GALOIS REPRESENTATIONS FOR EVEN GENERAL SPECIAL ORTHOGONAL GROUPS

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Abstract. We prove the existence of GSpin_{2n}-valued Galois representations corresponding to cohomological cuspidal automorphic representations of certain quasi-split forms of GSO_{2n} under the local hypotheses that there is a Steinberg component and that the archimedean parameters are regular for the standard representation. This is based on the cohomology of Shimura varieties of abelian type, of type $D^{\mathbb{H}}$, arising from forms of GSO_{2n}. As an application, under similar hypotheses, we compute automorphic multiplicities, prove meromorphic continuation of (half) spin L-functions, and improve on the construction of SO_{2n}-valued Galois representations by removing the outer automorphism ambiguity.

Contents

Date: November 9, 2023.
2020 *Mathematics Subject Classification.* Primary 11F80; Secondary 11R39, 11G18, 11F70.
Key words and phrases. Galois representations, Shimura varieties, Automorphic forms, Even orthogonal groups.

Introduction

Inspired by conjectures of Langlands and Clozel's work [18] for the group $G = \mathrm{GL}_n$, Buzzard–Gee [12, Conj. 5.16] formulate the following version of the Langlands correspondence (in one direction) for an arbitrary connected reductive group G over a number field F. Let $\mathbb{A}_F$ denote the ring of adèles over F. Write $\widehat{G}$ (resp. ${}^L G$) for the Langlands dual group (resp. L-group) of G over $\overline{\mathbb{Q}}_\ell$. When $g \in {}^L G(\overline{\mathbb{Q}}_\ell)$, let g_{ss} denote its semisimple part.

Conjecture 1. *Let ℓ be a prime number and fix an isomorphism $\iota\colon \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$. Let π be a cuspidal L-algebraic automorphic representation of $G(\mathbb{A}_F)$. Then there exists a Galois representation*

$$\rho_\pi = \rho_{\pi,\iota}\colon \mathrm{Gal}(\overline{F}/F) \to {}^L G(\overline{\mathbb{Q}}_\ell),$$

such that for all but finitely many primes $\mathfrak{q}$ of F (excluding $\mathfrak{q}|\ell$ and those such that $\pi_\mathfrak{q}$ are ramified), the $\widehat{G}$-conjugacy class of $\rho_\pi(\mathrm{Frob}_\mathfrak{q})_{\mathrm{ss}} \in {}^L G(\overline{\mathbb{Q}}_\ell)$ is the Satake parameter of $\pi_\mathfrak{q}$ via ι.

The conjecture of Buzzard-Gee is more precise (and does not assume cuspidality). They describe the image of each complex conjugation element and ℓ-adic Hodge-theoretic properties of ρ_π. Moreover they predict [12, Conj. 5.17] that the compatibility holds at every $\mathfrak{q}$ coprime to ℓ such that $\pi_\mathfrak{q}$ is unramified. In fact $\rho_\pi(\mathrm{Frob}_\mathfrak{q})$, instead of its semisimple part, appears in their conjecture. While $\rho_\pi(\mathrm{Frob}_\mathfrak{q})$ is expected to be always semisimple, this seems to be a problem of different nature and out of reach. Thus we state the conjecture with $\rho_\pi(\mathrm{Frob}_\mathfrak{q})_{\mathrm{ss}}$.

For most recent results on Conjecture 1 for GL_n (in the regular case), we refer to [35, 71] and the references therein. Arthur's endoscopic classification [1] (see [40, 62] for unitary groups)[1] provides a crucial input for constructing Galois representations as in the conjecture for symplectic, special orthogonal, and unitary groups by reducing the question to the case of general linear groups. When the group is SO_{2n}, however, such an approach proves only a weaker local-global compatibility up to outer automorphisms (see (SO-i) in Theorem 6.5 below), falling short of proving Conjecture 1 (even under local hypotheses); we will return to this point as an application of our main theorem.

Our goal is to prove Conjecture 1 for a quasi-split form G^* of GSO_{2n} over a totally real field under certain local hypotheses, as a sequel to our work [49] where we proved the conjecture for GSp_{2n} under similar local hypotheses. The group GSO_{2n} is closely related to the classical group SO_{2n}, just like GSp_{2n} is to Sp_{2n}, but the similitude groups may well be regarded as non-classical groups. An important reason is that the Langlands dual groups of GSO_{2n} and GSp_{2n}, namely the general spin groups GSpin_{2n} and GSpin_{2n+1}, do not admit standard *embeddings* (into general linear groups of proportional rank). This makes the problem both nontrivial and interesting. Furthermore, since the groups GSp_{2n} and GSO_{2n} appear as endoscopic groups of each other for varying n [81, Sect. 2.1], results for the one group likely have applications for the other, especially if one tries to prove cases of Conjecture 1 without local hypotheses.

To be more precise, we set up some notation. Let F be a totally real number field, and $n \in \mathbb{Z}_{\geq 3}$. Let GSO_{2n} denote the connected split reductive group over F which is the identity component of the orthogonal similitude group GO_{2n}. (See §2 below for an explicit definition.)

Our setup depends on the parity of n:

- (n **even**) $E = F$, and $G^* = \mathrm{GSO}_{2n}$ (the split form over F),
- (n **odd**) E is a totally imaginary quadratic extension of F, and G^* is a non-split quasi-split form of GSO_{2n} relative to E/F (explicitly given as (8.4)).

We write $\mathrm{GSO}_{2n}^{E/F}$ for the F-group G^* in either case. The setup is naturally designed so that there are Shimura varieties for (an inner twist of) $\mathrm{Res}_{F/\mathbb{Q}} G^*$. In particular $G^*(F_y)$ has discrete series at every infinite place y of F. (Indeed $G^*(F_y)$ has no discrete series if we swap the parity of n above.) There is a short exact sequence of F-groups

$$1 \to \mathrm{SO}_{2n}^{E/F} \longrightarrow \mathrm{GSO}_{2n}^{E/F} \xrightarrow{\mathrm{sim}} \mathbb{G}_m \to 1,$$

[1] The endoscopic classification is conditional in the following sense. At this time, the postponed articles [A25], [A26] and [A27] in the bibliography of [1] have not appeared. The proof of the weighted fundamental lemma for non-split groups has not become available yet either.

where $\mathrm{SO}_{2n}^{E/F}$ is a quasi-split form of SO_{2n}, defined similarly as $\mathrm{GSO}_{2n}^{E/F}$, and sim denotes the similitude character. It is convenient to use the version of L-group relative to E/F, with coefficients in either $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$:

$$^LG^* = \widehat{G}^* \rtimes \mathrm{Gal}(E/F) = \mathrm{GSpin}_{2n} \rtimes \mathrm{Gal}(E/F),$$

where the nontrivial element of $\mathrm{Gal}(E/F)$ acts non-trivially on GSpin_{2n}. (This identifies $^LG^*$ with GPin_{2n} if $[E:F]=2$.) An important feature of the (general) spin groups GSpin_m ($m \in \mathbb{Z}_{\geq 2}$) is their spin representation $\mathrm{spin}_m\colon \mathrm{GSpin}_m \to \mathrm{GL}_{2^{\lfloor m/2 \rfloor}}$. In case m is even, this representation is reducible and splits up into a direct sum $\mathrm{spin}_m = \mathrm{spin}_m^+ \oplus \mathrm{spin}_m^-$ of two irreducible representations of dimension $2^{\lfloor m/2 \rfloor -1}$. These representations $\mathrm{spin}_m^\pm$ are called the *half-spin* representations. Two other important representations are the *standard representation* and the *spinor norm* (see Lemma 3.1 for pr°)

$$\mathrm{std}\colon \mathrm{GSpin}_m \stackrel{\mathrm{pr}^\circ}{\to} \mathrm{SO}_m \to \mathrm{GL}_m, \quad \text{and} \quad \mathcal{N}\colon \mathrm{GSpin}_m \to \mathrm{GL}_1.$$

If m is odd, spin is faithful. In the even case $m=2n$, none of the representations $\mathrm{spin}^+, \mathrm{spin}^-, \mathrm{std}$, or $\mathcal{N}$ is faithful, but spin is faithful.

Let π be a cuspidal automorphic representation of $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$. Consider the following hypotheses on π, where $|\mathrm{sim}|$ denotes the composite $\mathrm{GSO}_{2n}^{E/F}(F \otimes_{\mathbb{Q}} \mathbb{R}) \stackrel{\mathrm{sim}}{\to} (F \otimes \mathbb{R})^\times \stackrel{|\cdot|}{\to} \mathbb{R}_{>0}^\times$:

- **(St)** There is a finite F-place $\mathfrak{q}_{\mathrm{St}}$ such that $\pi_{\mathfrak{q}_{\mathrm{St}}}$ is the Steinberg representation of $G^*(F_{\mathfrak{q}_{\mathrm{St}}})$ twisted by a character.
- **(L-coh)** $\pi_\infty |\mathrm{sim}|^{-n(n-1)/4}$ is ξ-cohomological for an irreducible algebraic representation $\xi = \otimes_{y: F \hookrightarrow \mathbb{C}} \xi_y$ of the group $(\mathrm{Res}_{F/\mathbb{Q}} G^*) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \prod_{y: F \hookrightarrow \mathbb{C}} (G^* \otimes_{F,y} \mathbb{C})$.
- **(std-reg)** The infinitesimal character of ξ_y for every $y: F \hookrightarrow \mathbb{C}$, which is a regular Weyl group orbit in the Lie algebra of $\widehat{G}^* = \mathrm{GSpin}_{2n}(\mathbb{C})$, remains regular under the standard representation $\mathrm{GSpin}_{2n} \to \mathrm{GL}_{2n}$.

In (L-coh), 'ξ-cohomological' means that the tensor product with ξ has nonvanishing relative Lie algebra cohomology in some degree (§1 below). Condition (L-coh) implies that π is L-algebraic. The other two conditions should be superfluous as they do not appear in Conjecture 1. Condition (St) plays an essential role in our argument, and would take significant new ideas and effort to get rid of. We assume (std-reg) for the reason that certain results for regular-algebraic self-dual cuspidal automorphic representations of GL_N, $N>2$, are missing in the non-regular case. However we need less than (std-reg) for our argument to work. The necessary input for us to proceed without (std-reg) is formulated as Hypothesis 6.11, which we expect to be quite nontrivial but within reach nonetheless. Thus we assume either (std-reg) or Hypothesis 6.11 in the main theorem, hoping that (std-reg) will be removed as soon as the hypothesis is verified.

Let $S_{\mathrm{bad}} = S_{\mathrm{bad}}(\pi)$ denote the finite set of rational primes p such that either $p=2$, p ramifies in F, or $\pi_{\mathfrak{q}}$ ramifies at a place $\mathfrak{q}$ of F above p. The following theorem assigns an ℓ-adic Galois representation to π for each prime number ℓ and each isomorphism $\iota\colon \mathbb{C} \stackrel{\sim}{\to} \overline{\mathbb{Q}}_\ell$.

Theorem A. *Assume that π satisfies conditions (St) and (L-coh). If (std-reg) does not hold for π, further assume Hypothesis 6.11 (for an $\mathrm{SO}_{2n}(\mathbb{A}_F)$-subrepresentation of π). Then there exists, up to $\widehat{G}$-conjugation, a unique semisimple Galois representation attached to π and ι*

$$\rho_\pi = \rho_{\pi,\iota}\colon \mathrm{Gal}(\overline{F}/F) \to {}^LG^*$$

such that the following hold.

- *(A1) For every prime $\mathfrak{q}$ of F not above $S_{\mathrm{bad}} \cup \{\ell\}$, $\rho_\pi(\mathrm{Frob}_{\mathfrak{q}})_{\mathrm{ss}}$ is $\widehat{G}^*$-conjugate to $\iota\phi_{\pi_{\mathfrak{q}}}(\mathrm{Frob}_{\mathfrak{q}})$, where $\phi_{\pi_{\mathfrak{q}}}$ is the unramified Langlands parameter of $\pi_{\mathfrak{q}}$.*
- *(A2) The composition*

$$\mathrm{Gal}(\overline{F}/F) \stackrel{\rho_\pi}{\to} {}^LG^* \stackrel{\mathrm{pr}^\circ}{\to} \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \mathrm{Gal}(E/F)$$

corresponds to a cuspidal automorphic $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$*-subrepresentation* $\pi^\flat$ *contained in* π *in that* $\mathrm{pr}^\circ(\rho_\pi(\mathrm{Frob}_\mathfrak{q})_{\mathrm{ss}})$ *is* $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$*-conjugate to the Satake parameter of* $\pi^\flat_\mathfrak{q}$ *via* ι *at every* $\mathfrak{q}$ *not above* $S_{\mathrm{bad}} \cup \{\ell\}$. *Further, the composition*

$$\mathrm{Gal}(\overline{F}/F) \overset{\rho_\pi}{\to} {}^L G^* \overset{\mathcal{N}}{\to} \mathrm{GL}_1(\overline{\mathbb{Q}}_\ell)$$

corresponds to the central character of π *via class field theory and* ι.

(A3) For every $\mathfrak{q}|\ell$, *the representation* $\rho_{\pi,\mathfrak{q}}$ *is de Rham (in the sense that* $r \circ \rho_{\pi,\mathfrak{q}}$ *is de Rham for all representations* r *of* $\widehat{G}^*$*). Moreover*
  - *(a) The Hodge–Tate cocharacter of* $\rho_{\pi,\mathfrak{q}}$ *is explicitly determined by* ξ. *More precisely, for all* $y\colon F \to \mathbb{C}$ *such that* ιy *induces* $\mathfrak{q}$, *we have*

$$\mu_{\mathrm{HT}}(\rho_{\pi,\mathfrak{q}}, \iota y) = \iota\mu_{\mathrm{Hodge}}(\xi_y) - \frac{n(n-1)}{4}\mathrm{sim}.$$

    (We still write sim *to mean the cocharacter of* GSpin_{2n} *dual to* $\mathrm{sim} : G^* \to \mathbb{G}_m$. *See §1 below for the Hodge–Tate and Hodge cocharacters* μ_{HT} *and* μ_{Hodge}.[2]*)*
  - *(b) If* $\pi_\mathfrak{q}$ *has nonzero invariants under a hyperspecial (resp. Iwahori) subgroup of* $G^*(F_\mathfrak{q})$ *then either* $\rho_{\pi,\mathfrak{q}}$ *or a quadratic character twist is crystalline (resp. semistable).*
  - *(c) If* $\ell \notin S_{\mathrm{bad}}$ *then* $\rho_{\pi,\mathfrak{q}}$ *is crystalline.*

(A4) For every $v|\infty$, $\rho_{\pi,v}$ *is odd (see §1 and Remark 12.6 below).*

(A5) The Zariski closure of the image of $\rho_\pi(\mathrm{Gal}(\overline{F}/E))$ *in* PSO_{2n} *maps onto one of the following four subgroups of* PSO_{2n}*:*
  - *(a)* PSO_{2n},
  - *(b)* PSO_{2n-1} *(as a reducible subgroup),*
  - *(c) the image of a principal* SL_2 *in* PSO_{2n}, *or*
  - *(d) (only when* $n = 4$*)* G_2 *(embedded in* $\mathrm{SO}_7 \subset \mathrm{PSO}_8$*) or* SO_7 *(as an irreducible subgroup via the projective spin representation).*

(A6) If $\rho'\colon \mathrm{Gal}(\overline{F}/F) \to {}^L G^*$ *is another semisimple Galois representation such that, for almost all finite* F*-places* $\mathfrak{q}$ *where* ρ' *and* ρ_π *are unramified, the semisimple parts* $\rho'(\mathrm{Frob}_\mathfrak{q})_{\mathrm{ss}}$ *and* $\rho_\pi(\mathrm{Frob}_\mathfrak{q})_{\mathrm{ss}}$ *are conjugate, then* ρ *and* ρ' *are conjugate.*

Remark 0.1. The proof of the above theorem relies crucially on the main results of Arthur's book [1], which are currently conditional as explained in footnote 2. In particular Theorem A, and in turn Theorems B, C and D are conditional on the same results mentioned in this footnote.

