma \circ w_{i-1}^{-1}$ and assume that σ fixes x_j for $j = 1, \dots, i-1$. For $m \geq 1$ there exists $w^m \in W$ such that $w^m(x_i s_{i-1}^m) = \sigma(x_i s_{i-1}^m) = \sigma(x_i) s_{i-1}^m$. Since W is finite, there exist integers $m < m'$ such that $w := w^m = w^{m'}$. Then $w(s_{i-1}^{m'-m}) = s_{i-1}^{m'-m}$. Since s_{i-1} is neat, the groups generated by s_{i-1} and $s_{i-1}^{m'-m}$ have the same Zariski closure, so this implies $w(s_{i-1}) = s_{i-1}$ and hence $w(x_j) = x_j = \sigma(x_j)$ for $j = 1, \dots, i-1$, and $w(x_i) = \sigma(x_i)$. Thus $w = w_i$ satisfies condition (2.2.2.1) above.

To construct s_i, note that, since W is a finite group, there exists $m > 0$ such that if $w \in W$ fixes $x_i^n s_{i-1}^{mn}$ then w fixes $x_i^n s_{i-1}^{m'n}$ for some $m' > m$. Then a calculation, as above, shows that any such w fixes x_i^n and s_{i-1} and hence $x_1, \dots, x_{i-1}, x_i$ Thus $s_i = x_i^n s_{i-1}^{mn}$ satisfies (2.2.2.2).

Finally, applying (2.2.2.1) with $i = s$, the proposition follows as H is Zariski dense in T. □

2.2.3. Let T be a torus over $\mathbb{Q}$ which satisfies the *Serre condition*. Recall that this means that the maximal $\mathbb{R}$-split sub-torus $T_1 \subset T$ is $\mathbb{Q}$-split. We say that a point $x \in T(\mathbb{Q})$ is a Weil point if it is a unit outside p. This means that for $v \neq p$ a finite prime of $\mathbb{Q}$, the subgroup of $T(\mathbb{Q})$ generated by x is bounded in $T(\mathbb{Q}_v)$, or equivalently, that for $\chi \in X^*(T)$, $\chi(x)$ is a p-unit. (See the proof of Lemma 2.2.4 below.)

We remark that the definition of a Weil point in $T(\mathbb{Q})$ makes sense, and Lemma 2.2.6 below holds without assuming that T satisfies the Serre condition. However without this assumption, such points do not deserve to be called Weil points - cf. Lemma 2.2.4 below.

Let $q = p^r$ for some integer $r \geq 1$ and $w \in \mathbb{Z}$. An $\alpha \in \bar{\mathbb{Q}}$, is called a q-Weil number of weight w if α is a p-unit and all its complex embeddings have absolute value $q^{w/2}$. We say that α is a Weil number if it is a q-Weil number of weight w for some q and w. Note that we do not insist that α is an algebraic integer.

We say that a point of $x \in \mathrm{GL}_n$, defined over some field of characteristic 0 is a Weil point if each eigenvalue of x is a Weil number (in particular algebraic over $\mathbb{Q}$). We say that x is a q-Weil point of weight w if each eigenvalue is a q-Weil number of weight w.

Lemma 2.2.4. *Let T be a torus over $\mathbb{Q}$ which satisfies the Serre condition. Then $x \in T(\mathbb{Q})$ is a Weil point if and only if for each $\chi \in X^*(T)$, $\chi(x)$ is a Weil number.*

Proof. Let F be a finite extension of $\mathbb{Q}$ such that T splits over F. For any finite place v of F, the subgroup $\langle x\rangle \subset T(\mathbb{Q})$ is bounded in $T(F_v)$ if and only if $\langle\chi(x)\rangle$ is bounded in $F_v^\times$ for any $\chi \in X^*(T)$. So x is a unit outside p if and only if $\chi(x)$ is a p-unit for all χ. This implies the "if" part of the lemma.

Now suppose that $x \in T(\mathbb{Q})$ is a Weil point. There is an isogeny $T \to T/T_1 \times T_1 = T'$ which induces an isogeny on T_1, and the identity on T/T_1. Let s be the degree of this isogeny. Then for every $\chi \in X^*(T)$, $\chi^s \in X^*(T')$. Let $y = (y_0, y_1) \in T/T_1 \times T_1$ be the image of x. Since $T/T_1(\mathbb{R})$ is compact, $\chi^s(y_0)$ has absolute value 1 for every complex embedding, while $\chi^s(y_1) \in \mathbb{Q}^\times$. It follows that $\chi(x)$ has the same absolute value under all complex embeddings.

Let $T_2 \subset T$ be the maximal subtorus such that $T_2(\mathbb{R})$ is compact. Then T_2 is defined over $\mathbb{Q}$. If we think of χ as defined over $\mathbb{C}$, then $\chi\bar{\chi}$ is trivial on T_2, and factors through T/T_2. Hence $\chi\bar{\chi}(x) \in \mathbb{Q}^\times$ is a p-unit and equal to p^w for some integer w. This shows that $\chi(x)$ has absolute value $p^{w/2}$ under all complex embeddings. $\square$

Corollary 2.2.5. *Let T be a torus over $\mathbb{Q}$ which satisfies the Serre condition, and $i : T \hookrightarrow GL_n$ an embedding defined over some field of characteristic 0. Then $x \in T(\mathbb{Q})$ is a Weil point if and only if $i(x)$ is a Weil point. If the maximal $\mathbb{Q}$-split subtorus $T_1 \subset T$ has rank at most 1, and $i(T_1)$ is central, then $i(x)$ is a Weil point if and only if $i(x)$ is a p-Weil point of weight w for some $w \in \mathbb{Z}$.*

Proof. The action of T on the $\mathbb{Q}$-vector space $\mathbb{Q}^n$ can be diagonalized over some extension. Thus there exist $\chi_1, \dots, \chi_n \in X^*(T)$ such that $\mathbb{Q}^n$ is a sum of the χ_j as a T-representation. Since i is injective the χ_j span $X^*(T)$. The eigenvalues of $i(x)$ are given by $\chi_j(x)$. Hence by Lemma 2.2.4, $x \in T(\mathbb{Q})$ is a Weil point if and only if $\chi_j(x)$ is a Weil number for each j.

Now suppose that $T_1 \subset T$ has rank at most 1, and $i(T_1)$ is central. As in the proof of Lemma 2.2.4, there is an integer $s > 0$ such that for each j, χ_j^s factors through $T/T_1 \times T_1$. Write $\chi_j^s = (\eta_{j,0}, \eta_{j,1}) \in X^*(T/T_1) \times X^*(T_1)$. Since $i(T_1)$ is central $\eta_{j,1}$ does not depend on j. Thinking of the χ_j as complex valued, we have $\chi_j^s\bar{\chi}_j^s = (1, \eta_{j,1}^2)$ which shows that the absolute value of $\chi_j(x)$ does not depend on j or the chosen complex embedding. $\square$

Lemma 2.2.6. *The set of Weil points forms a Zariski dense subgroup of T.*

Proof. It is clear that the Weil points form a subgroup. We show that it is Zariski dense. We may suppose that T has positive dimension.

First note that $T(\mathbb{Q}_p)$ is a Zariski dense subgroup of T - for example using the p-adic exponential map - with a finite torsion subgroup. Let $y_p \in T(\mathbb{Q}_p)$ be a non-torsion point and let $y \in T(\mathbb{A}_f)$ be the point with component y_p at p and trivial components away from p. For any compact open subgroup $K_T \subset T(\mathbb{A}_f)$ the quotient $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_T$ is finite. Hence there exists $x \in T(\mathbb{Q})$ and a positive integer m with $x = y^m \bmod K_T$. Then x is a Weil point, and for K_T sufficiently small x is not a torsion point, as y_p^m is not a torsion point.

Let T' be the connected component of the identity of the Zariski closure of the Weil points in T. If $T'' = T/T'$, then there exists an isogeny $\varepsilon : T' \times T'' \to T$ which is the identity on T', and induces an isogeny of T''. Note that T'' again satisfies the Serre condition and that an isogeny takes Weil points to Weil points. By what we saw above, if T'' is non-trivial, then there exists a Weil point $x \in T''(\mathbb{Q})$ of infinite order. Then $\varepsilon(x)$ is a Weil point of infinite order, contradicting, that some power of $\varepsilon(x)$ is contained in T'. This shows that T'' is trivial, and $T' = T$. $\square$

2.2.7. We now return to the assumptions and notation of 2.1.

Fix a point $s_0 \in \mathscr{S}_{K_p}(\bar{\mathbb{F}}_p)$ that is the pre-image of an $\mathbb{F}_q$-valued point on $\mathscr{S}_K$. Recall that we have defined for $\ell \neq p$ a subgroup $I_{\ell,s_0} \subset G_{\mathbb{Q}_\ell}$ as well as a map $i_\ell : I_{s_0} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to I_{\ell,s_0}$ which is an isomorphism for at least one ℓ by Proposition 2.1.6. (Such an ℓ exists by Cebotarev density.) We now define an analogue of I_{ℓ,s_0} for $\ell = p$.