As explained below Conjecture 1, the existence of Galois representations

$$\rho_{\pi^\flat}\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \mathrm{Gal}(E/F) \tag{0.1}$$

in a weaker form is known for cuspidal automorphic representations $\pi^\flat$ of $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ satisfying (coh°), (St°), and (std-reg°) (see Section 6 for these conditions), and possibly a larger class of representations though we have not worked it out. The main ingredients are Arthur's transfer [1, Thm. 1.5.2] from $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ to $\mathrm{GL}_{2n}(\mathbb{A}_F)$, and collective results on the Langlands correspondence for $\mathrm{GL}_{2n}(\mathbb{A}_F)$ in the self-dual case. Statements (SO-i)–(SO-v) of Theorem 6.5 below summarize what we know about $\rho_{\pi^\flat}$. A main drawback of Theorem 6.5 is that the conjugacy class of each $\rho_{\pi^\flat}(\mathrm{Frob}_\mathfrak{q})_{\mathrm{ss}}$ is determined only up to O_{2n}-conjugacy, rather than SO_{2n}-conjugacy.

Using Theorem A we can upgrade Theorem 6.5 and remove this "outer" ambiguity (coming from the outer automorphism) as long as $\pi^\flat$ can be extended to a cohomological representation π of $\mathrm{GSO}_{2n}^{E/F}$. If π is ξ-cohomological then ξ must satisfy condition (cent) of §9, so a necessary condition for such a cohomological extension to exist is the following condition (which is void for $F = \mathbb{Q}$):

[2] More precisely, the Hodge cocharacter is a half-integral cocharacter, but subtracting $n(n-1)/4$ times sim makes it integral. The two cocharacters in (A3)(a) are well-defined up to conjugacy (i.e., they are conjugacy classes of cocharacters), but the formula makes sense because sim is a central cocharacter.

(cent°) the central character $\{\pm 1\} = \mu_2(F_y) \to \mathbb{C}^\times$ of π_y at each infinite place y of F is independent of y.

Theorem B. *Let $\pi^\flat$ be a cuspidal automorphic representation of $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ satisfying (cent°), (coh°), (St°), and (std-reg°). Then Conjecture 1 holds (for every ℓ and ι). The associated Galois representation $\rho_{\pi^\flat}$ is characterized uniquely up to $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$-conjugation.*

See Theorem 13.1 below for a precise and stronger statement. The crux of the argument lies in showing that $\pi^\flat$ extends to an automorphic representation π of $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$ satisfying conditions of Theorem A. As Theorem A has no outer ambiguity, this yields Theorem B.

Theorem B offers a new perspective on the local Langlands correspondence for quasi-split forms of SO_{2n} over p-adic fields. By localizing the theorem at finite places, we get a candidate for the correspondence, not just up to O_{2n}-conjugacy as in [1]. More precisely, let H denote a quasi-split form of SO_{2n} over a p-adic field k, assumed to be split if n is even. Then we can find E/F as above (depending on the parity of n) and a prime $\mathfrak{q}$ of F such that $F_\mathfrak{q} \simeq k$ and $\mathrm{SO}_{2n,\mathfrak{q}}^{E/F} \simeq H$. If σ is an irreducible discrete series representation of $H(k)$ then a candidate for the L-parameter for σ is described by the following procedures.

(1) Find $\pi^\flat$ satisfying (cent°), (coh°), (St°), and (std-reg°) such that $\pi^\flat_\mathfrak{q} \simeq \sigma$.
(2) Obtain $\rho_{\pi^\flat}$ from Theorem B (which relies on Theorem A).
(3) Take $\mathrm{WD}(\rho_{\pi^\flat}|_{\Gamma_{F_\mathfrak{q}}})$, which can be viewed as an L-parameter for $H(k)$.

The globalization in (1) is possible by a standard trace formula argument proving the limit multiplicity formula. See §1 below for the definition of WD. The L-parameter resulting from the above is in the O_{2n}-orbit of the L-parameter in [1] by Theorem 6.5 (SO-i), but could a priori depend on various choices. It is an interesting problem to relate the global construction here to the purely local constructions by Kaletha [37, 39] and Fargues–Scholze [24]. In fact all this can be mimicked for GSO_{2n} in place of SO_{2n}, using Theorem A rather than Theorem B, so a similar question may be asked in the GSO_{2n}-case.

As another application of Theorem A, we compute the automorphic multiplicities $m(\pi)$ for certain automorphic representations π of $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$.

Theorem C. *Let π be a cuspidal automorphic representation of $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$ satisfying (L-coh), (St) and (std-reg). Then we have $m(\pi) = 1$.*

To compute $m(\pi)$ for $\mathrm{GSO}_{2n}^{E/F}$ we rely on Theorem A, Arthur's multiplicity formula [1] and a result of Bin Xu [81] to show that $m(\pi) = m(\pi^\flat)$ for $\pi^\flat \subset \pi$ a well-chosen $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$-subrepresentation. We remark that Arthur's multiplicity formula computes multiplicities up to an outer automorphism orbit, but $m(\pi)$ in the theorem is the honest multiplicity.

Our final application is meromorphic continuation of the (half) spin-L functions. Let π be a cuspidal automorphic representation of $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$ unramified away from a finite set of places S. To make uniform statements, define a set

$$\mathfrak{e} := \begin{cases} \{+,-\}, & \text{if } n \text{ is even (thus } E = F), \\ \{\emptyset\}, & \text{if } n \text{ is odd (thus } [E:F] = 2), \end{cases} \tag{0.2}$$

with the understanding that $\mathrm{spin}^\emptyset = \mathrm{spin}$. The partial (half-)spin L-function for π away from S is by definition

$$L^S(s, \pi, \mathrm{spin}^\varepsilon) := \prod_{\mathfrak{p} \notin S} \frac{1}{\det(1 - q_\mathfrak{p}^{-s}\mathrm{spin}^\varepsilon(\phi_{\pi_\mathfrak{p}}(\mathrm{Frob}_\mathfrak{p})))}, \qquad \varepsilon \in \mathfrak{e}, \tag{0.3}$$

where $q_\mathfrak{p} := \#(\mathcal{O}_F/\mathfrak{p})$ and $\phi_{\pi_\mathfrak{p}}$ is the unramified L-parameter of $\pi_\mathfrak{p}$. Consider the following hypothesis for L-parameters ϕ_{π_y} at infinite places y.

(spin-reg) $\mathrm{spin}^\varepsilon(\phi_{\pi_y})$ is regular for every infinite place y of F and every $\varepsilon \in \mathfrak{e}$.

When $n \geq 3$, (spin-reg) implies (std-reg). This hypothesis ensures that $\mathrm{spin}^{\varepsilon}(\rho_\pi)$ has distinct Hodge–Tate weights. Our construction and Theorem A allow us to apply the potential automorphy theorem of Barnet-Lamb–Gee–Geraghty–Taylor [3] to the weakly compatible system of $\mathrm{spin}^{\varepsilon}(\rho_\pi)$ (as ℓ and ι vary). Thereby we obtain the following.

Theorem D. *Assume $n \geq 3$. Let π be a cuspidal automorphic representation of $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$ satisfying (L-coh), (St) and (spin-reg). Then there exists a finite totally real extension F'/F (which can be chosen to be disjoint from any prescribed finite extension of F in $\overline{F}$) such that $\mathrm{spin}^{\varepsilon} \circ \rho_\pi|_{\mathrm{Gal}(\overline{F}/F')}$ is automorphic for each $\varepsilon \in \mathfrak{e}$. More precisely, there exists a cuspidal automorphic representation Π^{ε} of $\mathrm{GL}_{2^n/|\mathfrak{e}|}(\mathbb{A}_{F'})$ such that*

- *for each finite place $\mathfrak{q}'$ of F' not above $S_{\mathrm{bad}} \cup \{\ell\}$, the representation $\iota^{-1}\mathrm{spin}^{\varepsilon} \circ \rho_\pi|_{W_{F'_{\mathfrak{q}'}}}$ is unramified and its Frobenius semisimplification is the Langlands parameter for Π^{ε}_w,*
- *at each infinite place y' of F' above a place y of F, we have $\phi_{\Pi^{\varepsilon}_{y'}}|_{W_{\mathbb{C}}} \simeq \mathrm{spin}^{\varepsilon} \circ \phi_{\pi_y}|_{W_{\mathbb{C}}}$.*

In particular the partial spin L-function $L^S(s, \pi, \mathrm{spin}^{\varepsilon})$ admits a meromorphic continuation and is holomorphic and nonzero in an explicit right half plane (e.g., in the region $\Re(s) \geq 1$ if π has unitary central character).

We now give a sketch of the argument for Theorem A. For simplicity, we put ourselves in the *split* case (when n is even), and assume $F = \mathbb{Q}$ to simplify notation. We also ignore all character twists and duals in the following sketch and keep the isomorphism $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$ implicit. (See the main text for correct twists and duals.)

The basic idea is to construct ρ_π and prove its expected properties by understanding what should be $\mathrm{spin}^+ \circ \rho_\pi$, $\mathrm{spin}^- \circ \rho_\pi$, $\mathrm{std} \circ \rho_\pi$, and $\mathcal{N} \circ \rho_\pi$. One already has access to $\mathrm{std} \circ \rho_\pi$ via Arthur's endoscopic classification and known instances of the global Langlands correspondence. The seemingly innocuous $\mathcal{N} \circ \rho_\pi$ is not so trivial to combine with the other representations, but refer to the proof of Proposition 10.5. Most importantly, we realize $\mathrm{spin}^+ \circ \rho_\pi$ and $\mathrm{spin}^- \circ \rho_\pi$ in the cohomology of suitable Shimura varieties; this is the port of embarkation.

In fact ρ_π would not be recovered from $\mathrm{spin}^+ \circ \rho_\pi$, $\mathrm{spin}^- \circ \rho_\pi$, $\mathrm{std} \circ \rho_\pi$, and $\mathcal{N} \circ \rho_\pi$ in general due to essential group-theoretic difficulties (e.g., GSpin_{2n} is not acceptable in the sense of [55, 56]), but condition (St) mitigates the matter. Another important role of (St) is to remove complexity associated with endoscopy.

Our Shimura varieties are associated with an inner twist $G/\mathbb{Q}$ of the split group GSO_{2n} (unique up to isomorphism) which splits at all primes $p \neq p_{\mathrm{St}}$, and whose derived subgroup is isomorphic to the quaternionic orthogonal group $\mathrm{SO}^*(2n)$ over $\mathbb{R}$ (which is not isomorphic to $\mathrm{SO}(a, b)$ for any signature $a + b = 2n$). Concretely $G(\mathbb{R})$ is isomorphic to the group $\mathrm{GSO}_{2n}^J(\mathbb{R})$ in §8 below.

The group G admits two abelian-type Shimura data (G, X^{ε}) with $\varepsilon \in \{+, -\}$, corresponding to the two edges of the "fork" in the Dynkin diagram of type D_n (see Section 9). These two Shimura data are not isomorphic. (The analogous Shimura data *are* isomorphic via an outer automorphism when n is odd; see Lemma (ii) below. Even then, we distinguish the two data as the outer automorphism changes isomorphism classes of representations.)

Let π be as in Theorem A. Using a trace formula argument, we transfer π to a ξ-cohomological cuspidal automorphic representation $\pi^\natural$ of $G(\mathbb{A})$ with isomorphic unramified local components as π such that $\pi^\natural$ is Steinberg at a finite prime. Let $\rho_\pi^{\mathrm{Sh},\varepsilon}$ be the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation on the $\pi^{\natural,\infty}$-isotypical part of the (semisimplified) compact support cohomology of the ℓ-adic local system $\mathcal{L}_\xi/\mathrm{Sh}(G, X^{\varepsilon})$ attached to ξ. Conjecturally the two representations $\rho_\pi^{\mathrm{Sh},\varepsilon}$ should realize $\mathrm{spin}^{\varepsilon} \circ \rho_\pi$ up to semi-simplification (and up to a twist and a multiplicity that we ignore in this introduction), in the non-endoscopic case. In particular, if $\phi_{\pi_p} : W_{\mathbb{Q}_p} \to \mathrm{GSpin}_{2n}(\mathbb{C})$ is the unramified L-parameter of π_p at a prime $p \neq \ell$ where π_p is unramified, then $\rho_\pi^{\mathrm{Sh},\varepsilon}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ ought to be unramified and satisfy

$$\mathrm{Tr}\, \rho_\pi^{\mathrm{Sh},\varepsilon}(\mathrm{Frob}_{\mathfrak{p}}^j) = \mathrm{Tr}\,\mathrm{spin}^{\varepsilon}(\phi_{\pi_p}(\mathrm{Frob}_p)^j) \in \overline{\mathbb{Q}}_\ell, \qquad j \gg 1. \tag{0.4}$$

Employing Kisin's results on the Langlands–Rapoport conjecture [42] and the Langlands–Kottwitz method for Shimura varieties of abelian type in the forthcoming work of Kisin–Shin–Zhu [43], we prove (0.4) for almost all p.