Fix some neat level K. For any $m \in \mathbb{Z}_{>0}$, the crystalline realization of $\mathcal{A}_{s_0}$ is defined over $\mathbb{Q}_{q^m} = W(\mathbb{F}_{q^m})[p^{-1}]$; therefore, the isocrystal D_{s_0} has a natural descent to an F-isocrystal D_{m,s_0} over $\mathbb{Q}_{q^m}$, and the φ-invariant tensors $\{s_{\alpha,\mathrm{cris},s_0}\}$ belong to $D^{\otimes}_{m,s_0}$. Write $q = p^r$ and let $\gamma_{m,\mathrm{cris},s_0} = \varphi^{rm} : D_{m,s_0} \to D_{m,s_0}$ be the crystalline realization of γ_{m,s_0}. It is a φ-equivariant isomorphism fixing the tensors $\{s_{\alpha,\mathrm{cris},s_0}\}$.

As in 1.3.7, for m sufficiently divisible (which we now assume) we can find an isomorphism:

$$\mathbb{Q}_{q^m} \otimes V \xrightarrow{\simeq} D_{m,s_0} \tag{2.2.7.1}$$

carrying, for each α, $1 \otimes s_\alpha$ to $s_{\alpha,\mathrm{cris},s_0}$. Let $\delta_{s_0} \in G(\mathbb{Q}_{q^m})$ be such that $\varphi : D_{m,s_0} \to D_{m,s_0}$ pulls back to the automorphism $\delta_{s_0}(\sigma \otimes 1)$ of $\mathbb{Q}_{q^m} \otimes V$ under this isomorphism. Then, by construction, the class $[b_{s_0}] \in B(G_{\mathbb{Q}_p})$ associated with s_0 is exactly the σ-conjugacy class of δ_{s_0}.

Similarly, the automorphism $\gamma_{m,\mathrm{cris},s_0}$ of D_{m,s_0} pulls back to an element $\gamma_{m,p,s_0} \in G(\mathbb{Q}_{q^m})$, whose conjugacy class under $\varinjlim_m G(\mathbb{Q}_{q^m})$ is independent of all choices.

We have the relation:

$$\gamma_{m,p,s_0} = \delta_{s_0}\sigma(\delta_{s_0}) \cdots \sigma^{rm-2}(\delta_{s_0})\sigma^{rm-1}(\delta_{s_0}) \in G(\mathbb{Q}_{q^m}). \tag{2.2.7.2}$$

Define an algebraic group $I_{m,\delta_{s_0}}$ over $\mathbb{Q}_p$ as follows: For any $\mathbb{Q}_p$-algebra R, we have:

$$I_{m,\delta_{s_0}}(R) = \{g \in G(\mathbb{Q}_{q^m} \otimes_{\mathbb{Q}_p} R) : \ g\delta_{s_0} = \delta_{s_0}\sigma(g)\}.$$

Then $\mathbb{Q}_{q^m} \otimes_{\mathbb{Q}_p} I_{m,\delta_{s_0}}$ is naturally identified with the stabilizer in $G_{\mathbb{Q}_{q^m}}$ of γ_{m,p,s_0}. In particular, $I_{m,\delta_{s_0}}$ is a reductive group over $\mathbb{Q}_p$, and is connected for m sufficiently divisible. Set:

$$I_{p,s_0} = \varinjlim_m I_{m,\delta_{s_0}},$$

which is equal to $I_{m,\delta_{s_0}}$ for m sufficiently divisible. We have a canonical inclusion $i_p : I_{s_0} \otimes_{\mathbb{Q}} \mathbb{Q}_p \hookrightarrow I_{p,s_0}$, and an inclusion $I_{p,s_0} \hookrightarrow G$ defined over $\mathbb{Q}_{q^m}$ for m sufficiently divisible.

Let $J_{\delta_{s_0}}$ be the $\mathbb{Q}_p$-group defined in (1.1.4). For any m, we have the obvious inclusion $I_{m,\delta_{s_0}} \subset J_{\delta_{s_0}}$, and in particular $I_{p,s_0} \subset J_{\delta_{s_0}}$.

2.2.8. Given a connected reductive group H over a field F of characteristic 0, write $\mathrm{Conj}(H)$ for the scheme over F parameterizing conjugacy classes in H.

Following Noot [Noo09, 1.5], we will also define a certain quotient $\mathrm{Conj}'(H)$ of $\mathrm{Conj}(H)$ as follows: Let $\bar{F}$ be an algebraic closure of F; then $H^{\mathrm{der}}_{\bar{F}}$ is an almost direct product of simple reductive factors H_i with i in some indexing set I.

Write $I_D \subset I$ for the subset of indices i such that $H_i \simeq \mathrm{SO}(2n_i)$ for some $n_i \geq 4$. For each $i \in I_D$, set $H'_i = \mathrm{O}(2n_i)$. Since $I_D \subset I$ is $\mathrm{Gal}(\bar{F}/F)$-stable, the finite $\bar{F}$-group scheme

$$\mathrm{Out}'(H)_{\bar{F}} = \prod_{i \in I_D} H'_i/H_i$$

descends to a finite group scheme $\mathrm{Out}'(H)$ over F, which acts canonically on $\mathrm{Conj}(H)$. We will write $\mathrm{Conj}'(H)$ for the quotient of $\mathrm{Conj}(H)$ for this action.

Suppose that C is an algebraic group over F, and that we are given two maps $i_1, i_2 : C \to H$ defined over some extensions F_1, F_2 respectively, of F. We say that i_1 and i_2 are conjugate (resp. conjugate in $\mathrm{Conj}'(H)$) if there exists an extension F_3/F containing F_1 and F_2 and $g \in \mathrm{Conj}(H)(F_3)$ (resp. $g \in \mathrm{Conj}'(H)(F_3)$) such that $i_2 = g i_1 g^{-1}$.

Proposition 2.2.9. *Let $T \subset I_{s_0}$ be a maximal torus, and consider for each prime ℓ (including $\ell = p$) the composite*

$$i^T_\ell : T \hookrightarrow I_{s_0} \to G$$

defined over $\mathbb{Q}_\ell$ if $\ell \neq p$ and over $\mathbb{Q}_{q^m}$ for m sufficiently divisible if $\ell = p$. Then the maps i^T_ℓ are all conjugate in $\mathrm{Conj}'(G)$. In particular, if G^{ad} has no factors of type D then the i^T_ℓ are all conjugate.

Proof. We consider all maps of groups over an algebraically closed field k containing all $\mathbb{Q}_\ell$ for $\ell \neq p$ and $\mathbb{Q}_{q^m}$ for all m. We write $i_\ell = i^T_\ell$.

For any positive integer r denote by $[r]$ multiplication by r on $\mathcal{A}_{s_0}$. For any $\gamma \in I_{s_0}(\mathbb{Q})$, there exists an $r > 0$ such that $[r] \circ \gamma$ induces an endomorphism of $\mathcal{A}_{s_0}$, which has no eigenvalues equal to 1 on $\mathrm{Lie}\,\mathcal{A}_{s_0}$. This implies that $[r] \circ \gamma$ acting on $\mathcal{A}_{s_0}$ has isolated fixed points. It follows from the Lefschetz fixed point formula in étale and crystalline cohomology that the characteristic polynomial of $i_\ell(\gamma)$ as an element of $\mathrm{GL}(V)$ does not depend on ℓ. Here we are using that the for any of these cohomology theories and r large enough, the eigenvalues of $[r] \circ \gamma$ in different cohomological degrees are distinct.

Next consider a neat Weil point $\gamma \in T(\mathbb{Q})$. Since I_{s_0} is compact modulo the central subtorus consisting of scalars, T satisfies the conditions in the second part of Corollary 2.2.5 so γ is a p-Weil point of weight w for some $w \in \mathbb{Z}$. Hence, a result of Noot [Noo09, Lem. 2.6] implies that the elements $i_\ell(\gamma)$ in G are all conjugate by points of $\mathrm{Conj}'(G)(k)$. (Note that Noot's notion of Weil point corresponds to our notion of p-Weil point of weight w.) More precisely, Noot's result applies if the eigenvalues of $i_\ell(\gamma)$ are algebraic integers, hence it applies to $[p^i] \circ \gamma$ for i large enough, and the result for γ follows immediately from that for $[p^i] \circ \gamma$.

Now let $T_0 \subset G$ be a maximal torus over k. We conjugate each i_ℓ by an element of $G(k)$ to obtain a map $i'_\ell : T \xrightarrow{\simeq} T_0$. Fix primes ℓ, ℓ' and set $\sigma = i'_{\ell'} \circ i'^{-1}_\ell$. We denote by W the subgroup of $\mathrm{Aut}\,T$ consisting of automorphisms which are induced by an element of $\mathrm{Conj}'(G)(k)$. Then W is a finite group. Let $g \in \mathrm{Conj}'(G)(k)$ be an element which conjugates $i'_\ell(\gamma)$ to $i'_{\ell'}(\gamma)$, and let $M \subset G$ be the centralizer of $i'_{\ell'}(\gamma)$. Note that M is connected reductive, because γ is neat, and that $T_0, gT_0g^{-1} \subset M$ since $i'_{\ell'}(\gamma) \in T_0 \cap gT_0g^{-1}$. Hence there exists $m \in M(k)$ such that $mT_0m^{-1} = gT_0g^{-1}$. Replacing g by $m^{-1}g$, we may assume that $gT_0g^{-1} = T_0$, so that g induces a point of W. Thus, for any neat Weil point $\gamma \in T(\mathbb{Q})$ there exists $w \in W$ such that $w(\gamma) = \sigma(\gamma)$.