Let $\pi^\flat \subset \pi$ be an irreducible cuspidal automorphic $\mathrm{SO}_{2n}(\mathbb{A})$-subrepresentation. From the aforementioned weaker version of Conjecture 1 for SO_{2n}, we construct (see Theorem 6.5 below)

$$\rho_{\pi^\flat} \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell).$$

such that

$$\rho_{\pi^\flat}(\mathrm{Frob}_p)_{\mathrm{ss}} \overset{\circ}{\sim} \mathrm{pr}^\circ(\phi_{\pi_p}(\mathrm{Frob}_p)) \in \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell), \tag{0.5}$$

for all primes $p \neq \ell$ where $\pi^\flat$ is unramified. Here $\overset{\circ}{\sim}$ indicates $\mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$-conjugacy, and $\mathrm{pr}^\circ : \mathrm{GSpin}_{2n} \twoheadrightarrow \mathrm{SO}_{2n}$ is the natural surjection.

We expect ρ_π to lift $\rho_{\pi^\flat}$ (up to outer automorphism) and to sit inside $\rho^{\mathrm{Sh}} := \rho_\pi^{\mathrm{Sh},+} \oplus \rho_\pi^{\mathrm{Sh},-}$ as illustrated below. By $\overline{\mathrm{spin}}$ we mean the unique projective representation of SO_{2n} that the projectivization of spin factors through.

$$\begin{tikzcd}
\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r, dashed, "\rho_\pi"'] \arrow[rr, bend left, "\rho_\pi^{\mathrm{Sh}}"] \arrow[dr, bend right, "\rho_{\pi^\flat}"'] & \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell) \arrow[r, hook, "\mathrm{spin}"'] \arrow[d, "\mathrm{pr}^\circ"'] & \mathrm{GL}_{2^n}(\overline{\mathbb{Q}}_\ell) \arrow[d] \\
& \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \arrow[r, hook, "\overline{\mathrm{spin}}"'] & \mathrm{PGL}_{2^n}(\overline{\mathbb{Q}}_\ell).
\end{tikzcd} \tag{0.6}$$

We deduce from (0.4) and (0.5) that the outer diagram commutes, after a conjugation if necessary. In fact this is not straightforward because two PGL_{2^n}-valued Galois representations need not be conjugate even if they map each Frob_p into the same conjugacy class for almost all p. We get around the difficulty by using a classification of reductive subgroups of SO_{2n} containing a regular unipotent element by Saxl–Seitz [70]. This is applicable since (St) tells us that the image of $\rho_{\pi^\flat}$ contains a regular unipotent element. As a consequence, the Zariski closure of the image of $\rho_{\pi^\flat}$ is connected mod center. If it is connected, we have the commutativity of (0.6) after a conjugation, and it follows that there exists ρ_π completing the diagram. If the Zariski closure is connected only mod center, then we need a variant of (0.6) as explained in §10. A similar group-theoretic consideration shows that ρ_π is characterized up to isomorphism by the images of Frobenius elements at almost all primes, cf. (A6) of Theorem A.

Having constructed ρ_π, we verify that ρ_π enjoys the expected properties. Let us focus here on (A1). By construction,

$$\mathrm{spin}(\rho_\pi(\mathrm{Frob}_p)_{\mathrm{ss}}) \sim \mathrm{spin}(\phi_{\pi_p}(\mathrm{Frob}_p)), \qquad \text{for almost all } p.$$

The key point is to refine this, or break the symmetry, by showing the same relation with spin^+ and spin^- in place of spin (cf. proof of Proposition 10.5 below) with the help of (0.4). Roughly speaking, we are in a situation

$$\rho^{\mathrm{Sh},+} \oplus \rho^{\mathrm{Sh},-} \simeq \mathrm{spin}^+\rho_\pi \oplus \mathrm{spin}^-\rho_\pi$$

and want to match the $+$ and $-$ parts. The problem is easy enough if $\mathrm{spin}^+\rho_\pi \simeq \mathrm{spin}^-\rho_\pi$ as there is little to distinguish. If $\mathrm{spin}^+\rho_\pi \not\simeq \mathrm{spin}^-\rho_\pi$ then the idea is that the $+$ and $-$ parts do not overlap at sufficiently many places (by a Chebotarev type argument) to match the $+$ and $-$ parts unambiguously. If $\mathrm{spin}^+\rho_\pi$ or $\mathrm{spin}^-\rho_\pi$ is irreducible, it is quite doable to promote this idea to a robust argument. In general, the smaller image of ρ_π, the harder this problem becomes. On the other hand, in certain cases where the image is really small, such as contained in a principal PGL_2, the conjugacy classes $\rho_\pi(\mathrm{Frob}_p)_{\mathrm{ss}}$ are stable under outer conjugation, and there is no distinction between inner and outer conjugacy. As we also have a classification of the (Zariski closure) of the possible images of ρ_π, we can deal with each case via explicit group-theoretic computation. This finishes the sketch of proof for Theorem A.

Structure of the paper. The paper splits roughly into four parts consisting of Sections 1–8 (preparation), Sections 9–12 (the core argument), Sections 13–15 (applications), and the appendices. Let us go over these parts in more detail. In Sections 1–5 we define (variants of) orthogonal groups and spin groups along with subgroups containing regular unipotent elements and the outer automorphism. We define the spin groups and their spin representations through root data as well as Clifford algebras by fixing the underlying quadratic spaces, and clarify the relationship between them. The root-theoretic approach is natural in the context of Langlands correspondence whereas Clifford algebras have the advantage that various maps are determined and diagrams commute on the nose and not just up to conjugation. In Section 6 we construct Galois representations for certain cuspidal automorphic representations of quasi-split even orthogonal groups. This relies on Arthur's book [1] and the known construction of automorphic Galois representations, but a few extra steps are taken to get the information that we need later on. In particular we study what happens to the Steinberg representation under Arthur's transfer from $\mathrm{SO}_{2n}^{E/F}$ to GL_{2n} (this relies on Appendix B). In Section 7 we list a number of basic results on comparing representations of $\mathrm{SO}_{2n}^{E/F}$ with those of $\mathrm{GSO}_{2n}^{E/F}$. Section 8 discusses properties of the real points of $\mathrm{GSO}_{2n}^{E/F}$ and introduces certain global inner forms G of $\mathrm{GSO}_{2n}^{E/F}$. The core argument starts in Section 9, where we take the cohomology of Shimura varieties associated with two Shimura data $(G, X^{\pm})$ to find two Galois representations $\rho_\pi^{\mathrm{Sh},\pm}$ attached to π as in the main theorem. In Section 10 we construct a GSpin_{2n}-valued Galois representation ρ_π of $\mathrm{Gal}(\overline{F}/E)$ from $\rho_\pi^{\mathrm{Sh},\pm}$ and $\rho_{\pi^\flat}$. This representation is not quite the one of Theorem A: The image of Frobenius under ρ_π is controlled only outside an unspecified finite set of primes, and moreover ρ_π should be extended to a representation of $\mathrm{Gal}(\overline{F}/F)$. The two problems are resolved in Sections 11 and 12 respectively. We emphasize that neither of these arguments is formal, the first one relies on Bin Xu's work [81] and the second on a subtle global argument. The proof of Theorem A is also completed in Section 12. Sections 13–15 present applications of our main theorem to the construction of Galois representations for $\mathrm{SO}_{2n}^{E/F}$, automorphic multiplicity, and meromorphic continuation of (half)-spin L-functions.

Acknowledgments. We are very grateful for an anonymous referee for his or her comments and suggestions. SWS is partially supported by NSF grant DMS-1802039 and NSF RTG grant DMS-1646385. AK is partially supported by an NWO VENI and an NWO VIDI grant.

1. Notation and preliminaries

We fix the following notation.

- $n \geq 3$ is an integer.[3]
- If k is a field, $\overline{k}$ denotes an algebraic closure of k.
- When X is a square matrix, $\mathscr{E}\mathscr{V}(X)$ denotes the multi-set of eigenvalues of X.
- When A is a multi-set with elements in a ring R with $r \in R$, write $r \cdot A$ for the multi-set formed by the elements $ra \in A$ as a ranges over A. For $n \in \mathbb{Z}_{>0}$, write $A^{\oplus n}$ for the multi-set consisting of $a \in A$ whose multiplicity in A^n is n times that in A.
- F is a number field. (In the main text, F is a totally real field with a distinguished embedding into $\mathbb{C}$.)
- $\mathcal{O}_F$ is the ring of integers of F.
- $\mathbb{A}_F$ is the ring of adèles of F, $\mathbb{A}_F := (F \otimes \mathbb{R}) \times (F \otimes \widehat{\mathbb{Z}})$.
- If S is a finite set of F-places, then $\mathbb{A}_F^S \subset \mathbb{A}_F$ is the ring of adèles with trivial components at the places in S, and $F_S := \prod_{v \in S} F_v$; $F_\infty := F \otimes_{\mathbb{Q}} \mathbb{R}$.
- If $\mathfrak{q}$ is a finite F-place, we write $q_{\mathfrak{q}}$ for the cardinality of the residue field of $\mathfrak{q}$.
- $|\cdot| : \mathbb{A}_F^\times \to \mathbb{R}_{>0}^\times$ is the norm character on $\mathbb{A}_F^\times$ that is trivial on $F^\times$. Denote by $|\cdot|_v : F_v^\times \to \mathbb{R}_{>0}^\times$ the restriction of $|\cdot|$ to the v-component. Our normalization is that $|\cdot|_{\mathfrak{q}}$

[3] We should mention that if $n \leq 3$, there are exceptional isomorphisms of GSO_{2n} (and its outer forms) to other simpler groups; for instance for $n = 3$ the Shimura varieties that we obtain are (closely related to) Shimura varieties for unitary similitude groups, in particular more general results are already known.

sends a uniformizer of $F_{\mathfrak{q}}$ to $q_{\mathfrak{q}}^{-1}$, whereas $|\cdot|_v$ is the usual absolute value (resp. squared absolute value) when v is real (resp. complex).

- If S is a set of prime numbers we write S^F for the set of F-places above S.
- If p is a prime number, then $F_p := F \otimes_{\mathbb{Q}} \mathbb{Q}_p$.
- ℓ is a prime number (typically different from p).
- $\overline{\mathbb{Q}}_\ell$ is a fixed algebraic closure of $\mathbb{Q}_\ell$, and $\iota\colon \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$ is an isomorphism.
- For each prime number p we fix the positive root $p^{1/2} \in \mathbb{R}_{>0} \subset \mathbb{C}$. From ι we then obtain a choice for $p^{1/2} \in \overline{\mathbb{Q}}_\ell$. If q is a power of p, we obtain similarly a preferred choice $q^{1/2}$ in $\overline{\mathbb{Q}}_\ell$ and in $\mathbb{C}$.
- $\Gamma = \Gamma_F := \mathrm{Gal}(\overline{F}/F)$ is the absolute Galois group of F.
- For a finite extension E of F in $\overline{F}$, write $\Gamma_E := \mathrm{Gal}(\overline{F}/E)$ and $\Gamma_{E/F} := \mathrm{Gal}(E/F)$.
- $\Gamma_v = \Gamma_{F_v} := \mathrm{Gal}(\overline{F}_v/F_v)$ is (one of) the local Galois group(s) of F at the place v, $W_{F_v} \subset \Gamma_v$ is the corresponding Weil group.
- For each F-place v, choose an embedding $\iota_v : \overline{F} \hookrightarrow \overline{F}_v$, which induces $\Gamma_v \hookrightarrow \Gamma$ that is canonical up to conjugation.
- $\mathcal{V}_\infty := \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{R})$ is the set of infinite places of F.
- $c_y \in \Gamma$ is the complex conjugation (well-defined as a conjugacy class) induced by any embedding $\overline{F} \hookrightarrow \mathbb{C}$ extending $y \in \mathcal{V}_\infty$.
- If S is a finite set of F-places, write $\Gamma_{F,S}$ for the Galois group $\mathrm{Gal}(F(S)/F)$ where $F(S) \subset \overline{F}$ is the maximal extension of F that is unramified away from S. If S is a set of rational places we write $\Gamma_{F,S} := \Gamma_{F,S^F}$.
- $\mathrm{Frob}_{\mathfrak{q}}$ at a finite prime $\mathfrak{q}$ of F means the *geometric* Frobenius element in the quotient of $\Gamma_{\mathfrak{q}}$ by the inertia subgroup, or the image thereof in $\Gamma_{F,S}$. (The image in $\Gamma_{F,S}$ depends on the choice of $\iota_{\mathfrak{q}}$ but its conjugacy class is independent of the choice.)
- When G is a connected reductive group over F, write $\widehat{G}$ and ${}^LG = \widehat{G} \rtimes \Gamma_F$ for the Langlands dual group and the L-group, respectively (with coefficients in $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$, depending on the context). If G splits over a finite extension E/F in $\overline{F}$ then $\widehat{G} \rtimes \Gamma_{E/F}$ denotes the L-group with respect to E/F. (Namely such a semi-direct product is always understood with the L-action of $\Gamma_{E/F}$ on $\widehat{G}$.) Often we use ${}^LG$ to mean $\widehat{G} \rtimes \Gamma_{E/F}$.[4]
- When H is a reductive group over $\overline{\mathbb{Q}}_\ell$, we also use H to mean the topological group $H(\overline{\mathbb{Q}}_\ell)$ by abuse of notation. This should be clear from the context and not leading to confusion.
- When F is a p-adic field and G is the set of F-points of a reductive group over F, we write St_G for the Steinberg representation of G (defined in [8, X.4.6] for instance). Moreover, we write $\mathbf{1}_G$ for the trivial representation of G. In certain cases, when G is clear, we write $\mathrm{St} = \mathrm{St}_G$ or $\mathbf{1} = \mathbf{1}_G$. We also write sometimes St_n for $\mathrm{St}_{\mathrm{GL}_n(F)}$ (in case F is clear from the context).
- If G is an algebraic group, we write $Z(G)$ for its center.
- An inner twist of a reductive group G over a perfect field k means a reductive group G' over k together with an isomorphism $i : G_{\overline{k}} \to G'_{\overline{k}}$ such that the automorphism $i^{-1}\sigma(i)$ of $G_{\overline{k}}$ is inner for every $\sigma \in \mathrm{Gal}(\overline{k}/k)$. There is an obvious notion of isomorphism for inner twists (G', i), cf. [38, 2.2]. We often say G' is an inner twist of G, keeping i implicit. If we forget i and only remember the k-group G' and the existence of i, we refer to it as an inner form of G.

Fix G and E/F as above. We introduce some notions on the Galois side. By an (ℓ-adic) *Galois representation* of Γ_F (with values in $\widehat{G} \rtimes \Gamma_{E/F}$), we mean a continuous morphism

$$\rho\colon \Gamma_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$$

which factors through $\Gamma_{F,S}$ for some finite set S and commutes with the obvious projections onto $\Gamma_{E/F}$. Similarly we define a Galois representation with the source $\Gamma_{\mathfrak{q}}$ or with values in ${}^LG(\overline{\mathbb{Q}}_\ell)$.

[4]This is harmless for us as the inflation map induces a bijection of isomorphism classes of ${}^LG$-valued Galois representations when $\Gamma_{E/F}$ is replaced with Γ_F in the semi-direct product.

Two Galois representations are considered isomorphic if they are conjugate by an element of $\widehat{G}(\overline{\mathbb{Q}}_\ell)$. We say that ρ as above is *(totally) odd* if for every real place y of F, the following holds: writing Ad for the adjoint action of ${}^LG$ on $\mathrm{Lie}\, G(\overline{\mathbb{Q}}_\ell)$, which preserves the Lie algebra of the derived subgroup G_{der}, the image of c_y under the composite

$$\Gamma_y \hookrightarrow \Gamma \xrightarrow{\rho} {}^LG(\overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathrm{Lie}\, G_{\mathrm{der}}(\overline{\mathbb{Q}}_\ell))$$

has trace equal to the rank of $\widehat{G}_{\mathrm{der}}$. (Compare with [27].)

An ${}^LG$-valued *Weil–Deligne representation* of $W_{F_\mathfrak{q}}$ is a pair (r, N) consisting of a morphism

$$r \colon W_{F_\mathfrak{q}} \to \widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E_\mathfrak{p}/F_\mathfrak{q}}$$

which has open kernel on the inertia subgroup and commutes with the canonical projections onto $\Gamma_{E_\mathfrak{p}/F_\mathfrak{q}}$, and a nilpotent operator $N \in \mathrm{Lie}\, \widehat{G}(\overline{\mathbb{Q}}_\ell)$ such that $\mathrm{Ad}(r(w))N = |w|N$ for $w \in W_{F_\mathfrak{q}}$, where $|\cdot| : W_{F_\mathfrak{q}} \to \|\mathfrak{q}\|^{\mathbb{Z}}$ is the homomorphism sending a geometric Frobenius element to $\|\mathfrak{q}\|^{-1}$; here $\|\mathfrak{q}\| \in \mathbb{Z}_{>0}$ denotes the norm of $\mathfrak{q}$. The Frobenius-semisimplification (r^{ss}, N) is obtained by replacing r with its semisimplification. We say (r, N) is Frobenius-semisimple if $r = r^{\mathrm{ss}}$.

Let $\rho \colon \Gamma_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$ be a Galois representation. Write $\mathfrak{p}$ for the prime of E induced by $\iota_\mathfrak{q} : \overline{F} \hookrightarrow \overline{F}_\mathfrak{q}$. Then the restriction (via $\iota_\mathfrak{q}$)

$$\rho|_{\Gamma_\mathfrak{q}} \colon \Gamma_{F_\mathfrak{q}} \to \widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E_\mathfrak{p}/F_\mathfrak{q}}$$

gives rise to an ${}^LG$-valued Weil–Deligne representation, to be denoted by $\mathrm{WD}(\rho|_{\Gamma_{F_\mathfrak{q}}})$. The construction follows from the case of $G = \mathrm{GL}_n$ by the Tannakian formalism via algebraic representations of $\widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E_\mathfrak{p}/F_\mathfrak{q}}$. (The case $\mathfrak{q}|\ell$ is more subtle than $\mathfrak{q} \nmid \ell$. In the former case, a detailed explanation is given in the proof of [49, Lem. 3.2], where $\widehat{G}$ is denoted by H. In *loc. cit.* $\Gamma_{E_\mathfrak{p}/F_\mathfrak{q}}$ is trivial but the same argument extends.) When $\mathfrak{q} \nmid \ell$, one can alternatively appeal to Grothendieck's ℓ-adic monodromy theorem to construct $\mathrm{WD}(\rho|_{\Gamma_{F_\mathfrak{q}}})$ directly (without going through general linear groups).

A local L-parameter $\phi : W_{F_\mathfrak{q}} \times \mathrm{SL}(2) \to \widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E_\mathfrak{p}/F_\mathfrak{q}}$ is associated with a Frobenius-semisimple ${}^LG$-valued Weil–Deligne representation (r, N) given by the following recipe:

$$r(w) = \phi\left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right), \qquad \text{and} \qquad N = \phi\left(1, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right).$$

This induces a bijection on the sets of equivalence classes of such objects [28, Prop. 2.2]. In practice (where only equivalence classes matter), we will use them interchangeably.

We introduce some further notation and conventions in representation theory. If π is a representation on a complex vector space then we set $\iota\pi := \pi \otimes_{\mathbb{C},\iota} \overline{\mathbb{Q}}_\ell$. Similarly if ϕ is a local L-parameter of a connected reductive group G over a nonarchimedean local field so that ϕ maps into ${}^LG(\mathbb{C})$, then $\iota\phi$ is the parameter with values in ${}^LG(\overline{\mathbb{Q}}_\ell)$ obtained from ϕ via ι. If G is a locally profinite group equipped with a Haar measure, then we write $\mathcal{H}(G)$ for the *Hecke algebra* of locally constant, complex valued functions with compact support. We write $\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(G)$ for the same algebra, but now consisting of $\overline{\mathbb{Q}}_\ell$-valued functions. We normalize every parabolic induction by the half power of the modulus character as in [7, 1.8], so that it preserves unitarity.

Let G be a real reductive group, K a maximal compact subgroup of $G(\mathbb{R})$, and $\tilde{K} := K \cdot Z(G)(\mathbb{R})$. Let ξ be an irreducible algebraic representation of G over $\mathbb{C}$. An irreducible admissible representation π of $G(\mathbb{R})$ is said to be *ξ-cohomological* if $H^i(\mathrm{Lie}\, G(\mathbb{C}), \tilde{K}, \pi \otimes_\mathbb{C} \xi) \neq 0$ for some $i \geq 0$. If this is the case, we assign a Hodge cocharacter over $\mathbb{C}$ (well-defined up to $\widehat{G}$-conjugacy) as in [49, Def 1.14]:

$$\mu_{\mathrm{Hodge}}(\xi) \colon \mathbb{G}_{\mathrm{m}} \to \widehat{G}.$$

Let L be a finite extension of $\mathbb{Q}_\ell$. Let H be a possibly disconnected reductive group over $\overline{\mathbb{Q}}_\ell$ (e.g., an L-group relative to a finite Galois extension), and $\rho \colon \mathrm{Gal}(\overline{L}/L) \to H(\overline{\mathbb{Q}}_\ell)$ a continuous morphism. If ρ is Hodge–Tate with respect to each $\mathbb{Q}_\ell$-embedding $i \colon L \hookrightarrow \overline{\mathbb{Q}}_\ell$, we define a

Hodge–Tate cocharacter over $\overline{\mathbb{Q}}_\ell$ (well-defined up to H-conjugacy) as in [12, §2.4] (cf. [49, Def 1.10]):

$$\mu_{\mathrm{HT}}(\rho, i) : \mathbb{G}_m \to H.$$

We recall the following lemma that can be easily deduced from the Chebotarev density theorem, as it will be needed in §10. Let F be a number field. The *density* of a set S consisting of primes of F is defined to be the limit $d(S) = \lim_{n\to\infty} a_n(S)/a_n(F)$, where $a_n(F)$ is the number of primes $\mathfrak{q}$ with bounded norm $\|\mathfrak{q}\| < n$ and $a_n(S)$ is the number of $\mathfrak{q} \in S$ with $\|\mathfrak{q}\| < n$ [72, Sect. I.2.2]. Depending on S, the limit $d(S)$ may or may not exist — in the former case, we say S has density $d(S)$, and otherwise we leave the density undefined.

Lemma 1.1. *Let S be a finite set of places of a number field F. Let $G/\overline{\mathbb{Q}}_\ell$ be a linear algebraic group and let $r\colon \Gamma_{F,S} \to G(\overline{\mathbb{Q}}_\ell)$ be a Galois representation with Zariski dense image. Let $X \subset G$ be a closed subvariety that is invariant by G-conjugation and such that* $\dim(X) < \dim(G)$. *Then the set of F-places $\mathfrak{q} \notin S$ with $r(\mathrm{Frob}_\mathfrak{q}) \in X(\overline{\mathbb{Q}}_\ell)$ has density* 0.

Proof. Let μ be the Haar measure on $\Gamma_S = \Gamma_{F,S}$ with total volume 1. We write X to also mean $X(\overline{\mathbb{Q}}_\ell)$ to simplify notation. Then $Y = r^{-1}(X)$ is a closed subset of Γ_S (hence measurable) and stable under Γ_S-conjugation. If we further have that $\mu(Y) = 0$, then the Chebotarev density theorem [72, I-8 Cor. 2b] implies that the set of places $\mathfrak{q} \notin S$ such that $\mathrm{Frob}_\mathfrak{q} \in Y$ has measure 0, so we will be done.

So it suffices to prove that $\mu(Y) = 0$. We induct on $\dim(X) \in \{0, 1, \ldots, \dim(G)-1\}$. We may assume that X is irreducible by induction. When $\dim X = 0$ then X is a point and the preimage Y of X is a torsor under $\ker(r)$. We then have $\mathrm{vol}(Y) = \mathrm{vol}(\ker(r)) = 0$, since $\ker(r) \subset \Gamma_S$ is a closed subgroup of infinite index by hypothesis. Now assume that the assertion is known whenever $\dim(X) < d$ and consider the case $\dim(X) = d < \dim(G)$. There exists an infinite sequence $\gamma_1, \gamma_2, \ldots \in \Gamma_S$ such that the subset $r(\gamma_i)X$ are mutually distinct. (If the choice were impossible after $i = r$, then multiplication by $r(g)$ preserves $\bigcup_{i<=r} r(\gamma_i)X$ for every $g \in \Gamma_S$. This can't happen because r has Zariski dense image, and the union has dimension $d < \dim(G)$.) Consider

$$\Gamma_S \supset \bigcup_{i=1}^{\infty} \gamma_i r^{-1}(X).$$

The volume of Γ_S is finite. Each term on the right hand side is closed (so measurable), and the volumes of $\gamma_i r^{-1}(X)$ are all equal. We claim that their pairwise intersections have volume 0. If this is true, then we deduce that $\mathrm{vol}(\gamma_i r^{-1}(X)) = \mathrm{vol}(r^{-1}(X)) = 0$, completing the proof.

It remains to verify the claim. Observe that the intersection

$$(*) \qquad \gamma_i r^{-1}(X) \cap \gamma_j r^{-1}(X), \qquad i \neq j,$$

maps into the intersection $r(\gamma_i)X \cap r(\gamma_j)X$ in G, which has dimension less than d, so indeed (*) has measure 0 by induction hypothesis. This completes the proof. $\square$

We also record a lemma on projective algebraic representations, which will be usefull later on.

Lemma 1.2. *Let G be a connected simply-connected semi-simple group over $\mathbb{C}$. Let $T \subset G$ be a maximal torus. Let $r_1, r_2\colon G \to \mathrm{PGL}_N$ be two projective representations whose restrictions to T are conjugate. Then r_1, r_2 are conjugate.*

Proof. We claim that any $r\colon G \to \mathrm{PGL}_N$ can be lifted to a representation $\widetilde{r}\colon G \to \mathrm{SL}_N$. Let $H := (G \times_{\mathrm{PGL}_N} \mathrm{SL}_N)^0$, then $f\colon H \to G$ is a central isogeny, and hence is an isomorphism as G is simply connected [61, Prop. 18.8]. The composition $G \to H \to \mathrm{SL}_N$ is the desired lift.

After conjugating, we may assume that $r_1|_T = r_2|_T$. By the preceding paragraph, we can choose lifts $\widetilde{r}_i$ of r_i for $i = 1, 2$. Define a morphism of varieties $\chi : G \to \mathrm{SL}_N$ by $\chi(g) := \widetilde{r}_1(g)\widetilde{r}_2(g)^{-1}$. The image of $\chi|_T$ lies in μ_N since $r_1|_T = r_2|_T$. Hence the image is trivial as T is connected, that is, $\widetilde{r}_1|_T = \widetilde{r}_2|_T$. Hence $\widetilde{r}_1$, $\widetilde{r}_2$ are GL_N-conjugate because the trace functions coincide on semisimple elements. It follows that r_1, r_2 are PGL_N-conjugate. $\square$

2. Root data of GSO_{2n} and GSpin_{2n}

Let $\mathrm{GO}_{2n}/\mathbb{Q}$ be the algebraic group such that for all $\mathbb{Q}$-algebras R we have
(2.1)
$$\mathrm{GO}_{2n}(R) = \left\{ g \in \mathrm{GL}_{2n}(R) \;\middle|\; \exists \operatorname{sim}(g) \in R^\times \;:\; g^{\mathrm{t}} \cdot \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix} \cdot g = \operatorname{sim}(g) \cdot \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix} \right\}.$$
(in the above formula 1_n is the $n \times n$ identity matrix.) The group GO_{2n} is disconnected; its neutral component $\mathrm{GSO}_{2n} \subset \mathrm{GO}_{2n}$ is defined by the condition $\det(g) = \operatorname{sim}(g)^n$. The groups GO_{2n}, GSO_{2n} are split and defined by a quadratic form of signature (n, n). An element t of the diagonal torus $\mathrm{T}_{\mathrm{GSO}} \subset \mathrm{GSO}_{2n}$ is of the form

$$t = \operatorname{diag}(t_i)_{i=1}^{2n} = \operatorname{diag}(t_1, t_2, \ldots, t_n, t_0 t_1^{-1}, t_0 t_2^{-1}, \ldots, t_0 t_n^{-1}), \quad t_0 := \operatorname{sim}(t) \tag{2.2}$$

hence $\mathrm{T}_{\mathrm{GSO}} \simeq \mathbb{G}_{\mathrm{m}}^{n+1}$ by sending t to $(t_0, t_1, \ldots, t_n)$. We identify $X^*(\mathrm{T}_{\mathrm{GSO}}) = \bigoplus_{i=0}^n \mathbb{Z} \cdot e_i$ and $X_*(\mathrm{T}_{\mathrm{GSO}}) = \bigoplus_{i=0}^n \mathbb{Z} \cdot e_i^*$ accordingly. We let $\mathrm{B}_{\mathrm{GSO}}$ be the Borel subgroup of GSO_{2n} of matrices of the form

$$g = \begin{pmatrix} A & AB \\ 0 & cA^{\mathrm{t},-1} \end{pmatrix}, \quad A \in B_{\mathrm{GL}_n},\ B \in \mathrm{M}_n,\ B^{\mathrm{t}} = -B \quad \text{and} \quad c = \operatorname{sim}(g), \tag{2.3}$$

where $B_{\mathrm{GL}_n} \subset \mathrm{GL}_n$ is the upper triangular Borel subgroup. (To see that $\mathrm{B}_{\mathrm{GSO}}$ is indeed a Borel subgroup, notice that any block matrix $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with $C = 0$ is of the above form if and only if $g \in \mathrm{GSO}_{2n}$, and moreover the displayed group is solvable of dimension $n^2 + 1$).