Now Lemma 2.2.6 tells us that the set of Weil points in $T(\mathbb{Q})$ is a Zariski dense subgroup. Hence the subgroup of neat Weil points is also Zariski dense, by the argument in the proof of Lemma 2.2.2. It follows by Lemma 2.2.2 that $\sigma \in W$. That is, i'_ℓ and $i'_{\ell'}$ are conjugate in $\mathrm{Conj}'(G)$. □

Corollary 2.2.10. *For every ℓ, the map*

$$i_\ell : \mathbb{Q}_\ell \otimes I_{s_0} \to I_{\ell,s_0}$$

is an isomorphism.

Proof. Fix a maximal torus T in I_{s_0} containing γ_{m,s_0}. Then $I_{\ell,s_0} \subset G_{\mathbb{Q}_\ell}$ is the centralizer of $i_\ell(\gamma_{m,\ell,s_0}) = i^T_\ell(\gamma_{m,s_0})$ for m sufficiently divisible. By Proposition 2.2.9, for two choices of ℓ, the elements $i^T_\ell(\gamma_{m,s_0})$ differ by an automorphism of G (defined over some sufficiently large field). In particular, the quantity $\dim I_{\ell,s_0}$ is independent of ℓ. Since i_ℓ is an isomorphism for *some* $\ell \neq p$ by Proposition 2.1.6, and since i_ℓ is injective for every ℓ, the corollary follows.

We remark that it is possible to prove this corollary by applying [Noo09, Thm 1.8] directly, rather than Proposition 2.2.9. □

Corollary 2.2.11. *Let $T \subset I_{s_0}$ be a maximal torus, and suppose that G is quasi-split at p and has no factors of type D. Then there is an embedding $i^T : T \hookrightarrow G$ which is conjugate to each of the embeddings i^T_ℓ. In particular, for each $m > 0$, there is an element $\gamma_{m,0,s_0} \in G(\mathbb{Q})$ conjugate to γ_{m,ℓ,s_0} in $G(\bar{\mathbb{Q}}_\ell)$ for each ℓ.*

Proof. Let G^* be the quasi-split inner form of G, and choose an inner twisting $G \xrightarrow{\simeq} G^*$ over $\bar{\mathbb{Q}}$. Let $i^{T*}_\ell : T \hookrightarrow G^*$ be the embedding over $\bar{\mathbb{Q}}_\ell$ induced by i^T_ℓ and the chosen inner twisting. Choose an embedding $\bar{i}^T : T \hookrightarrow G^*$ defined over $\bar{\mathbb{Q}}$ and conjugate to each of the i^{T*}_ℓ. For $\ell \neq p$, i^T_ℓ is defined over $\mathbb{Q}_\ell$ so the conjugacy class of i^{T*}_ℓ is invariant by $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. Hence, by Cebotarev density, the conjugacy class of $\bar{i}^T$ is invariant by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. It follows by [Kot82, Cor 2.2] that $\bar{i}^T$ is conjugate to an embedding $i^{T*} : T \hookrightarrow G^*$ defined over $\mathbb{Q}$. We view T as a subgroup of G^* via i^{T*}.

Now T transfers to G at every prime $\ell \neq p, \infty$ as i^T_ℓ is defined over $\mathbb{Q}_\ell$. It transfers to G at p, since G is quasi-split at p, and it transfers to G at infinity as the image of T in G^{ad} is anisotropic at infinity. Hence T transfers to G by [LR87, Lemma 5.6].

For the final statement, writing $i^T : T \hookrightarrow G$ for the transfer, we take $\gamma_{m,0,s_0} = i^T(\gamma_{m,s_0})$. □

2.3. CM lifts and the conjugacy class of Frobenius.

2.3.1. We again return to the notation and assumptions of §2.1. Let $s_0, s'_0 \in \mathscr{S}_{K_p}(\bar{\mathbb{F}}_p)$. Then s_0, s'_0 are defined over $\bar{\mathbb{F}}_q$ for some q, and we use the notation of 2.2.7.

Write $\underline{\mathrm{Hom}}_{\mathbb{Q}}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$ for the scheme over $\mathbb{Q}$ that assigns to any $\mathbb{Q}$-algebra R, the group $R \otimes \mathrm{Hom}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$. (Here the Hom-spaces are taken in the prime-to-p isogeny categories.) For any $\mathbb{Q}$-algebra R, an *R-isogeny from $\mathcal{A}_{s_0}$ to $\mathcal{A}_{s'_0}$* is an element

$$f \in \underline{\mathrm{Hom}}_{\mathbb{Q}}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})(R)$$

such that there exists

$$f' \in \underline{\mathrm{Hom}}_{\mathbb{Q}}(\mathcal{A}_{s'_0}, \mathcal{A}_{s_0})(R)$$

with $f' \circ f \in \underline{\mathrm{Aut}}_{\mathbb{Q}}(\mathcal{A}_{s_0})(R)$.

Let $\underline{\mathrm{Isog}}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$ be the functor on $\mathbb{Q}$-algebras that assigns to any $\mathbb{Q}$-algebra R the set of R-isogenies from $\mathcal{A}_{s_0}$ to $\mathcal{A}_{s'_0}$. Note that this functor is either empty or representable by a torsor over $\mathbb{Q}$ under $\underline{\mathrm{Aut}}_{\mathbb{Q}}(\mathcal{A}_{s_0})$.

2.3.2. For any prime $\ell \neq p$, denote by $V_\ell(\mathcal{A}_{s_0})$ be the ℓ-adic Tate module of $\mathcal{A}_{s_0}$, and let $\underline{\mathrm{Isog}}_\ell(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$ be the $\mathbb{Q}_\ell$-scheme that assigns to any $\mathbb{Q}_\ell$-algebra R the set of R-linear isomorphisms

$$R \otimes_{\mathbb{Q}_\ell} V_\ell(\mathcal{A}_{s_0}) \xrightarrow{\simeq} R \otimes_{\mathbb{Q}_\ell} V_\ell(\mathcal{A}_{s'_0})$$

that carry $1 \otimes \gamma_{m,\ell,s_0}$ to $1 \otimes \gamma_{m,\ell,s'_0}$ for all m sufficiently divisible.

For any $\ell \neq p$, cohomological realization gives us a natural map of $\mathbb{Q}_\ell$-schemes:

$$i_\ell(s_0, s'_0) : \mathbb{Q}_\ell \otimes \underline{\mathrm{Isog}}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0}) \to \underline{\mathrm{Isog}}_\ell(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0}).$$

Similarly, let $\underline{\mathrm{Isog}}(D_{s_0}, D_{s'_0})$ be the $\mathbb{Q}_p$-scheme that assigns to every $\mathbb{Q}_p$-algebra R the set of $1 \otimes \varphi$-equivariant, $R \otimes L$-linear isomorphisms $R \otimes_{\mathbb{Q}_p} D_{s'_0} \xrightarrow{\simeq} R \otimes_{\mathbb{Q}_p} D_{s_0}$ which carries D_{m,s_0} to D_{m,s'_0} for m sufficiently large. We have a natural map of $\mathbb{Q}_p$-schemes:

$$i_p(s_0, s'_0) : \mathbb{Q}_p \otimes \underline{\mathrm{Isog}}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0}) \to \underline{\mathrm{Isog}}(D_{s_0}, D_{s'_0}).$$

By Tate's theorem on endomorphisms of abelian varieties and its crystalline analogue, $i_\ell(s_0, s'_0)$ is an isomorphism for all ℓ.

2.3.3. For $\ell \neq p$ let $P_\ell(s_0, s'_0) \subset \underline{\mathrm{Isog}}_\ell(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$ (resp. $P_p(s_0, s'_0) \subset \underline{\mathrm{Isog}}(D_{s_0}, D_{s'_0})$) be the closed subscheme parameterizing isomorphisms that carry, for each α, $1 \otimes s_{\alpha,\ell,s_0}$ to $1 \otimes s_{\alpha,\ell,s'_0}$ (resp. $1 \otimes s_{\alpha,\mathrm{cris},s'_0}$ to $1 \otimes s_{\alpha,\mathrm{cris},s_0}$). Let $P(s_0, s'_0) \subset \underline{\mathrm{Isog}}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$ be the largest closed subscheme (defined over $\mathbb{Q}$) that maps into $P_\ell(s_0, s'_0)$ for every ℓ, including $\ell = p$. Note that $P(s_0, s'_0)$ is either empty or an I_{s_0}-torsor.

We make the following

Conjecture 2.3.4. *For every ℓ, the map*

$$P(s_0, s'_0) \otimes \mathbb{Q}_\ell \to P_\ell(s_0, s'_0)$$

induced by i_ℓ is an isomorphism.

When $s'_0 = s_0$ this is simply Corollary 2.2.10.

Lemma 2.3.5. *The schemes $P(s_0, s'_0)$ and $P_\ell(s_0, s'_0)$ depend only on s_0 and s'_0 and not on the choice of the collection of Hodge cycles $\{s_\alpha\}$. In particular, the truth of Conjecture 2.3.4 depends only on s_0, s'_0 and not on $\{s_\alpha\}$.*

If (G, X) is PEL of type A or C then Conjecture 2.3.4 holds.