We realize the split forms of even (special) orthogonal groups in $\mathrm{GO}_{2n}/\mathbb{Q}$. Namely we write O_{2n} (resp. SO_{2n}) for the subgroup of GO_{2n} (resp. GSO_{2n}) where sim is trivial.

Lemma 2.1. *The root datum of* GSO_{2n} *with respect to* $\mathrm{B}_{\mathrm{GSO}}$ *is described as follows.*
- *(i) The set of roots (resp. coroots) consists of* $\pm(e_i - e_j)$ *and* $\pm(e_i + e_j - e_0)$ *(resp.* $\pm(e_i^* - e_j^*)$ *and* $\pm(e_i^* + e_j^*)$*) with* $1 \le i < j \le n$.
- *(ii) The positive roots are* $\{e_i + e_j - e_0\}_{1 \le i < j \le n} \cup \{e_i - e_j\}_{1 \le i < j \le n}$ *and the positive coroots are* $\{e_i^* \pm e_j^*\}_{1 \le i < j \le n}$.
- *(iii) The simple roots are* $\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n$*, and* $\alpha_n = e_{n-1} + e_n - e_0$.
- *(iv) The simple coroots* $\Delta^\vee$ *are* $\alpha_1^\vee = e_1^* - e_2^*$, $\alpha_2^\vee = e_2^* - e_3^*$, $\ldots$, $\alpha_{n-1}^\vee = e_{n-1}^* - e_n^*$*, and* $\alpha_n^\vee = e_{n-1}^* + e_n^*$.

Remark 2.2. The root datum of SO_{2n} is described similarly. Putting $\mathrm{T}_{\mathrm{SO}} := \mathrm{T}_{\mathrm{GSO}} \cap \mathrm{SO}_{2n}$ and $\mathrm{B}_{\mathrm{SO}} := \mathrm{B}_{\mathrm{GSO}} \cap \mathrm{SO}_{2n}$, we have $\mathrm{T}_{\mathrm{SO}} = \{t \in \mathrm{T}_{\mathrm{GSO}} : t_0 = 1\}$ as well as $X^*(\mathrm{T}_{\mathrm{SO}}) = \oplus_{i=1}^n e_i \cdot \mathbb{Z}$ and $X_*(\mathrm{T}_{\mathrm{SO}}) = \oplus_{i=1}^n e_i^* \cdot \mathbb{Z}$. To describe (positive or simple) roots and coroots, we only need to formally set $e_0 = 0$ in the lemma above.

Proof. The standard computation for SO_{2n} as in [25, 18.1] can be easily adapted to GSO_{2n}. $\square$

We define the following element (over any $\mathbb{Q}$-algebra point of O_{2n})[5]

$$\vartheta^\circ := -\left(\begin{array}{cc|cc} 1_{n-1} & & & \\ & 0 & & 1 \\ \hline & & 1_{n-1} & \\ & 1 & & 0 \end{array}\right) \in \mathrm{O}_{2n}. \tag{2.4}$$

Since $\det(\vartheta^\circ) = -1$ we have $\vartheta^\circ \notin \mathrm{SO}_{2n}$. We write $\theta^\circ \in \mathrm{Aut}(\mathrm{GSO}_{2n})$ for the automorphism given by ϑ°-conjugation.

Lemma 2.3. *The automorphism* θ° *stabilizes* $\mathrm{B}_{\mathrm{GSO}}$ *and* $\mathrm{T}_{\mathrm{GSO}}$*, and acts on* $\mathrm{T}_{\mathrm{GSO}}$ *by*
$$(t_0, t_1, \ldots, t_n) \mapsto (t_0, t_1, \ldots, t_{n-1}, t_0 t_n^{-1}).$$
Furthermore $\theta^\circ(\alpha_i) = \alpha_i$ *for* $i < n-2$, $\theta^\circ(\alpha_{n-1}) = \alpha_n$*, and* $\theta^\circ(\alpha_n) = \alpha_{n-1}$.

Proof. By a direct computation, $\theta^\circ(\mathrm{T}_{\mathrm{GSO}}) = \mathrm{T}_{\mathrm{GSO}}$ and $\theta^\circ(\mathrm{B}_{\mathrm{GSO}}) = \mathrm{B}_{\mathrm{GSO}}$. Since θ° only switches t_n and $t_{2n} = t_0 t_n^{-1}$, its action on $\mathrm{T}_{\mathrm{GSO}}$ is explicitly described as in the lemma. Thus $\theta^\circ(e_i) = e_i$ for $1 \le i \le n-1$ and $\theta^\circ(e_n) = e_0 - e_n$, from which the last assertion follows. $\square$

[5] The minus sign for ϑ° makes it compatible with $\vartheta \in \mathrm{GSpin}_{2n}$ to be introduced above Lemma 3.6.

We define GSpin_{2n} to be the Langlands dual group $\widehat{\mathrm{GSO}_{2n}}$ over $\mathbb{C}$ (or later over $\overline{\mathbb{Q}}_\ell$ via $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$). That is, GSpin_{2n} is the connected reductive group over $\mathbb{C}$, equipped with a Borel subgroup B_{GSpin} and a maximal torus $\mathrm{T}_{\mathrm{GSpin}}$, whose based root datum is dual to the one of GSO_{2n} that we described above. In particular

$$X_*(\mathrm{T}_{\mathrm{GSpin}}) = X^*(\mathrm{T}_{\mathrm{GSO}}) \quad \text{and} \quad X^*(\mathrm{T}_{\mathrm{GSpin}}) = X_*(\mathrm{T}_{\mathrm{GSO}}).$$

Via the identification $X^*(\mathrm{T}_{\mathrm{GSO}}) = \mathbb{Z}^{n+1}$, we represent elements $s \in \mathrm{T}_{\mathrm{GSpin}}$ as $(s_0, s_1, \dots, s_n)$. In Section 3 we will also define an explicit model of GSpin_{2n} over $\mathbb{Q}$ using Clifford algebras.

Lemma 2.4. *There is a unique $\theta \in \mathrm{Aut}(\mathrm{GSpin}_{2n})$ that fixes $\mathrm{T}_{\mathrm{GSpin}}$ and $\mathrm{B}_{\mathrm{GSpin}}$, switches $\alpha_{n-1}^\vee$ and $\alpha_n^\vee$, leaves the other $\alpha_i^\vee$ invariant, and induces the trivial automorphism of the cocenter of GSpin_{2n}. We have $\theta^2 = 1$, and on the torus $\mathrm{T}_{\mathrm{GSpin}}$ the involution θ is given by*

$$(s_0, s_1, \dots, s_n) \mapsto (s_0 s_n, s_1, \dots, s_{n-1}, s_n^{-1}). \tag{2.5}$$

Proof. We have $\theta(e_i^* - e_{i+1}^*) = e_i^* - e_{i+1}^*$ $(1 \le i < n)$ and $\theta(e_{n-1}^* - e_n^*) = e_{n-1}^* + e_n^*$. Thus

$$\theta(e_i^*) = e_i^* \;\; (1 \le i < n) \quad \text{and} \quad \theta(e_n^*) = -e_n^*. \tag{2.6}$$

The center of GSO_{2n} is the image of $\mathbb{G}_{\mathrm{m}} \ni z \mapsto (z^2, z, \dots, z) \in \mathrm{T}_{\mathrm{GSO}}$. The dual map is

$$\mathrm{T}_{\mathrm{GSpin}} \to \mathbb{G}_{\mathrm{m}}, \quad (s_0, s_1, \dots, s_n) \mapsto s_0^2 s_1 \cdots s_n. \tag{2.7}$$

Thus $\theta(2e_0^* + e_1^* + \dots + e_n^*) = 2e_0^* + e_1^* + \dots + e_n^*$, so $\theta(2e_0^*) - e_n^* = 2e_0^* + e_n^*$ and $\theta(e_0^*) = e_0^* + e_n^*$. □

Lemma 2.5. *We have $Z(\mathrm{GSpin}_{2n}) = \{(s_0, \dots, s_n) : s_1 = s_2 = \dots = s_n \in \{\pm 1\}\}$, which is isomorphic to $\mathbb{G}_{\mathrm{m}} \times \{\pm 1\}$ via $(s_0, \dots, s_n) \mapsto (s_0, s_1)$. In the latter coordinate, $\theta(s_0, s_1) = (s_0 s_1, s_1)$.*

Proof. Let $s \in \mathrm{T}_{\mathrm{GSpin}}$. Then $s \in Z(\mathrm{GSpin}_{2n})$ if and only if $\alpha^\vee(t) = 1$ for all $\alpha^\vee \in \Delta^\vee$. From Lemma 2.1(*iii*) we obtain $s_i/s_{i+1} = 1$ $(i \le n-1)$, and $s_{n-1}s_n = 1$. Hence $s \in Z(\mathrm{GSpin}_{2n})$ if and only if $s_1 = \dots = s_n \in \{\pm 1\}$. By (2.5) we get $\theta(s_0, s_1) = (s_0 s_1, s_1)$. □

The Weyl group of GSO_{2n} (and GSpin_{2n}) is equal to $\{\pm 1\}^{n,\prime} \rtimes \mathfrak{S}_n$, where $\{\pm 1\}^{n,\prime}$ is the group of $a \in \{\pm 1\}^n$ such that $\prod_1^n a(i) = 1$. The action of $\mathrm{W}_{\mathrm{GSO}}$ on $\mathrm{T}_{\mathrm{GSO}}$ is determined by

$$\begin{cases} \sigma \cdot (t_0, t_1, \dots, t_n) = (t_0, t_{\sigma(1)}, \dots t_{\sigma(n)}) & \sigma \in \mathfrak{S}_n \\ a \cdot (t_0, t_1, \dots, t_n) = (t_0, t_0 t_1^{-1}, t_0 t_2^{-1}, t_3, \dots, t_n) & a = (-1, -1, 1 \dots, 1) \in \{\pm 1\}^{n,\prime}. \end{cases} \tag{2.8}$$

We define, for $\varepsilon \in \{\pm 1\}$ the following cocharacter

$$\mu_\varepsilon := \begin{cases} (1, 1, \dots, 1, 1) & \text{if } \varepsilon = (-1)^n \\ (1, 1, \dots, 1, 0) & \text{if } \varepsilon = (-1)^{n+1} \end{cases} \in \mathbb{Z}^{n+1} = X_*(\mathrm{T}_{\mathrm{GSO}}) = X^*(\mathrm{T}_{\mathrm{GSpin}}). \tag{2.9}$$

Then μ_ε is a minuscule cocharacter of GSO_{2n} with $\langle \alpha_i, \mu_\varepsilon \rangle = 1$ if and only if $i = n$ (for $\varepsilon = (-1)^n$) and $i = n-1$ (for $\varepsilon = (-1)^{n+1}$).

Definition 2.6. For $\varepsilon \in \{+, -\}$, define the *half-spin representation* $\mathrm{spin}^\varepsilon = \mathrm{spin}_{2n}^\varepsilon$ to be the irreducible representation of GSpin_{2n} whose highest weight is equal to μ_ε in $X^*(\mathrm{T}_{\mathrm{GSpin}})$. By the *spin representation* of GSpin_{2n} we mean $\mathrm{spin} := \mathrm{spin}^+ \oplus \mathrm{spin}^-$.

These representations will be realized explicitly via Clifford algebras. Our sign convention is natural in that spin^+ (resp. spin^-) accounts for even (resp. odd) degree elements. See (4.2) and Lemma 4.1 below.

The minuscule μ_ε has 2^{n-1} translates under the Weyl group action. Thus each half-spin representation has dimension 2^{n-1}. More precisely the weights of $\mathrm{spin}_{2n}^{(-1)^n}$ are

$$\mathrm{T}_{\mathrm{GSpin}} \ni (s_0, s_1, \dots, s_n) \mapsto \left(s_0 \prod_{i \in U} s_i \right)_{\substack{U \subset \{1,2,\dots,n\}, \\ 2 \mid \# U}} \in \mathbb{Z}^{n+1} = X_*(\mathrm{T}_{\mathrm{GSO}}) = X^*(\mathrm{T}_{\mathrm{GSpin}}) \tag{2.10}$$

and $\mathrm{spin}_{2n}^{(-1)^{n+1}}$ has similar weights, except that the cardinality of U is now required to be odd. By computing the θ-action on weights, we verify that (see Lemma 4.4 for an explicit intertwiner)

$$\mathrm{spin}^+ \circ \theta \simeq \mathrm{spin}^- \quad \text{and } \mathrm{spin}^- \circ \theta \simeq \mathrm{spin}^+.$$

Lemma 2.7. *The kernel Z^ε of $\mathrm{spin}^\varepsilon$ is central in GSpin_{2n} and finite of order 2. The non-trivial element z_ε of Z^ε equals $(\varepsilon, -1) \in \mathbb{G}_{\mathrm{m}} \times \{\pm 1\}$. The spin representation of GSpin_{2n} is faithful.*

Proof. Since GSpin_{2n} is simple modulo the center, the kernel $Z^\varepsilon \subset \mathrm{GSpin}_{2n}$ must be central. The central character is the restriction of $\mu_\varepsilon \colon \mathrm{T}_{\mathrm{GSpin}} \to \mathbb{G}_{\mathrm{m}}$ to the center $Z(\mathrm{GSpin}_{2n}) \subset \mathrm{T}_{\mathrm{GSpin}}$. Let $s = (s_0, s_1, \ldots, s_n) = (a, b) \in Z(\mathrm{GSpin}_{2n}) \subset \mathrm{T}_{\mathrm{GSpin}}$. Then (see proof of Lemma 2.5)

$$\mu_\varepsilon(s) = \begin{cases} s_0 s_1 \cdots s_n = ab^n & \text{if } \varepsilon = (-1)^n \\ s_0 s_1 \cdots s_{n-1} = ab^{n-1} & \text{if } \varepsilon = (-1)^{n+1}. \end{cases} \tag{2.11}$$

The first assertion follows by considering the 4 different cases where n even or odd and $\varepsilon = \pm 1$. For the second point, it suffices to observe that $Z^+ \cap Z^- = \{1\}$. □

Later on the following fact on SO_{2n-1} will be needed, so we record it here.

Lemma 2.8. *Let $n \geq 3$. Up to isomorphism the group SO_{2n-1} has exactly one faithful representation of dimension $2n$, namely $\mathrm{std}_{2n-1} \oplus \mathbf{1}$.*

Proof. We use the root system notation and conventions from [10, Ch. 4, p. 253]. Assume V_λ is a non-trivial irreducible representation of SO_{2n-1} with highest weight $\lambda \neq \omega_1$. We show that $\dim(V_\lambda) > 2n$. Write $\lambda = \sum_{i=1}^{n-1} x_i \omega_i$ with $x_i \geq 0$. If $x_i \neq 0$ for some i with $n-1 > i > 1$, then $\dim(V_\lambda) \geq \dim(V_{\omega_i})$, and $\dim(V_{\omega_i}) = \dim(\wedge^i \mathrm{std}) > 2n$. We thus assume $\lambda = x_1\omega_1 + x_{n-1}\omega_{n-1}$. If $x_{n-1} = 0$, then, as $\lambda \neq \omega_1$, we have $\dim(V_\lambda) \geq \dim(V_{2\omega_1}) = (n-1)(2n+1) > 2n$ by the Weyl dimension formula. Assume $x_{n-1} \neq 0$. We can't have $x_{n-1} = 1$, because then V_λ does not descend to SO_{2n-1}. Thus $\dim(V_\lambda) \geq \dim(V_{2\omega_{n-1}})$ which equals 10 if $n = 3$, 35 if $n = 4$, and if $n > 4$ then $\dim(V_{2\omega_{n-1}}) \geq \dim(V_{\omega_{n-1}}) = 2^{n-1} > 2n$. □

3. Clifford algebras and Clifford groups

We recall how GSpin_{2n} is realized using the Clifford algebra, and define a number of fundamental maps such as $i_{\mathrm{std}} \colon \mathrm{GSpin}_{2n-1} \hookrightarrow \mathrm{GSpin}_{2n}$ and the projections from GSpin_{2n} to GSO_{2n} and SO_{2n}. We also give a concrete definition of outer automorphisms θ of GSpin_{2n} and θ° of GSO_{2n}. Our main reference is [4], which introduces Clifford algebras over arbitrary commutative rings (with unity). Other useful references are [9, §9] and [25, §20].

Let V be a quadratic space over $\mathbb{Q}$ with quadratic form Q, giving rise to the groups $\mathrm{O}(V)$, $\mathrm{GO}(V)$, $\mathrm{SO}(V)$ and $\mathrm{GSO}(V)$. The *Clifford algebra* $C(V)$ is a universal map $V \to C(V)$ which is initial in the category of $\mathbb{Q}$-linear maps $f \colon V \to A$ into associative $\mathbb{Q}$-algebras A with unity 1_A such that $f(v)^2 = Q(v) \cdot 1_A$ for all $v \in V$. (See [4, (2.3)] or [9, §9.1].)

We define $\langle x, y \rangle := Q(x+y) - Q(x) - Q(y)$ for $x, y \in V$, and similarly $\langle x, y \rangle = (x+y)^2 - x^2 - y^2$ for $x, y \in C(V)$. In particular $\langle x, y \rangle$ measures if x and y anti-commute in $C(V)$:

$$\langle x, y \rangle = (x+y)^2 - x^2 - y^2 = xy + yx \in C(V). \tag{3.1}$$

The map $V \to C(V)$ induces a map $V \to C(V)^{\mathrm{opp}}$ (sending each $v \in V$ to the same element), where $C(V)^{\mathrm{opp}}$ is the opposite algebra. The latter factors through a unique $\mathbb{Q}$-algebra map $\beta \colon C(V) \to C(V)^{\mathrm{opp}}$. It is readily checked that β^2 is the identity on $C(V)$. By the universal property β is the unique involution of $C(V)$ that is the identity on V.

The universal property also yields a surjection from the tensor algebra

$$\bigoplus_{d \in \mathbb{Z}_{\geq 0}} V^{\otimes d} \twoheadrightarrow C(V).$$

Define $C^+ = C(V)^+$ (resp. $C^- = C(V)^-$) to be the image of $\oplus_{d \in \mathbb{Z}_{\geq 0}} V^{\otimes 2d}$ (resp. $\oplus_{d \in \mathbb{Z}_{\geq 0}} V^{\otimes 2d+1}$) so that $C(V) = C(V)^+ \oplus C(V)^-$. In fact the discussion of Clifford algebras so far works when V is replaced with a quadratic space on a module over an arbitrary commutative ring, in a way compatible with base change: in particular if R is a (commutative) $\mathbb{Q}$-algebra then $C(V \otimes_{\mathbb{Q}} R) = C(V) \otimes_{\mathbb{Q}} R$ [9, §9.1, Prop 2]. By scalars in $C(V \otimes_{\mathbb{Q}} R)$ we mean R times the multiplicative unity. We keep using β to denote the main involution of $C(V \otimes_{\mathbb{Q}} R)$.

The *Clifford group* $\mathrm{GPin}(V)$ is the $\mathbb{Q}$-group such that for every $\mathbb{Q}$-algebra R,

$$\mathrm{GPin}(V)(R) = \{x \in C(V \otimes_{\mathbb{Q}} R)^{\times} : x(V \otimes_{\mathbb{Q}} R)x^{-1} = V \otimes_{\mathbb{Q}} R,\ x \text{ is homogeneous}\},$$

where homogeneity of x means that $x \in C(V \otimes_{\mathbb{Q}} R)^{\varepsilon}$ for some sign ε. The *special Clifford group* $\mathrm{GSpin}(V)$ is defined similarly with C^+ in place of C. The embedding of invertible scalars in $C(V \otimes_{\mathbb{Q}} R)$ induces a central embedding

$$\mathbb{G}_m \to \mathrm{GSpin}(V). \tag{3.2}$$

Since $x\beta(x) \in R$ for $x \in C(V \otimes_{\mathbb{Q}} R)$ by [4, Prop 3.2.1 (a)], we have the *spinor norm* morphism

$$\mathcal{N} \colon \mathrm{GPin}(V) \to \mathbb{G}_{\mathrm{m}}, \qquad x \mapsto x\beta(x)$$

over $\mathbb{Q}$. (The involution in *loc. cit.* differs from our β by $C(-1_P)$ in their notation, so our $\mathcal{N}$ does not coincide with their N, but $\mathcal{N}$ and N have the same kernel.) Evidently, composing $\mathcal{N}$ with (3.2) yields the squaring map.

Define $\mathrm{Spin}(V)$ by the following exact sequence of algebraic groups:

$$1 \to \mathrm{Spin}(V) \to \mathrm{GSpin}(V) \xrightarrow{\mathcal{N}} \mathbb{G}_m \to 1.$$

Lemma 3.1. *The following are true, where kernels and surjectivity are always meant in the category of algebraic groups over* $\mathbb{Q}$.

(i) The map $\mathrm{pr}^\circ = \mathrm{pr}^\circ_V \colon \mathrm{GPin}(V) \to \mathrm{O}(V)$, $x \mapsto (v \mapsto xvx^{-1})$ *is surjective for* $\dim V$ *even, and* $\mathrm{pr}^\circ \colon \mathrm{GPin}(V) \to \mathrm{SO}(V)$ *is surjective when* $\dim V$ *is odd.*

(ii) We have $\ker(\mathrm{pr}^\circ) = \mathbb{G}_{\mathrm{m}}$ *via* (3.2).

(iii) $\mathrm{pr} \colon \mathrm{GPin}(V) \to \mathrm{GO}(V)$, $x \mapsto (v \mapsto xv\beta(x))$ *is a surjection, and* $\mathrm{sim} \circ \mathrm{pr} = \mathcal{N}^2$.

(iv) The map pr *factors as* $\mathrm{GPin}(V) \xrightarrow{(\mathrm{pr}^\circ, \mathcal{N})} \mathrm{O}(V) \times \mathrm{GL}_1 \xrightarrow{\text{mult.}} \mathrm{GO}(V)$, *where the latter is the multiplication map. The map* $(\mathrm{pr}^\circ, \mathcal{N})$ *has kernel* μ_2 *(scalars* $\{\pm 1\}$ *in* $C(V)$*) and image* $\mathrm{O}(V) \times \mathrm{GL}_1$ *(resp.* $\mathrm{SO}(V) \times \mathrm{GL}_1$*) for n even (resp. odd).*

(v) The multiplication map $\mathrm{Spin}(V) \times \mathbb{G}_m \to \mathrm{GSpin}(V)$ *is a surjection with kernel* $\{\pm(1,1)\}$ *(diagonally embedded* μ_2*), where* $\{\pm 1\} \hookrightarrow \mathrm{Spin}(V)$ *via* (3.2).

Proof. (i) The surjectivity can be checked on field-valued points. This is proved in [9, §9.5, Thm. 4].

(ii) As $V \subset C(V)$ generates the Clifford algebra, the identity $xvx^{-1} = v$ implies $xyx^{-1} = y$ for all $y \in C(V)$, and the analogue holds for $C(V \otimes_{\mathbb{Q}} R)$ for $\mathbb{Q}$-algebras R. Thus $\ker(\mathrm{pr}^\circ)(R)$ consists of invertible elements in the center of $C(V \otimes_{\mathbb{Q}} R)$. Let $W \subset V$ be an isotropic subspace. Then $C(V \otimes_{\mathbb{Q}} R) \simeq \mathrm{End}(\bigwedge(W \otimes_{\mathbb{Q}} R))$ as super R-algebras by [4, (2.4) Thm.], so the center of $C(V \otimes_{\mathbb{Q}} R)$ is R, implying that $\ker(\mathrm{pr}^\circ) = \mathbb{G}_{\mathrm{m}}$.

(iii) We observe that $\mathrm{pr}(x)$ preserves V: as $x(V \otimes_{\mathbb{Q}} R)x^{-1} = V \otimes_{\mathbb{Q}} R$ and $x\beta(x) \in R^\times$ imply that $x(V \otimes_{\mathbb{Q}} R)\beta(x) = V \otimes_{\mathbb{Q}} R$. Moreover $\mathrm{pr}(x) \in \mathrm{GO}(V)$ as

$$Q(xv\beta(x)) = xv\beta(x)xv\beta(x) = \mathcal{N}(x)^2 Q(v). \tag{3.3}$$

Moreover pr and pr° coincide on $\mathrm{Pin}(V)$, so $\mathrm{(S)O}(V)$ is in the image of pr. On the other hand, $\mathcal{N}$ is seen to be surjective by considering scalar elements, telling us that the image of pr also contains $\mathbb{G}_{\mathrm{m}}$ (scalar matrices in $\mathrm{GO}(V)$). Since $\mathbb{G}_{\mathrm{m}}$ and $\mathrm{(S)O}(V)$ generate $\mathrm{G(S)O}(V)$, the surjectivity of pr follows. The equality $\mathrm{sim} \circ \mathrm{pr} = \mathcal{N}^2$ follows from (3.3).

(iv) The first part follows from $\mathrm{pr}(x)(v) = xv\beta(x) = xvx^{-1}x\beta(x) = \mathrm{pr}^\circ(x)(v)\mathcal{N}(x)$ when $x \in \mathrm{GPin}(V)$ and $v \in V$. The second part is easily seen from (i) and (ii).

(v) This readily follows from the preceding points. □

If V is odd dimensional then $\mathrm{SO}(V) \times \{\pm 1\} = \mathrm{O}(V)$, and the group $\mathrm{GO}(V)$ is connected. For convenience we define $\mathrm{GSO}(V) := \mathrm{GO}(V)$ in this case. If $\dim(V)$ is even, then $\mathrm{O}(V)$ (resp. $\mathrm{GO}(V)$) has two connected components but does not admit a direct product decomposition into $\mathrm{O}(V)$ (resp. $\mathrm{GSO}(V)$) and $\{\pm 1\}$.

Assume that we have an orthogonal sum decomposition $\varphi\colon W_1 \oplus W_2 \xrightarrow{\sim} V$ of non-degenerate quadratic spaces over $\mathbb{Q}$. As super algebras we have ([4, (2.3)] or [9, §9.3, Cor. 3, Cor. 4])

$$C_\varphi\colon C(W_1)\widehat{\otimes}C(W_2) \xrightarrow{\sim} C(V), \quad w_1\widehat{\otimes}w_2 \mapsto w_1w_2.$$

By definition, the algebra given by $\widehat{\otimes}$ on the left side has underlying vector space $C(W_1)\otimes C(W_2)$ and product

$$(a\widehat{\otimes}b)\cdot(c\widehat{\otimes}d) := (-1)^{k_b\cdot k_c} ac\widehat{\otimes}bd,$$

if $a, c \in C(W_1)$, $b, d \in C(W_2)$ are homogeneous elements of degree $k_a, k_b, k_c, k_d \in \mathbb{Z}/2\mathbb{Z}$. The sign is there to make C_φ compatible with products since $bc = (-1)^{k_bk_c}cb$ in $C(V)$.

In fact C_φ intertwines the involution β on $C(V)$ with the involution

$$\beta'\colon C(W_1)\widehat{\otimes}C(W_2) \to C(W_1)\widehat{\otimes}C(W_2), \quad \beta'(a\widehat{\otimes}b) = (-1)^{k_ak_b}\beta_1(a)\widehat{\otimes}\beta_2(b),$$

for homogeneous elements $a \in C(W_1)$, $b \in C(W_2)$ of degree $k_a, k_b \in \mathbb{Z}/2\mathbb{Z}$, where β_1, β_2 are the involutions of $C(W_1)$ and $C(W_2)$ (see below (3.1)). To verify that β is compatible with β', observe that β on $C(V)$ restricts to β_1, β_2 via the obvious inclusions $C(W_1) \hookrightarrow C(V)$ and $C(W_2) \hookrightarrow C(V)$ induced by $W_1 \subset V$ and $W_2 \subset V$ (since β acts as the identity on both W_1 and W_2), and use the property that β_1, β_2, and β are preserving degrees. It follows that

$$\beta(ab) = \beta(b)\beta(a) = (-1)^{k_ak_b}\beta(a)\beta(b) = (-1)^{k_ak_b}\beta_1(a)\beta_2(b).$$

Lemma 3.2. *The mapping C_φ induces a morphism* $\mathrm{GSpin}(W_1) \times \mathrm{GSpin}(W_2) \to \mathrm{GSpin}(V)$.