Proof. From the definitions it suffices to prove the first statement for $P_\ell(s_0, s'_0)$ for each ℓ. If $\{t_\beta\}$ is another collection of Hodge cycles defining G, it suffices to consider the case $\{s_\alpha\} \subset \{t_\beta\}$. If $P_{\ell,1}(s_0, s'_0)$ is the analogue of $P_\ell(s_0, s'_0)$ defined using $\{t_\beta\}$ then $P_{\ell,1}(s_0, s'_0) \subset P_\ell(s_0, s'_0)$ and it suffices to show that if one scheme is non-empty then so is the other, as then each is an I_ℓ-torsor. However each scheme is non-empty if and only if $\gamma_{m,s_0,\ell}$ and $\gamma_{m,s'_0,\ell}$ are conjugate in $G(\bar{\mathbb{Q}}_\ell)$ (even for $\ell = p$).

Now suppose that (G, X) is PEL of type A or C. In this case G is the group preserving a collection of endomorphisms $\{t_\beta\}$ together with the polarization ψ up to a scalar. (Note that ψ does not have weight 0, so does not quite fit into our formalism involving $\{s_\alpha\}$.) Then ψ induces a pairing

$$V_\ell(\mathcal{A}_{s_0}) \times V_\ell(\mathcal{A}_{s_0}) \to \mathbb{Q}_\ell(1),$$

well defined up to a non-zero scalar, and similarly for D_{m,s_0}. We refer to these pairings as *polarizations.*

Define $P_1(s_0, s'_0)$ to be the subscheme of $\underline{\mathrm{Isog}}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$ which preserves the $\{t_\beta\}$ and polarizations up to a scalar. For $\ell \neq p$ let $P_{\ell,1}(s_0, s'_0) \subset \underline{\mathrm{Isog}}_\ell(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$ (resp. $P_{p,1}(s_0, s'_0) \subset \underline{\mathrm{Isog}}(D_{s_0}, D_{s'_0})$) be the closed subscheme parameterizing isomorphisms that carry, for each β,

$1 \otimes t_{\beta,\ell,s_0}$ to $1 \otimes t_{\beta,\ell,s'_0}$ (resp. $1 \otimes t_{\beta,\mathrm{cris},s'_0}$ to $1 \otimes t_{\beta,\mathrm{cris},s_0}$) and which preserve polarizations up to a scalar.

By Tate's theorem, for each ℓ the map

$$P_1(s_0, s'_0) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} P_{\ell,1}(s_0, s'_0)$$

is an isomorphism. An argument as in the proof of the first part of the lemma shows that this map can be identified with the map of Conjecture 2.3.4. □

2.3.6. The restriction that (G, X) be of type A or C in the lemma above is in some sense a question of definitions. When (G, X) is PEL of type D, one cannot actually define $G \subset \mathrm{GSp}(V)$ using endomorphisms and polarizations. Instead, there is a collection $\{t_\beta\} \subset V^{\otimes}$ of a polarization and endomorphisms which define a group $G' \subset \mathrm{GSp}(V)$ whose connected component is G [Kot92, p393]. An analogue of the last statement of the Lemma then holds for G'.

We will say that s_0 and s'_0 are $\bar{\mathbb{Q}}$-*isogenous* if the space $P(s_0, s'_0)$ of (2.3.3) is non-empty. We will say that they are *isogenous* if $P(s_0, s'_0)(\mathbb{Q})$ is non-empty. If $s_0, s'_0 \in \mathscr{S}_{K_p}(\bar{\mathbb{F}}_p)$ we will say that s'_0 and s_0 are $\bar{\mathbb{Q}}$-isogenous (resp. isogenous) if this condition holds when s_0, s'_0 are viewed as $\mathbb{F}_q$ points for some $q = p^r$.

2.3.7. Let $s_0 \in \mathscr{S}_{K_p}(\bar{\mathbb{F}}_p)$. Suppose that $T \subset I_{s_0}$ is a maximal torus. Let $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m \to G_\mathbb{R}$ be an $\mathbb{R}$-morphism. Let $\mathrm{Sh}_{K_{T,p}}(h)$ be the pro-Shimura variety associated with $(T, \{h\})$ and $K_{T,p} = K_p \cap T(\mathbb{Q}_p)$. An *isogeny CM lift* (resp. *a* $\bar{\mathbb{Q}}$-*isogeny CM lift*) of s_0 *with respect to* T will consist of a triple (j, x, s'_0), where:

- $j : T \hookrightarrow G$ is an embedding defined over $\mathbb{Q}$, such that for each ℓ, j is conjugate over $\bar{\mathbb{Q}}_\ell$ to the embedding
$$i_\ell : T_{\mathbb{Q}_\ell} \hookrightarrow I_{\ell,s_0} \hookrightarrow G_{\mathbb{Q}_\ell};$$
- $x \in X$ is a point with h_x factoring through $j(T_\mathbb{R})$; and
- $s'_0 \in \mathscr{S}_{K_p}(\bar{\mathbb{F}}_p)$ is a point admitting a lift to $\mathrm{Sh}_{K_{T,p}}(h_x)$;

such that s'_0 is isogenous (resp. $\bar{\mathbb{Q}}$-isogenous) to s_0.

Of course isogeny CM lifts can exist only when the i_ℓ are conjugate for all ℓ. We make the following conjecture:

Conjecture 2.3.8. *If G is quasi-split at p, then for any $s_0 \in \mathscr{S}_{K_p}(\bar{\mathbb{F}}_p)$ and any maximal torus $T \subset I_{s_0}$, s_0 admits an isogeny CM lift with respect to T.*

When K_p is hyperspecial this conjecture is proved in [Kis10]. The main point of this section is to show that Conjecture 2.3.4 implies a version of Conjecture 2.3.8 with $\bar{\mathbb{Q}}$-isogenies, when G^{ad} has no factors of type D. In particular, we will show a $\bar{\mathbb{Q}}$-version of this conjecture holds for (G, X) of PEL type A or C.

2.3.9. Let $T \subset G$ be a maximal torus and $x \in X$ with h_x factoring through T. Let $s_0 \in \mathscr{S}_{K_p}(\bar{\mathbb{F}}_p)$ be defined over $\mathbb{F}_q$ for some $q = p^r$. Suppose that s_0 is a reduction of a $\bar{\mathbb{Q}}_p$-valued point s of $\mathrm{Sh}_{K_{T,p}}(h_x)$.

For any $m \in \mathbb{Z}_{>0}$, the q^m-Frobenius acts on $\mathcal{A}_{s_0}$, and the corresponding automorphism $\gamma_{m,s_0} \in \underline{\mathrm{Aut}}_\mathbb{Q}(\mathcal{A}_{s_0})$ lies in $I_{s_0}(\mathbb{Q})$. Since T contains the Mumford-Tate group of $\mathcal{A}_s$ (defined via some embedding $\bar{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$), there are natural embeddings:

$$T \hookrightarrow \underline{\mathrm{Aut}}_\mathbb{Q}(\mathcal{A}_s) \hookrightarrow \underline{\mathrm{Aut}}_\mathbb{Q}(\mathcal{A}_{s_0}).$$

It follows from the definitions that this embedding exhibits T as a subtorus of I_{s_0}.

Recall that an element $\gamma \in T(\mathbb{Q})$ is called a p-unit if $\{\gamma^r\}_{r\in\mathbb{Z}}$ is contained in a compact subset of $T(\mathbb{Q}_\ell)$ for all $\ell \neq p$.

Lemma 2.3.10. *The element γ_{m,s_0} lies in $T(\mathbb{Q}) \subset I_{s_0}(\mathbb{Q})$. It has the following properties:*

(2.3.10.1) γ_{m,s_0} *is a p-unit.*

(2.3.10.2) Set $\mu = \mu_x^{-1} \in X_*(T)$. Under the composition

$$T(\mathbb{Q}) \to T(\mathbb{Q}_p) \to B(T) \xrightarrow[\simeq]{\kappa_T} X_*(T)_{\Gamma_p},$$

γ_{m,s_0} *is mapped to $m \log_p q \cdot \mu^\sharp$.*

Given any other element $\gamma \in T(\mathbb{Q})$ satisfying the two conditions above, there exists $r \in \mathbb{Z}_{>0}$ such that $\gamma_{m,s_0}^r = \gamma^r$.

Proof. Since γ_{m,s_0} is the q^m-Frobenius automorphism of $\mathcal{A}_{s_0}$, it is central in $I_{s_0}(\mathbb{Q})$. Hence $\gamma_{m,s_0} \in T(\mathbb{Q})$. Now we prove (2.3.10.1). For each $\ell \neq p$, the eigenvalues of γ_{m,s_0} acting on $V_\ell(\mathcal{A}_{s_0})$ are q^m-Weil numbers, determined independently of ℓ. In particular, these eigenvalues have ℓ-adic valuation 0. Thus $\{\gamma^r\}_{r\in\mathbb{Z}}$ lies in a compact subgroup of $\mathrm{GL}(V_\ell(\mathcal{A}_{s_0}))$ and hence is a compact subset of $T(\mathbb{Q}_\ell)$.

Let us show (2.3.10.2). First, we note that, for m sufficiently large, the embedding:

$$T_{\mathbb{Q}_p} \hookrightarrow \mathbb{Q}_p \otimes I_{s_0} \hookrightarrow \underline{\mathrm{Aut}}(D_{m,s_0})$$

arises from an isomorphism $\mathbb{Q}_{q^m} \otimes V \xrightarrow{\simeq} D_{m,s_0}$. We can choose this isomorphism so that the semi-linear map $\varphi : D_{m,s_0} \to D_{m,s_0}$ is identified with the automorphism $\delta_{s_0}(\sigma \otimes 1)$ of $\mathbb{Q}_{q^m} \otimes V$, for some element $\delta_{s_0} \in T(\mathbb{Q}_{q^m})$. By (1.3.8), the image of δ_{s_0} in $X_*(T)_{\Gamma_p}$ is $\mu^\sharp$. The assertion now follows from (2.2.7.2).

For the final assertion, note that, since $(T/\mathbb{G}_m)_\mathbb{R}$ is compact, $T(\mathbb{Q})$ is a discrete subgroup of $T(\mathbb{A}_f)$. Given γ satisfying (2.3.10.1) and (2.3.10.2), set $\beta = \gamma^{-1}\gamma_{m,s_0}$. We have to show that $\beta^r = 1$ for some $r \in \mathbb{Z}_{>0}$.

For $\ell \neq p$, the eigenvalues of β acting on $V_\ell(\mathcal{A}_{s_0})$ all belong to $\mathbb{Z}_\ell^\times$; therefore, β lies in a compact subgroup of $T(\mathbb{A}_f^p)$. Moreover, β is in the kernel of $T(\mathbb{Q}_p) \to B(T)$, and so it lies in the compact subgroup in $T(\mathbb{Q}_p)$ consisting of elements σ-conjugate to 1 over L. In sum, we find that β lies in both the discrete subgroup $T(\mathbb{Q})$ and a compact subgroup of $T(\mathbb{A}_f)$, and must therefore be of finite order. □

Proposition 2.3.11. *Suppose that G is quasi-split at p, that G^{ad} has no factors of type D, and that Conjecture 2.3.8 holds for (G, X). Then for any maximal torus $T \subset I_{s_0}$, s_0 admits a $\bar{\mathbb{Q}}$-isogeny CM lift with respect to T.*

Proof. We can view $T_{\mathbb{Q}_p}$ as a maximal torus in I_{p,s_0}. By (1.1.17), there exists a cocharacter $\mu_T \in X_*(T)$ defined over $\bar{\mathbb{Q}}$ whose image in $G_{\bar{\mathbb{Q}}}$ lies in the conjugacy class $\{\mu_X\}_p$, and such that $\nu_{\delta_{s_0}} = N\mu_T \in X_*(T)_{\mathbb{Q}}^{\Gamma_p}$.

By Corollary 2.2.11 there is an embedding $i : T \hookrightarrow G$ such that for all ℓ, i is $G(\bar{\mathbb{Q}}_l)$-conjugate to the embeddings

$$i_\ell : T_{\mathbb{Q}_\ell} \hookrightarrow I_{\ell,s_0} \hookrightarrow G_{\mathbb{Q}_\ell}.$$

The cocharacter

$$\mu_{T,\infty} : \mathbb{G}_{m,\mathbb{C}} \to T_\mathbb{C}$$

obtained from μ_T via the embedding $\iota_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ is $G(\mathbb{C})$-conjugate to μ_y, for $y \in X$. By modifying i within its $G(\bar{\mathbb{Q}})$-conjugacy class, as in (1.2.5), we can assume that $\mu_{T,\infty}$ is $G(\mathbb{R})$-conjugate to μ_y, and so arises from a homomorphism $h_x : \mathbb{S} \to T_\mathbb{R}$, for $x \in X$.

Let $s_0' \in \mathscr{S}_{K_p}(\bar{\mathbb{F}}_p)$ be the reduction of a point of $\mathrm{Sh}_{K_{T,p}}(h_x)$. Recall from the preceding lemma that the q^m-Frobenius $\gamma_{m,s_0} \in I_{s_0}(\mathbb{Q})$ is contained in $T(\mathbb{Q})$. We claim that for m sufficiently divisible,

$$\gamma_{m,s_0} = \gamma_{m,s_0'} \in T(\mathbb{Q}).$$

Here we view $s_0, s_0' \in \mathscr{S}_{K_p}(\mathbb{F}_{q^m})$. Assuming this, we see that, since i and i_ℓ are conjugate for any ℓ, γ_{m,ℓ,s_0} and $\gamma_{m,\ell,s_0'}$ are conjugate in $G(\bar{\mathbb{Q}}_\ell)$. This implies that $P_\ell(s_0, s_0')$ is non-empty, and hence $P(s_0, s_0')$ is non-empty by Conjecture 2.3.8, which implies that s_0' is a $\bar{\mathbb{Q}}$-isogeny CM lift of s_0 with respect to T.

To see the claim, note that the eigenvalues of γ_{m,s_0} acting on $V_\ell(\mathcal{A}_{s_0})$ for $\ell \neq p$ are q^m-Weil numbers. So $\gamma_{m,s_0} \in T(\mathbb{Q})$ is a p-unit as in Lemma 2.3.10. We have

$$\gamma_{m,p,s_0} = \delta_{s_0}\sigma(\delta_{s_0}) \cdots \sigma^{rm-2}(\delta_{s_0})\sigma^{rm-1}(\delta_{s_0})$$

so using (1.1.2.4) we see that the image of γ_{m,p,s_0} under the composite

$$T(\mathbb{Q}_p) \xrightarrow{\kappa} X_*(T)_{\Gamma_p} \to X_*(T)_{\Gamma_p} \otimes \mathbb{Q} \xrightarrow[N]{\simeq} X_*(T)_{\mathbb{Q}}^{\Gamma_p}.$$

is equal to the image of $rm\nu_{\delta_{s_0}}(p) = rmN\mu_T(p)$, which is just the image of $rm\mu_T = m\log_p q \cdot \mu_T$ in $X_*(T)_{\Gamma_p} \otimes \mathbb{Q}$ by (1.1.2.4). Hence for m divisible enough the image of γ_{m,s_0} in $X_*(T)_{\Gamma_p}$ is $m \log_p q \cdot \mu^\sharp$. It follows by Lemma 2.3.10 that $\gamma_{m,s_0} = \gamma_{m,s_0'}$ for m sufficiently divisible. □

2.3.12. We will show that in some cases, the result of Proposition 2.3.11 can be improved to produce $\mathbb{Q}$-isogeny lifts of s_0. To do that we need the following.

Lemma 2.3.13. *Suppose that $s_0 \in \mathscr{S}_{K_p}(\bar{\mathbb{F}}_p)$, $T \subset I_{s_0}$ a maximal torus, and that s_0 admits a $\bar{\mathbb{Q}}$-isogeny CM lift (j, x, s_0') with respect to T. Let $P^T = P^T(s_0, s_0')$ be the subscheme of $P(s_0, s_0')$ consisting of isomorphisms which respect the action of T. Then P^T is a T-torsor, whose class in $H^1(\mathbb{Q}_v, G)$ is trivial for every place v of G.*

Proof. By construction $\mathcal{A}_{s_0}$ and $\mathcal{A}_{s_0'}$ are equipped with an action of T, so the subscheme P^T is well defined. For each ℓ, we denote by $P_\ell^T(s_0, s_0')$ the subscheme of $P_\ell(s_0, s_0')$ consisting of isomorphisms which respect the action of T. Since j is conjugate to i_ℓ by an element of $G(\bar{\mathbb{Q}}_\ell)$, $P_\ell^T(s_0, s_0')$ is non-empty. Hence by Tate's theorem $P^T := P^T(s_0, s_0')$ is non-empty, and thus is a T-torsor, which is a reduction of the I_{s_0}-torsor $P(s_0, s_0')$.

Let I_{s_0}' denote the group of automorphisms of $\mathcal{A}_{s_0}$ respecting polarizations up to a $\mathbb{Q}^\times$-scalar. If F is a number field, then a point of $P(F)$ corresponds to an isogeny $\mathcal{A}_{s_0} \otimes F \to \mathcal{A}_{s_0'} \otimes F$ respecting polarizations up to a $\mathbb{Q}^\times$-scalar. Hence by [Kot92, Lem. 17.1], the class of P in $H^1(\mathbb{R}, I_{s_0}')$ is trivial, so the class of P^T in $H^1(\mathbb{R}, T)$ is trivial by [Kis17, Lem. 4.4.5]. In particular, the class of P^T in $H^1(\mathbb{R}, G)$ is trivial.

Next for $\ell \neq p$ a finite prime, consider the scheme of isomorphisms $\underline{\mathrm{Isom}}_{\{s_\alpha\}}(V_\ell(\mathcal{A}_{s_0}), V_\ell(\mathcal{A}_{s_0'}))$ which take s_{α,ℓ,s_0} $s_{\alpha,\ell,s_0'}$. (Note that we do not require that the isomorphisms respect Frobenius.) This scheme is a G-torsor over $\mathbb{Q}_\ell$, obtained from P^T via the natural map $T \to G$ over $\mathbb{Q}_\ell$. If $\tilde{s}_0, \tilde{s}_0' \in \mathscr{S}_{K_p}$ are lifts of $\tilde{s}_0, \tilde{s}_0'$ then this G-torsor may also be identified with $\underline{\mathrm{Isom}}_{\{s_\alpha\}}(V_\ell(\mathcal{A}_{\tilde{s}_0}), V_\ell(\mathcal{A}_{\tilde{s}_0'}))$. However, from the definition of the universal abelian scheme over $\mathscr{S}_{K_p}$, one sees that this last torsor is trivial.