Proof. We check that the image of C_φ is in $\mathrm{GSpin}(V)$. Let $g \in \mathrm{GSpin}(W_1)$, $h \in \mathrm{GSpin}(W_2)$. Note that $C_\varphi(g\widehat{\otimes}h) = gh \in C^+(V)$. Let $w_1 + w_2 \in V$ with $w_i \in W_i$, $i = 1, 2$. To verify that $gh \in \mathrm{GSpin}(V)$, since homogeneous elements of even degree commute with each other if they are perpendicular, we see that

$$gh(w_1 + w_2)h^{-1}g^{-1} = gw_1g^{-1} + hw_2h^{-1} \in V.$$

$\square$

Lemma 3.3. *The diagram*

$$\begin{array}{ccc} \mathrm{GSpin}(W_1) \times \mathrm{GSpin}(W_2) & \xrightarrow{C_\varphi} & \mathrm{GSpin}(V) \\ \downarrow{\scriptstyle \mathrm{pr}^\circ_{W_1}\times\mathrm{pr}^\circ_{W_2}} & & \downarrow{\scriptstyle \mathrm{pr}^\circ_V} \\ \mathrm{SO}(W_1) \times \mathrm{SO}(W_2) & \xrightarrow{i_{W_1,W_2}} & \mathrm{SO}(V) \end{array}$$

commutes, where i_{W_1,W_2} is the block diagonal embedding.

Proof. Immediate from the computation in the proof of the preceding lemma. $\square$

In later chapters we will carry out explicit computations. It will then be convenient to work with fixed bases and quadratic forms. For this reason we now fix quadratic forms on the vector spaces $V_{2n} = \mathbb{C}^{2n}$ and $V_{2n-1} = \mathbb{C}^{2n-1}$. We take the following quadratic forms:

$$Q_{2n}\colon x_1x_{n+1} + x_2x_{n+2} + \ldots + x_nx_{2n} \text{ on } \mathbb{C}^{2n}$$

$$Q_{2n-1}\colon y_1y_{n+1} + \ldots + y_{n-2}y_{2n-2} + y_{2n-1}^2 \text{ on } \mathbb{C}^{2n-1}. \tag{3.4}$$

Using them, we write $\mathrm{SO}_m = \mathrm{SO}(V_m)$, $\mathrm{GSO}_m = \mathrm{GSO}(V_m)$, and likewise for O_m, GO_m, for $m = 2n$ and $m = 2n-1$. This is identical to the convention of §2 for m even. Similarly we write $\mathrm{pr}^\circ_{2n-1} = \mathrm{pr}^\circ_{V_{2n-1}}$ and $\mathrm{pr}^\circ_{2n} = \mathrm{pr}^\circ_{V_{2n}}$.

Now we claim that $\mathrm{GSpin}(V_{2n})$ is isomorphic to GSpin_{2n} of §8 that is, the Clifford algebra definition is compatible with the root-theoretic definition as the Langlands dual of GSO_{2n}. (An analogous argument shows that GSpin_{2n-1} is dual to GSp_{2n-2}.) As this is a routine exercise, we only sketch the argument. First, pr° restricts to a connected double covering $\mathrm{Spin}(V_m) \to \mathrm{SO}(V_m)$ ([25, Prop. 20.38]), which must then be the unique (up to isomorphism) simply connected covering. This determines the root datum of $\mathrm{Spin}(V_m)$. From this, we compute the root datum of $\mathrm{GSpin}(V_m)$ via the central isogeny $\mathrm{Spin}(V_m) \times \mathbb{G}_m \to \mathrm{GSpin}(V_m)$ of Lemma

3.1. Finally when $m = 2n$, we deduce that the outcome is dual to the root datum of GSO_{2n} in Lemma 2.1. Therefore $\mathrm{GSpin}(V_{2n})$ is isomorphic to GSpin_{2n} of §8. Henceforth we identify

$$\mathrm{GSpin}(V_{2n}) = \mathrm{GSpin}_{2n}. \tag{3.5}$$

In fact we may and will choose $\mathrm{B}_{\mathrm{GSpin}}$ and $\mathrm{T}_{\mathrm{GSpin}}$ to be the preimages of B_{SO} and T_{SO} via $\mathrm{pr}^\circ : \mathrm{GSpin}_{2n} \to \mathrm{SO}_{2n}$. We fix pinnings of GSpin_{2n}, GSO_{2n}, and SO_{2n} (which are Γ_F-equivariant if (V_{2n}, Q_{2n}) is defined over F) compatibly via pr and pr°.

Lemma 3.4. *Via* (3.5), *the central embedding of scalar matrices* $\mathrm{cent}^\circ : \mathbb{G}_m \to \mathrm{GSO}_{2n}$ *and* $\mathrm{sim} : \mathrm{GSO}_{2n} \to \mathbb{G}_m$ *are dual to* $\mathcal{N} : \mathrm{GSpin}_{2n} \to \mathbb{G}_m$ *and the central embedding* $\mathrm{cent} : \mathbb{G}_m \to \mathrm{GSpin}_{2n}$ *of* (3.2), *respectively.*

Remark 3.5. The dual map of cent° was made explicit in (2.7). According to the present lemma, (2.7) gives an explicit formula for $\mathcal{N}$ restricted to T_{GSpin}.

Proof. Write Z^0 for the identity component of the center of GSpin_{2n}, consisting of $(s_0, 1, \dots, 1)$ with $s_0 \in \mathbb{G}_m$ in the notation of Lemma 2.5. The dual of $\mathrm{sim} : \mathrm{GSO}_{2n} \to \mathbb{G}_m$ is calculated as the central cocharacter $\mathbb{G}_m \to Z^0 \subset \mathrm{GSpin}_{2n}$, $z \mapsto (z, 1, \dots, 1)$. The inclusion $\mathrm{cent} : \mathbb{G}_m \to \mathrm{GSpin}_{2n}$ identifies $\mathbb{G}_m$ with Z^0. Thus cent is dual to sim.

Both $\mathcal{N} \circ \mathrm{cent}$ and $\mathrm{sim} \circ \mathrm{cent}^\circ$ are the squaring map on $\mathbb{G}_m$. Using the hat symbol to denote a dual morphism, we see that

$$\mathcal{N} \circ \mathrm{cent} = \widehat{\mathrm{cent}^\circ} \circ \widehat{\mathrm{sim}} = \widehat{\mathrm{cent}^\circ} \circ \mathrm{cent}$$

and that they are all equal to the squaring map. It follows that $\mathcal{N}$ is dual to cent°. □

We have the morphism of quadratic spaces

$$\varphi \colon (\mathbb{C}^{2n-1}, Q_{2n-1}) \to (\mathbb{C}^{2n}, Q_{2n}), \quad y \mapsto (y_1, y_2, \dots, y_{n-1}, y_{2n-1}, y_n, y_{n+1}, \dots, y_{2n-1}).$$

Indeed, $Q_{2n}\varphi = Q_{2n-1}$ as readily checked. We have the complementary embedding:

$$\varphi' \colon \mathbb{C} \to \mathbb{C}^{2n}, \quad u \mapsto x, \quad \text{where} \quad \begin{cases} x_k = 0 & k \neq n, 2n \\ x_k = (-1)^{k/n} u & \text{if } k = n \text{ or } k = 2n. \end{cases}$$

Write $U := \varphi'(\mathbb{C}) = (e_n - e_{2n}) \cdot \mathbb{C}$ for the image. The induced quadratic form on U is then $a \cdot (e_n - e_{2n}) \mapsto -a^2$. This gives us an orthogonal decomposition of quadratic spaces $\mathbb{C}^{2n} = \mathbb{C}^{2n-1} \widehat{\oplus} U$. Let PO_m denote the adjoint group of O_m. The decomposition induces morphisms (cf. Lemmas 3.2, 3.3)

$$\begin{aligned} i_{\mathrm{std}} &:= C_{\varphi,\varphi'} \colon \mathrm{GSpin}_{2n-1} \times \mathrm{GSpin}_1 \to \mathrm{GSpin}_{2n}, \\ i^\circ_{\mathrm{std}} &:= i_{\mathbb{C}^{2n-1},\mathbb{C}} \colon \mathrm{O}_{2n-1} \times \mathrm{O}_1 \to \mathrm{O}_{2n}, \text{ and} \\ \overline{i_{\mathrm{std}}} &:= \mathrm{PO}_{2n-1} \to \mathrm{PO}_{2n}, \end{aligned} \tag{3.6}$$

where $\overline{i_{\mathrm{std}}}$ is induced from $i_{\mathrm{std}} \colon \mathrm{GSpin}_{2n-1} \times \mathrm{GSpin}_1 \to \mathrm{GSpin}_{2n} \twoheadrightarrow \mathrm{PSO}_{2n} \subset \mathrm{PO}_{2n}$. By Lemma 3.3, we have $\mathrm{pr}^\circ \circ i_{\mathrm{std}} = i^\circ_{\mathrm{std}} \circ (\mathrm{pr}^\circ_{2n-1} \times \mathrm{pr}^\circ_U)$.

Let $1_{2n-1}, 1_U$ denote the identity map on $\mathbb{C}^{2n-1}, U$. Then (cf. (2.4))

$$i^\circ_{\mathrm{std}}(-1_{2n-1}, 1_U) = -\left(\begin{array}{cc:cc} 1_{n-1} & & & \\ & 0 & & 1 \\ \hdashline & & 1_{n-1} & \\ & 1 & & 0 \end{array}\right) = \vartheta^\circ \in \mathrm{O}_{2n}.$$

Fix $\sqrt{-1} \in \mathbb{G}_m = Z(\mathrm{GPin}_{2n})$. Define

$$\vartheta := \sqrt{-1} \cdot i_{\mathrm{std}}(1_{C(\mathbb{C}^{2n-1})} \widehat{\otimes} (e_n - e_{2n})) = \sqrt{-1}(e_n - e_{2n}) \in \mathrm{GPin}_{2n} \backslash \mathrm{GSpin}_{2n}. \tag{3.7}$$

Lemma 3.6. *We have*

(i) $\mathrm{pr}^\circ_{2n}(\vartheta) = \vartheta^\circ$ *and* $\vartheta^2 = 1$.

(ii) The conjugation action of ϑ *(resp.* ϑ°*) fixes the subgroup* $i_{\mathrm{std}}(\mathrm{GSpin}_{2n-1} \times \mathrm{GSpin}_1) \subset \mathrm{GSpin}_{2n}$ *via* i_{std} *(resp.* $\mathrm{SO}_{2n-1} \times \mathrm{SO}_1 \subset \mathrm{SO}_{2n}$ *via* i°_{std}*) and induces the identity automorphism on that subgroup.*

(iii) The conjugation action of ϑ (resp. ϑ°) defines the outer automorphism θ of GSpin_{2n} *(resp. θ° of* GSO_{2n}*) in Lemmas 2.3 and 2.4.*

Proof. (i) Let $w_1 \in \mathbb{C}^{2n-1}$ and $w_2 := e_n - e_{2n} \in U$. All of w_1, w_2, ϑ have degree 1 in $C(\mathbb{C}^{2n})$. In either $C(\mathbb{C}^{2n})$ or $C(U)$, we have $w_2^2 = Q_{2n}(w_2) = -1$ and $\vartheta^2 = -w_2^2 = 1$. Thus $\vartheta w_1 \vartheta^{-1} = -w_1 \vartheta\vartheta^{-1} = -w_1$ and $\vartheta w_2 \vartheta^{-1} = w_2$. Hence $\mathrm{pr}^\circ_{2n}(\vartheta) = \vartheta^\circ$.

(ii) This is obvious for ϑ°. The conjugation by ϑ is the identity on $C^+(\mathbb{C}^{2n-1})$ and $C^+(U)$, since $\vartheta \perp \mathbb{C}^{2n-1}$ and $C^+(U)$ is commutative, respectively. The assertion for ϑ follows.

(iii) This is true by definition for θ°. Since θ and the conjugation by ϑ act trivially on the center of GSpin_{2n}, it suffices to check that their actions are identical on the adjoint group. This reduces to the fact that θ° is given by the ϑ°-conjugation, as θ and θ° (resp. ϑ and ϑ°) induce the same action on the adjoint group (thanks to part (i)). □

We have fixed pinnings of GSpin_{2n}, GSO_{2n}, and SO_{2n} compatibly via pr. They are fixed by $\theta \in \mathrm{Aut}(\mathrm{GSpin}_{2n})$ and $\theta^\circ \in \mathrm{Aut}(\mathrm{GSO}_{2n})$. It is easy to see that θ and θ° induce automorphisms of based root data, which correspond to each other via duality of the two based root data. Thus letting E/F be a quadratic extension of fields of characteristic 0, and $\mathrm{GSO}_{2n}^{E/F}$ an outer form of GSO_{2n} over F with respect to the Galois action $\Gamma_{E/F} = \{1, c\} \xrightarrow{\sim} \{1, \theta\}$, we can identify

$$^L(\mathrm{GSO}_{2n}^{E/F}) = \mathrm{GSpin}_{2n} \rtimes \{1, c\} = \mathrm{GPin}_{2n},$$

where the semi-direct product is given by $cgc^{-1} = \theta(g)$. (Of course $c = c^{-1}$.) The second identification above is via $c \mapsto \vartheta$. Similarly, for $\mathrm{SO}_{2n}^{E/F}$ an outer form of SO_{2n} with respect to $\Gamma_{E/F} = \{1, c\} \xrightarrow{\sim} \{1, \theta^\circ\}$, we have

$$^L(\mathrm{SO}_{2n}^{E/F}) = \mathrm{SO}_{2n} \rtimes \{1, c\} = \mathrm{O}_{2n} \qquad \text{via} \quad c \mapsto \vartheta^\circ.$$

Let us describe the center $Z(\mathrm{Spin}_{2n})$ of $\mathrm{Spin}_{2n} = \mathrm{Spin}(V_{2n})$ explicitly as this is going to be useful for classifying inner twists of (quasi-split forms of) SO_{2n} and GSO_{2n} in §8. In what follows, we identify $Z(\mathrm{GSpin}_{2n}) = \{(s_0, s_1) : s_0 \in \mathbb{G}_m,\ s_1 \in \{\pm 1\}\}$ as in Lemma 2.5 and write $1, -1$ for $(1, 1), (1, -1) \in Z(\mathrm{GSpin}_{2n})$.

Lemma 3.7. *Let ζ_4 be a primitive fourth root of unity. Recall the elements $z_\pm$ defined in Lemma 2.7. Then we have $Z(\mathrm{Spin}_{2n}) \subset Z(\mathrm{GSpin}_{2n})$ via $\mathrm{T}_{\mathrm{Spin}} \subset T_{\mathrm{GSpin}}$ explained above, and the following are true.*

(i) If n is even, $Z(\mathrm{Spin}_{2n}) = \{1, -1, z_+, z_-\}$ and is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. If n is odd, $Z(\mathrm{Spin}_{2n}) = \{1, -1, \zeta, -\zeta = \zeta^{-1}\}$ and is isomorphic to $\mathbb{Z}/4\mathbb{Z}$, where $\zeta = (\zeta_4, -1)$.

(ii) The action of θ is trivial on $\{1, -1\}$ and permuting $\{z_+, z_-\}$ (resp. $\{\zeta, -\zeta\}$).