It remains to check that the image of P^T in $H^1(\mathbb{Q}_p, G)$ is trivial. Fix q such that s_0, s_0' are defined over $\mathbb{F}_q$. As above, by Steinberg's theorem, for m sufficiently large, we may fix isomorphisms

$$D_{m,s_0} \simeq \mathbb{Q}_{q^m} \otimes V \simeq D_{m,s_0'}$$

which take s_α to $s_{\alpha,\mathrm{cris},s_0}$ and $s_{\alpha,\mathrm{cris},s_0'}$ respectively and respect the action of T. Then φ on D_{m,s_0} and $D_{m,s_0'}$ are given by $\delta_{s_0}(\sigma \otimes 1), \delta_{s_0'}(\sigma \otimes 1)$ respectively for δ_{s_0}, $\delta_{s_0'} \in T(\mathbb{Q}_{q^m})$.

Recall that for any reductive group H over $\mathbb{Q}_p$ we have isomorphisms [Kot85]

$$H^1(\mathbb{Q}_p, H) \simeq H^1(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p), H) \simeq (\pi_1(H)_{\Gamma_p})_{\mathrm{tors}}.$$

Here the first isomorphism is given by Steinberg's theorem, and the second isomorphism takes a cocycle c to $\kappa_H(c_\sigma)$, σ the Frobenius.

The class of P^T in $H^1(\mathbb{Q}_p, T)$ corresponds to the cocycle sending σ to $\delta_{s_0'}\delta_{s_0}^{-1}$. By Lemma 1.3.8, δ_{s_0} and $\delta_{s_0'}$ have the same image in $\pi_1(G)_{\Gamma_p}$, so that the class of this torsor in $H^1(\mathbb{Q}_p, G)$ is trivial, as required. $\square$

Corollary 2.3.14. *With the assumptions of Proposition 2.3.11, suppose that G^{der} is simply connected and that G^{ab} satisfies the Hasse principle:*

$$\ker^1(\mathbb{Q}, G^{\mathrm{ab}}) := \ker(H^1(\mathbb{Q}, G^{\mathrm{ab}}) \to \prod_v H^1(\mathbb{Q}_v, G^{\mathrm{ab}})) = 0.$$

Then for any maximal torus $T \subset I_{s_0}$, s_0 admits a $\mathbb{Q}$-isogeny CM lift with respect to T.

Proof. By Corollary 2.3.11 s_0 admits a $\bar{\mathbb{Q}}$-isogeny CM lift with respect to T, say (j, x, s_0'). Let P^T be as in Lemma 2.3.13. For every place v of $\mathbb{Q}$, the class of P^T is trivial in $H^1(\mathbb{Q}_v, G)$ and hence in $H^1(\mathbb{Q}_v, G^{\mathrm{ab}})$. Since G^{ab} satisfies the Hasse principle the class of P^T in $H^1(\mathbb{Q}, G^{\mathrm{ab}})$ is trivial.

As P^T has trivial image in $H^1(\mathbb{R}, G)$ and $H^1(\mathbb{Q}, G^{\mathrm{ab}})$, and G^{der} is simply connected, P^T has trivial image in $H^1(\mathbb{Q}, G)$ by [Bor98, Thm. 5.12], so P^T arises from a point $\omega \in (G/T)(\mathbb{Q})$. Now let $j' = \omega^{-1} j \omega$. Then $j' : T \to G$ is defined over $\mathbb{Q}$. Since the image of ω in $H^1(\mathbb{R}, T)$ is trivial, $\omega^{-1} h_x \omega$ corresponds to a point $x' \in X$ and factors through $j'(T_{\mathbb{R}})$ (cf. [Kis17, 4.2.2]). If $s_0'' \in \mathscr{S}_{K_p}(\bar{\mathbb{F}}_p)$ is a point admitting a lift to $\mathrm{Sh}_{K_{T,p}}(h_x)$, then $P(s_0, s_0'')$ is a trivial I_{s_0}-torsor by [Kis17, Prop. 4.2.6], so (j', x', s_0'') is an isogeny CM lift with respect to T. $\square$

Corollary 2.3.15. *Suppose that G is quasi-split at p, and that (G, X) is of PEL type A or C. Then for any maximal torus $T \subset I_{s_0}$, s_0 admits a $\bar{\mathbb{Q}}$-isogeny CM lift with respect to T. Moreover, s_0 admits a $\mathbb{Q}$-isogeny CM lift with respect to T unless G is of type A_n with n even.*

Proof. The first statement follows from Proposition 2.3.11, and Lemma 2.3.5. For the second statement note that if (G, X) is of PEL type A or C then G^{der} is simply connected, and G^{ab} satisfies the Hasse principle unless G is of type A_n with n even [Kot92, §7]. Hence the second statement follows from Corollary 2.3.14. $\square$

Remarks 2.3.16. (1) In fact the corollary can be shown for certain groups G of type A_n with n even. Namely if it is a unitary similitude group (in $n+1$ variables) arising from a CM quadratic extension F of a totally real field F^+ with $[F^+ : \mathbb{Q}]$ odd, then the Hasse principle holds for G by the proof of Lemma 3.1.1 of [Shi11], so the above proof goes through.

(2) As in 2.3.6, one can extend the proof of the first statement of the last corollary to the case of type D if one works with the disconnected group G'. For an algebraically closed field k, two points of $G(k)$ give rise to the same point of $\mathrm{Conj}'(G)$ if and only if they are conjugate in $G'(k)$. Using this one can deduce a version of Corollary 2.2.11 fromProposition 2.2.9, and use it to deduce an analog of the first part of Corollary 2.3.15, but where $\bar{\mathbb{Q}}$-isogeny is defined using the tensors $\{t_\beta\}$. We leave this as an exercise for the reader.

(3) In [Zin83], Zink proves that for PEL Shimura varieties, and primes of good reduction, every point has an isogeny CM lift with respect to T. However, his definition of isogeny is required to respect only endomorphisms and not polarizations. In that case the analogue of $P(s_0, s_0')$ is a torsor under the group of units in a product of (possibly skew) fields. Any such torsor is trivial, for example because a $\mathbb{Q}$-vector space has a Zariski dense set of rational points, or alternatively because in this case the group is a product of inner forms of GL_n.

Thus, the first part of Corollary 2.3.15 recovers Zink's result in this case. However, the second part is really stronger. Even for the moduli space of principally polarized abelian varieties the deduction of this statement using Honda-Tate theory does not quite seem to be in the literature. Although it is a special case of a result of [Kis17], the techniques used there are quite different.

(4) The condition on G^{ab} in Corollary 2.3.14 and the second part of Corollary 2.3.15 is used to show that the class of P^T in $H^1(\mathbb{Q}, G^{\mathrm{ab}})$ is trivial. In fact this should follow from the

fact that s_0, s_0' lie on the same Shimura variety, since the motive obtained from $\mathcal{A}_{s_0}$ and any representation of G which factors through G^{ab} should be constant; for example this holds in characteristic 0 at the level of variations of Hodge structure. Even when G^{der} is not simply connected, there is a corresponding cohomology group $H^1(\mathbb{Q}, G/\tilde{G})$, in which the image of P^T should be trivial (here $\tilde{G}$ is the simply connected cover of G^{der}), which would be enough for the argument of Corollary 2.3.14. Unfortunately we do not know how to make these motivic heuristics rigorous.

Appendix A. Construction of isocrystals with G-structure

The purpose of this appendix is to prove Proposition 1.3.11. The main tool is Faltings' comparison theorem [Fal02] p. 62, as well as de Jong's theorem on alterations [Jon96] and a result of Ogus on proper descent for convergent isocrystals [Ogu84].

A.1. Let k be a perfect field of characteristic p, and $W = W(k)$. We equip k and W with the trivial log structure.

Let X be a scheme over W, equipped with a fine saturated log structure. A p-adic formal log scheme T over W, is a p-adic formal scheme T/W together with the data of a compatible system of log structures on $T_n = T \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ for $n \geq 1$, such that the inclusions $T_n \hookrightarrow T_{n+1}$ are exact.

An *enlargement* of X is a triple (T, I, i_T) consisting of a p-adic formal log scheme T over W, an ideal of definition I of T, and a map of log schemes $i_T : T_0 \to X$, where T_0 is the subscheme of T defined by I. We say that (T, I, i_T) is reduced if T_0 is reduced. We say that (T, I, i_T) is a PD-enlargement if I is equipped with divided powers extending the divided powers on pW.

As in [Ogu84, 2.7], using the definition of an enlargement we can define the category of convergent log isocrystals (cf. [Ogu95, §3]). This category does not change if we allow I to be any p-adically closed ideal as in [Fal02, p. 258]. Indeed, the value of a convergent log isocrystal on such an enlargement can be defined as the inverse limit of its values on $(T, (I, p^n), i_{T,n})$ for $n \geq 1$, where $I_{T,n}$ is the composite $T_n \hookrightarrow T_0 \xrightarrow{i_T} X$ and T_n is defined by (I, p^n).

The category of convergent log isocrystals also does not change if we define it using only reduced enlargements. In particular it depends only on $X \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$, and not on X, and is equipped with a Frobenius pullback functor F^*. Thus, we have the notion of a convergent log F-isocrystal (again cf. [Ogu95, §3]). When the log structure on X is trivial, this agrees with the definition of convergent isocrystal and F-isocrystal in [Ogu84].

The log crystalline site of X is the site whose objects consist of PD-enlargements. As in [MP15, 1.3.3] a log Dieudonné crystal over X is a crystal M in the log crystalline site of X together with maps $F^*M \to M$ and $M \to F^*M$ whose composite in either order is multiplication by p. As in [Ogu84, 2.18] or [Ogu95, Rem. 16], a log Dieudonné crystal over X gives rise to a convergent F-isocrystal on X.

A.2. Let S be a flat, normal, finite type W-scheme, $D \subset S$ a relative Cartier divisor, and $j : U = D - S \hookrightarrow S$, the inclusion. We consider S as a log scheme equipped with the log structure $j_*\mathscr{O}_U$, and for $n \geq 1$, we give $S \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ the induced log structure.

Let $\pi : \mathcal{A} \to U$ be an abelian scheme, which extends to a semi-abelian scheme over S. We denote by $\mathbb{L}$ the étale local system $R^1\pi_*\mathbb{Q}_p$ on $U_{K,\text{ét}}$. We denote by $\mathcal{E}$ the convergent F-isocrystal on U attached to the p-divisible group $\mathcal{A}[p^\infty]$.

By [MP15, 1.3.5] there is a log Dieudonné crystal on S attached to $\mathcal{A}$, and hence a convergent log F-isocrystal $\mathcal{E}^{\log}$ on S, whose restriction to U is $\mathcal{E}$, and whose formation is compatible with Cartier duality.

A.3. Let $K_0 = W[1/p]$, and K/K_0 a finite extension. Fix an algebraic closure $\bar{K} \supset K$, and let $G_K = \mathrm{Gal}(\bar{K}/K)$. We keep the above notation, but we now assume that S is semi-stable over $\mathscr{O}_K$, and that $S_0 \cup D \subset S$ is a normal crossing divisor. Here $S_0 = S \otimes_{\mathscr{O}_K} k$.

Above we considered S with the log structure given by D. We denote by S^{vlog} the scheme S considered with the log structure given by $S_0 \cup D$. There is a map of log schemes $i : S^{\mathrm{vlog}} \to S$. We set $\mathcal{E}^{\mathrm{vlog}} = i^*(\mathcal{E}^{\log})$, a convergent F-isocrystal on S^{vlog}.

Lemma A.4. *With the above notation, $\mathbb{L}$ and $\mathcal{E}^{\mathrm{vlog}}$ are associated in the sense of [Fal02, p. 258].*

Proof. As already remarked in [Fal02], $\mathcal{E}^{\mathrm{vlog}}$ gives rise to a convergent isocrystal in the sense of *loc. cit.* p. 258. The proof of the lemma is entirely analogous to the argument given in [Fal99, §6], cf. also [MP15, A2.2] for the case of log schemes. □

A.5. We now return to the assumptions of A.2, so we no longer assume that S is semi-stable.

Let $s : \mathbf{1} \to \mathbb{L}^{\otimes}$ be a map of étale local systems over U. That is, s is a global section of $\mathbb{L}^{\otimes}$. For any finite K'/K in $\bar{K}$, with residue field k', and any $\xi \in U(\mathscr{O}_{K'})$, $\xi^*(s)$ corresponds to a section

$$s_{0,\xi} = D_{\mathrm{cris}}(\xi^*(s)) : \mathbf{1} \to \xi^*(\mathcal{E})^{\otimes}.$$

Proposition A.6. *If S is proper and semi-stable over $\mathscr{O}_K$, and $S_0 \cup D \subset S$ is a normal crossing divisor, then there is a morphism of convergent log F-isocrystals $s_0 : \mathbf{1} \to \mathcal{E}^{\log\otimes}$ over S such that $\xi^*(s_0)(W(k')) = s_{0,\xi}$ for all K'/K, k', and ξ as above*

Proof. Let $\pi \in \mathscr{O}_K$ be a uniformizer and $E(T)$ an Eisenstein polynomial for π. Let $R = W[\![T]\!]$, and for $n \geq 1$ let R_n be the p-adic completion of $W[T, E(T)^{ni}/i!]$. We view $\mathscr{O}_K$ as an R_n-algebra, and so an R-algebra via $T \mapsto \pi$. It suffices to construct s_0 étale locally on S.

Let $\mathrm{Spec}\, A$ be an étale neighborhood of S, which admits an étale map

$$\varpi : \mathrm{Spec}\, A \to \mathscr{O}_K[t_1, \dots, t_d]/(t_1 \cdots t_e - \pi)$$

for some $e \leq d$, and such that the log structure on $\mathrm{Spec}\, A$ is given by the preimage of the Cartier divisor defined by $t_1 \cdots t_r$ for some $e \leq r \leq d$. Let $\hat{A}$ be the p-adic completion of A. Thus $\mathrm{Spf}\, \hat{A}$ is a p-adic formal log scheme over $\mathscr{O}_K$, which is formally smooth when $\mathscr{O}_K$ is equipped with the log structure $\mathscr{O}_K - \{0\}$. Lift $\mathrm{Spf}\, \hat{A}$, to a formally smooth (p, T)-adic formal log scheme $Y_R = \mathrm{Spf}\, \hat{A}_R$ over R (defined as in the p-adic case). Thus, $\hat{A}_R$ is formally étale over the (p, T)-adic completion of $R[t_1, \cdots, t_d]/(t_1 \cdots t_e - T)$, with the the log structure given by the preimage of the Cartier divisor defined by $t_1 \cdots t_r$.

We consider the Frobenius lift F on $\hat{A}_R$ induced by $t_i \mapsto t_i^p$, and $T \mapsto T^p$. Let Y_n be the base change of Y_R to R_n. Then F induces a lift of Frobenius on Y_n. Note that Y_n is an enlargement of S^{vlog}, and so we may evaluate $\mathcal{E}^{\mathrm{vlog}}$ on it.

By (A.4) and [Fal02, §5, Cor. 4, Rem. 1)], s gives rise to a Frobenius invariant, parallel section s_0 of $\mathcal{E}^{\mathrm{vlog}}(Y_1)^{\otimes}$. Note that the result of *loc. cit* applies, because $\mathcal{E}^{\mathrm{vlog}}$ arises from a log Dieudonné crystal on S^{vlog}. Hence for any m, n we can apply that result to the log F-crystal obtained by multiplying the Frobenius on $\mathcal{E}^{\mathrm{vlog}\otimes n} \otimes \mathcal{E}^{\mathrm{vlog}*\otimes m}$ by a high enough power of p, and replacing $\mathbb{L}^{\otimes n} \otimes \mathbb{L}^{*\otimes m}$ by a suitable Tate twist. Since s_0 is Frobenius invariant, it gives rise to a section of $\mathcal{E}^{\mathrm{vlog}}(Y_n)^{\otimes}$ for any $n \geq 1$.

Now let Y_n^h be the p-adic formal log scheme with the same underlying formal scheme as Y_n, but with the log structure defined by $t_{e+1} \cdots t_r$. Then Y_n^h is an enlargement of S, and from the definitions we have $\mathcal{E}^{\log}(Y_n^h) = \mathcal{E}^{\mathrm{vlog}}(Y_n)$. Since Y_R is formally smooth over W, as in [Ogu84, Thm 2.11], the sections $s_0 \in \mathcal{E}^{\log}(Y_n^h)^{\otimes}$ give rise to a morphism of convergent F-isocrystals $s_0 : \mathbf{1} \to \mathcal{E}^{\log\otimes}$ over $\mathrm{Spec}\, A$. The relation $\xi^*(s_0)(W(k')) = s_{0,\xi}$ follows from the functoriality of the map constructed in [Fal02]. □

Corollary A.7. *For any S (not assumed proper or semi-stable), and $s : \mathbf{1} \to \mathbb{L}^{\otimes}$ as above, there exists a unique morphism of convergent F-isocrystals over U*

$$s_0 : \mathbf{1} \to \mathcal{E}^{\otimes}$$

such that for every K'/K finite, and ξ as above $\xi^*(s_0)(W(k')) = s_{0,\xi}$.

Proof. By [Jon96, Thm 6.5], after replacing K by a finite extension, there exists a proper truncated hypercovering

$$U_1 \rightrightarrows U_0 \to U$$

such that for $i = 0, 1$ there is a dense open immersion $U_i \hookrightarrow S_i$, with S_i proper and semi-stable, and $(S_i \backslash U_i) \cup S_i \otimes_{\mathscr{O}_K} k$ is a normal crossings divisor in S_i. By proper descent for convergent isocrystals [Ogu84, Thm 4.6], it suffices to prove the proposition with U_i in place of U. Thus we may replace U by U_i, and S by S_i, and assume that S is proper and semi-stable, and $S_0 \cup D \subset S$ is a normal crossing divisor. Then the required map is obtained by restricting the map $s_0 : \mathbf{1} \to \mathcal{E}^{\log\otimes}$ of Proposition A.6 to U. The uniqueness is easily deduced from [Ogu84, Thm 4.1]. □

References

[BO83] P. Berthelot and A. Ogus. "F-isocrystals and de Rham cohomology. I". *Invent. Math.* 72.2 (1983), pp. 159–199 (cit. on p. 15).

[Bla94] D. Blasius. "A p-adic property of Hodge classes on abelian varieties". In: *Motives*. Vol. 55. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1994, pp. 293–308 (cit. on p. 15).

[Bor98] M. Borovoi. *Abelian Galois Cohomology of Reductive Groups*. Vol. 132. Mem. Amer. Math. Soc. 626. American Mathematical Society, 1998 (cit. on pp. 4, 12, 29).

[CKV15] M. Chen, M. Kisin, and E. Viehmann. "Connected components of affine Deligne-Lusztig varieties in mixed characteristic". *Compos. Math.* 151.9 (2015), pp. 1697–1762 (cit. on p. 9).

[Del79] P. Deligne. "Valeurs de fonctions L et périodes d'intégrales". In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*. Proc. Sympos. Pure Math., XXXIII. With an appendix by N. Koblitz and A. Ogus. Amer. Math. Soc., Providence, R.I., 1979, pp. 313–346 (cit. on p. 16).

[Fal99] G. Faltings. "Integral crystalline cohomology over very ramified valuation rings". *J. Amer. Math. Soc.* 12.1 (1999), pp. 117–144 (cit. on p. 31).

[Fal02] G. Faltings. "Almost étale extensions". *Astérisque* 279 (2002). Cohomologies p-adiques et applications arithmétiques, II, pp. 185–270 (cit. on pp. 30, 31).

[HK17] P. Hamacher and W. Kim. "l-adic étale cohomology of Shimura varieties of Hodge type with non-trivial coefficients" (2017). URL: `https://arxiv.org/abs/1711.07123` (cit. on p. 2).

[Har66] G. Harder. "Über die Galoiskohomologie halbeinfacher Matrizengruppen. II". *Math. Z.* 92 (1966), pp. 396–415. ISSN: 0025-5874. DOI: `10.1007/BF01112219`. URL: `http://dx.doi.org/10.1007/BF01112219` (cit. on p. 12).

[HR17] X. He and M. Rapoport. "Stratifications in the reduction of Shimura varieties". *Manuscripta Math.* 152.3-4 (2017), pp. 317–343 (cit. on p. 2).

[Jon95] A. J. de Jong. "Crystalline Dieudonné module theory via formal and rigid geometry". *Inst. Hautes Études Sci. Publ. Math.* 82.1 (1995), pp. 5–96 (cit. on p. 15).

[Jon96] A. J. de Jong. "Smoothness, semi-stability and alterations". *Inst. Hautes Études Sci. Publ. Math.* 83 (1996), pp. 51–93 (cit. on pp. 30, 32).

[Kat79] N. Katz. "Slope filtration of F-crystals". In: *Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. I.* Vol. 63. Astérisque. Soc. Math. France, Paris, 1979, pp. 113–163 (cit. on p. 17).

[Kis10] M. Kisin. "Integral models for Shimura varieties of abelian type". *J. Amer. Math. Soc.* 23.4 (2010), pp. 967–1012 (cit. on pp. 14, 15, 17, 26).

[Kis17] M. Kisin. "mod p points on Shimura varieties of abelian type". *J. Amer. Math. Soc.* 30.3 (2017), pp. 819–914 (cit. on pp. 3, 4, 9, 19, 20, 28, 29).

[KP15] M. Kisin and G. Pappas. "Integral models of Shimura varieties with parahoric level structure". 2015. URL: `http://www.math.harvard.edu/~kisin/dvifiles/parahoric.pdf` (cit. on p. 18).

[Kot82] R. Kottwitz. "Rational conjugacy classes in reductive groups". *Duke Math. J.* 49.4 (1982), pp. 785–806 (cit. on p. 24).

[Kot85] R. Kottwitz. "Isocrystals with additional structure". *Compos. Math.* 56.2 (1985), pp. 201–220 (cit. on pp. 5–7, 10, 28).

[Kot86] R. Kottwitz. "Stable trace formula: Elliptic singular terms". *Math. Ann.* 275.3 (1986), pp. 365–399 (cit. on pp. 11, 12).

[Kot92] R. Kottwitz. "Points on Some Shimura Varieties Over Finite Fields". *J. Amer. Math. Soc.* 5.2 (1992), pp. 373–444 (cit. on pp. 3, 26, 28, 29).

[Kot97] R. Kottwitz. "Isocrystals with additional structure. II". *Compos. Math.* 109.3 (1997), pp. 255–339 (cit. on pp. 6, 11).

[Lan83] R. Langlands. *Les débuts d'une formule des traces stable.* Vol. 13. Publications Mathématiques de l'Université de Paris VII. Université de Paris VII, U. E. R. de Mathématiques, Paris, 1983 (cit. on p. 12).

[Lan89] R. Langlands. "On the classification of irreducible representations of real algebraic groups". In: *Representation Theory and Harmonic Analysis on Semisimple Lie Groups.* Vol. 31. Math. Surveys Monogr. Providence, Rhode Island: Amer. Math. Soc., 1989, pp. 101–170 (cit. on p. 11).

[LR87] R. Langlands and M. Rapoport. "Shimuravarietäten und Gerben". *J. Reine Angew. Math.* 378 (1987), pp. 113–220 (cit. on pp. 2, 9, 13, 24).

[Lee18] Dong Uk Lee. "Nonemptiness of Newton strata of Shimura varieties of Hodge type". *Algebra Number Theory* 12.2 (2018), pp. 259–283 (cit. on p. 2).

[MP15] K. Madapusi Pera. "Toroidal compactifications of integral models of Shimura varieties of Hodge type". 2015. URL: `http://www.math.uchicago.edu/~keerthi/papers/toroidal_new.pdf` (cit. on pp. 17, 30, 31).

[Noo09] R. Noot. "Classe de conjugaison du Frobenius d'une variete abelienne sur un corps de nombres". *J. Lond. Math. Soc.* 79.1 (2009), pp. 53–71 (cit. on pp. 23, 24).

[Ogu84] A. Ogus. "F-isocrystals and de Rham cohomology II-Convergent isocrystals". *Duke Math. J.* 51.4 (1984), pp. 765–850 (cit. on pp. 30–32).

[Ogu95] A. Ogus. "F-crystals on schemes with constant log structure". *Compositio Math.* 97.1-2 (1995). Special issue in honour of Frans Oort, pp. 187–225 (cit. on p. 30).

[Rap05] M. Rapoport. "A guide to the reduction modulo p of Shimura varieties". *Astérisque* 298 (2005). Automorphic forms. I, pp. 271–318 (cit. on p. 2).

[RR96] M. Rapoport and M. Richartz. "On the classification and specialization of F-isocrystals with additional structure". *Compos. Math.* 103.2 (1996), pp. 153–181 (cit. on pp. 1, 5–7, 9, 10, 16, 17).

[RV14] M. Rapoport and E. Viehmann. "Towards a theory of local Shimura varieties". *Münster J. Math.* 7.1 (2014), pp. 273–326 (cit. on pp. 2, 10).

[Shi11] Sug Woo Shin. "Galois representations arising from some compact Shimura varieties". *Ann. of Math. (2)* 173.3 (2011), pp. 1645–1741 (cit. on p. 29).

[Tat66] J. Tate. "Endomorphisms of abelian varieties over finite fields". *Invent. Math.* 2 (1966), pp. 134–144 (cit. on pp. 3, 4).

[VW13] Eva Viehmann and Torsten Wedhorn. "Ekedahl-Oort and Newton strata for Shimura varieties of PEL type". *Math. Ann.* 356.4 (2013), pp. 1493–1550 (cit. on p. 2).

[Wed99] T. Wedhorn. "Ordinariness in good reductions of Shimura varieties of PEL-type". *Ann. Sci. École Norm. Sup. (4)* 32.5 (1999), pp. 575–618 (cit. on p. 2).

[Win97] J-P. Wintenberger. "Propriétés du groupe tannakien des structures de Hodge p-adiques et torseur entre cohomologies cristalline et étale". *Ann. Inst. Fourier (Grenoble)* 47.5 (1997), pp. 1289–1334 (cit. on p. 16).

[Win05] J-P. Wintenberger. "Existence de F-cristaux avec structures supplémentaires". *Adv. Math.* 190.1 (2005), pp. 196–224 (cit. on p. 9).

[Wor13] D. Wortmann. "The μ-ordinary locus for Shimura varieties of Hodge type". 2013. URL: `http://arxiv.org/abs/1310.6444` (cit. on pp. 2, 17).

[Zho17] R. Zhou. "Mod-p Isogeny classes on Shimura varieties with parahoric level structure" (2017). URL: `https://arxiv.org/abs/1707.09685` (cit. on pp. 2, 3).

[Zin83] T. Zink. "Isogenieklassen von Punkten von Shimuramannigfaltigkeiten mit Werten in einem endlichen Körper". *Math. Nachr.* 112 (1983), pp. 103–124 (cit. on pp. 3, 29).

Department of Mathematics, Harvard University, 1 Oxford St, Cambridge, MA 02138, USA
E-mail address: `kisin@math.harvard.edu`

Department of Mathematics, Maloney Hall, Boston College, Chestnut Hill, MA 02467, USA
E-mail address: `madapusi@bc.edu`

Department of Mathematics, UC Berkeley, Berkeley, CA 94720, USA// Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 130-722, Republic of Korea
E-mail address: `sug.woo.shin@berkeley.edu`