Proof. We have $Z(\mathrm{Spin}_{2n}) = Z(\mathrm{GSpin}_{2n}) \cap \mathrm{Spin}_{2n} = \{z \in Z(\mathrm{GSpin}_{2n}) : \mathcal{N}(z) = 1\}$, where $\mathcal{N}$ is described by (2.7) (Remark 3.5). It follows from Lemma 2.5 that

$$Z(\mathrm{Spin}_{2n}) = \{(s_0, s_1) : s_0^2 = s_1^n\},$$

which is alternatively described as in (i). Assertion (ii) is also clear from that lemma. □

4. The spin representations

We recollect how to construct the spin representations via Clifford algebras, and show that they coincide with the highest weight representations in Section 2. We also check some compatibility of maps that will become handy.

Consider the quadratic space $V_{2n} := \mathbb{C}^{2n}$ from (3.4) with standard basis $\{e_1, .., e_{2n}\}$ and quadratic form Q_{2n}. Define $W_{2n} := \oplus_{i=1}^n \mathbb{C} e_i$ and $W'_{2n} := \oplus_{i=n+1}^{2n} \mathbb{C} e_i$. We often omit the subscript $2n$ to lighten notation, when there is no danger of confusion. Since W is isotropic we obtain a morphism $\bigwedge W \xrightarrow{\sim} C(W) \hookrightarrow C(V)$. Through this injection we view $\bigwedge W$ as a subspace of $C(V)$. The space $\bigwedge W$ carries an $C(V)$-module structure

$$\mathrm{spin}\colon C(V) \to \mathrm{End}(\bigwedge W)$$

that is uniquely characterized by the following:

- $w \in W \subset V$ acts through left multiplication,
- and $w' \in W' \subset V$ acts as

$$w'(w_1 \wedge w_2 \wedge \cdots \wedge w_r) = \sum_{i=1}^{r} (-1)^{i+1} \langle w', w_i \rangle (w_1 \wedge w_2 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_r), \tag{4.1}$$

  on $w_1 \wedge \cdots \wedge w_r \in \bigwedge^r W \subset \bigwedge W$.

The subspaces $\bigwedge^+ W := \bigwedge_{i \in 2\mathbb{Z}_{\geq 0}} W$ and $\bigwedge^- W := \bigwedge_{i \in 1+2\mathbb{Z}_{\geq 0}} W$ of $\bigwedge W$ are stable under $C^+(V)$. By restriction we obtain the spin representations

$$\mathrm{spin}\colon \mathrm{GPin}_{2n} \to \mathrm{GL}\left(\bigwedge W\right) \quad \text{and} \quad \mathrm{spin}^{\pm}\colon \mathrm{GSpin}_{2n} \to \mathrm{GL}\left(\bigwedge^{\pm} W\right). \tag{4.2}$$

We recall that the representations $\mathrm{spin}^{\pm}$ are irreducible. In (4.5) and (4.6) below, we will choose (ordered) bases for $\bigwedge W$ and $\bigwedge^{\pm} W$ coming from $\{e_1, \ldots, e_n\}$ to view spin and $\mathrm{spin}^{\pm}$ as GL_{2^n} and $\mathrm{GL}_{2^{n-1}}$-valued representations, respectively. We had another definition of $\mathrm{spin}^{\varepsilon}$ as the representation with highest weight μ_{ε} (Definition 2.6), $\varepsilon \in \{+, -\}$. Let us check that the two definitions coincide via (3.5).

Lemma 4.1. *The highest weight of the half-spin representation* $\mathrm{spin}^{\varepsilon}$ *of* GSpin_{2n} *on* $\bigwedge^{\varepsilon} W$ *is equal to* μ_{ε}.

Proof. We may compare μ_{ε} and the highest weight of $\mathrm{spin}^{\varepsilon}$ after pulling back along $\mathrm{Spin}_{2n} \times \mathbb{G}_m \twoheadrightarrow \mathrm{GSpin}_{2n}$. They coincide on Spin_{2n} by [25, Prop. 20.15] and evidently restrict to the weight 1 character on $\mathbb{G}_m$. The lemma follows. □

Let us introduce a bilinear pairing on $\bigwedge W$ which is invariant under the spin representation up to scalars. Let $\mathrm{pr}_n : \bigwedge W \to \mathbb{C}$ denote the projection onto $\bigwedge^n W$, identified with $\mathbb{C}$ via $e_1 \wedge \cdots \wedge e_n \mapsto 1$. Write $\tau : \bigwedge W \xrightarrow{\sim} \bigwedge W$ for the $\mathbb{C}$-linear anti-automorphism $w_1 \wedge \cdots \wedge w_r \mapsto w_r \wedge \cdots \wedge w_1$ for $r \geq 1$ and $w_1, \ldots, w_r \in W$. Define

$$(\!(\dot{w}_1, \dot{w}_2)\!) := \mathrm{pr}_n(\tau(\dot{w}_1) \wedge \dot{w}_2), \qquad \dot{w}_1, \dot{w}_2 \in \bigwedge W.$$

We write $\mathrm{spin}^{\vee}$ and $\mathrm{spin}^{\varepsilon,\vee}$ for the dual representations of spin and $\mathrm{spin}^{\varepsilon}$. By the preceding lemma, the highest weight of $\mathrm{spin}^{\varepsilon,\vee}$ is in the Weyl group orbit of $(\mu_{\varepsilon})^{-1}$.

Lemma 4.2. *The pairing* $(\!(\ ,\ )\!)$ *is nondegenerate; it is alternating if* $n \equiv 2, 3 \pmod 4$ *and symmetric if* $n \equiv 0, 1 \pmod 4$. *The restriction of* $(\!(\ ,\ )\!)$ *to each of* $\bigwedge^+ W$ *and* $\bigwedge^- W$ *is nondegenerate if n is even, and identically zero if n is odd. We have*

$$(\!(\mathrm{spin}(g)\dot{w}_1, \mathrm{spin}(g)\dot{w}_2)\!) = \mathcal{N}(g)(\!(\dot{w}_1, \dot{w}_2)\!), \qquad g \in \mathrm{GPin}_{2n}(\mathbb{C}),\ \dot{w}_1, \dot{w}_2 \in \bigwedge W. \tag{4.3}$$

In particular, we have $\mathrm{spin}^{\varepsilon} \simeq \mathrm{spin}^{(-1)^n \varepsilon, \vee} \otimes \mathcal{N}$.

Proof. The first two assertions are elementary and left to the reader. The last assertion follows from the rest. For the equality (4.3), we claim that

$$(\!(c\dot{w}_1, \dot{w}_2)\!) = (\!(\dot{w}_1, \beta(c)\dot{w}_2)\!), \qquad c \in C(V),\ \dot{w}_1, \dot{w}_2 \in \bigwedge W. \tag{4.4}$$

Since $\mathrm{GPin}_{2n} \subset C(V)$, this implies (4.3) as

$$(\!(\mathrm{spin}(g)\dot{w}_1, \mathrm{spin}(g)\dot{w}_2)\!) = (\!(\dot{w}_1, \mathrm{spin}(\beta(g)g)\dot{w}_2)\!) = \beta(g)g(\!(\dot{w}_1, \dot{w}_2)\!).$$

It remains to prove the claim. The proof of (4.4) reduces to the case $c \in V$, then to the two cases $c \in W$ and $c \in W'$ by linearity. In both cases, (4.4) follows from the explicit description of the $C(V)$-action as in (4.1). Indeed, (4.4) is obvious if $c \in W$. When $c \in W'$, it is enough to show that for $0 \leq r, s \leq n$, $1 \leq i_1 < \cdots < i_r \leq n$, $1 \leq j_1 < \cdots < j_s \leq n$, and $1 \leq k \leq n$,

$$\tau(e_{n+k}(e_{i_1} \wedge \cdots \wedge e_{i_r})) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_s}) = \tau(e_{i_1} \wedge \cdots \wedge e_{i_r}) \wedge (e_{n+k}(e_{j_1} \wedge \cdots \wedge e_{j_s})).$$

(This implies (4.4) by taking pr_n.) The equality is simply $0 = 0$ unless $k = r_0 = s_0$ for some $1 \leq r_0 \leq r$ and $1 \leq s_0 \leq s$. In the latter case, the equality boils down to

$$(-1)^{r_0+1} e_{i_r} \wedge \cdots \wedge \widehat{e}_{i_{r_0}} \wedge \cdots \wedge e_{i_1} \wedge e_{j_1} \wedge \cdots \wedge e_{j_s} = (-1)^{s_0+1} e_{i_r} \wedge \cdots \wedge e_{i_1} \wedge e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_{s_0}} \wedge \cdots \wedge e_{j_s},$$

which is clear. The proof is complete. □

We also discuss the odd case. Equip $V_{2n-1} = \mathbb{C}^{2n-1}$ with standard basis $\{f_1, \dots, f_{2n-1}\}$ and quadratic form Q_{2n-1} of (3.4). As in [25, p.306], we decompose

$$V_{2n-1} := \mathbb{C}^{2n-1} = W_{2n-1} \oplus W'_{2n-1} \oplus U_{2n-1},$$

where $W_{2n-1} := \oplus_{i=1}^{n-1} \mathbb{C} f_i$, $W'_{2n-1} := \oplus_{i=n}^{2n-2} \mathbb{C} f_i$, and $U_{2n-1} := \mathbb{C} f_{2n-1}$. Again we omit the subscript $2n-1$ when it is clear from the context. Then W and W' are $(n-1)$-dimensional isotropic subspaces, and U is a line perpendicular to them. As in the even case, each of $\bigwedge W$ and $\bigwedge^{\pm} W$ can be viewed as a subspace of $C(V)$ and has a unique structure of left $C(V)$-module where:

- $w \in W \subset V$ acts on $\bigwedge W$ through left multiplication,
- $w' \in W' \subset V$ acts as in (4.1) (cf. [25, 20.16]),
- f_{2n-1} acts trivially on $\bigwedge^+ W$ and as -1 on $\bigwedge^- W$.

Consider the bijection

$$\psi \colon \bigwedge W_{2n-1} \xrightarrow{\sim} \bigwedge^+ W_{2n}, \quad w_1 \wedge \cdots \wedge w_r \mapsto \begin{cases} w_1 \wedge \cdots \wedge w_r \wedge e_n, & r \text{ odd} \\ w_1 \wedge \cdots \wedge w_r, & r \text{ even.} \end{cases}$$

Lemma 4.3. *For all $g \in \mathrm{GSpin}_{2n-1}$ and all $w \in \bigwedge W_{2n-1}$ we have $i_{\mathrm{std}}(g)\psi(w) = \psi(gw)$, where $i_{\mathrm{std}}(g)$ and g act by spin^+ of GSpin_{2n} and spin of GSpin_{2n-1}, respectively.*

Proof. We keep writing $W = W_{2n-1}$, $W' = W'_{2n-1}$, $U = U_{2n-1}$. We identify $V_{2n} = (W \oplus U^1 \oplus W' \oplus U^2$ via $W_{2n} = W \oplus U^1$ and $W'_{2n} = W' \oplus U^2$ with $U^1 = \mathbb{C} e_n$ and $U^2 = \mathbb{C} e_{2n}$, mapping the basis of W (resp. W') onto the first $n-1$ elements in the basis of W_{2n} (resp. W'_{2n}). This also gives the embedding $V_{2n-1} \subset V_{2n}$, with U diagonally embedded in $U^1 \oplus U^2$ (so f_{2n-1} maps to $e_n + e_{2n}$), as in the formula below (3.4).

There is an obvious embedding $\iota^+ : \bigwedge W \hookrightarrow \bigwedge(W \oplus U^1)$. We also have $\iota^- : \bigwedge W \hookrightarrow \bigwedge(W \oplus U^1)$ by $(\cdot) \wedge e_n$. Both ι^+ and ι^- are $C(W \oplus W')$-equivariant, by using that left and right multiplications commute and that e_n is orthogonal to $W \oplus W'$. Furthermore, ι^- intertwines the f_{2n-1}-action on $\bigwedge^- W$, which is by multiplication by -1, and the $e_n + e_{2n}$-action on $\bigwedge^+(W \oplus U^1)$, since $w \wedge e_n = -e_n \wedge w$ if $w \in \bigwedge^- W$ and since $W \perp e_{2n}$ with respect to Q_{2n}.

Now we claim that ψ is $C^+(W \oplus W' \oplus U)$-equivariant, which implies the lemma by restricting from $C^+(W \oplus W' \oplus U)$ to GSpin_{2n-1}. It suffices to verify equivariance of ψ under $C^+(W \oplus W')$ and $C^-(W \oplus W') \otimes f_{2n-1}$. But ψ is ι^+ on $\bigwedge^+ W$ and ι^- on $\bigwedge^- W$. Thus the claim is deduced by putting together the equivariance in the preceding paragraph. □

Lemma 4.4. *Let $\vartheta \in \mathrm{GPin}_{2n}$ be the element from (3.7). We have $\bigwedge^+ W_{2n} \xrightarrow{\sim} \bigwedge^- W_{2n}$, $x \mapsto \vartheta x$. We have $\mathrm{spin}^+ \circ \theta = \mathrm{spin}^-$ via this isomorphism, i.e., $\vartheta(\mathrm{spin}^+(g)x) = \mathrm{spin}^-(\theta(g))\vartheta x$ for each $g \in \mathrm{GSpin}_{2n}$.*

Proof. Henceforth we omit the symbol $\wedge$ for the wedge product in W_{2n}. Consider $v = e_{k_1} \cdots e_{k_r} \in \bigwedge^+ W_{2n}$, with $k_1 < k_2 < \ldots < k_r$ and r is even. Then

$$\vartheta v = \sqrt{-1}(e_n e_{k_1} \cdots e_{k_r} - e_{2n} \cdot e_{k_1} \cdots e_{k_r}) \in \bigwedge W_{2n},$$

where e_{2n} acts by (4.1). Thus the isomorphism follows from the following computations.

$$e_n e_{k_1} \cdots e_{k_r} = \begin{cases} 0, & k_r = n, \\ e_{k_1} \cdots e_{k_r} e_n, & k_r \neq n, \end{cases}$$

$$e_{2n} e_{k_1} \cdots e_{k_r} = \sum_{i=1}^{r} (-1)^{i+1} \langle e_{2n}, e_{k_i} \rangle e_{k_1} \cdots \widehat{e_{k_i}} \cdots e_{k_r} = \begin{cases} -e_{k_1} \cdots e_{k_{r-1}}, & k_r = n, \\ 0, & k_r \neq n. \end{cases}$$

The last assertion comes down to showing that $\vartheta g x = \theta(g)\vartheta x$, where $\vartheta g, \theta(g)\vartheta \in C(V)$ act through the $C(V)$-module structure on $x \in \bigwedge W_{2n}$. But this is clear since $\theta(g) = \vartheta g \vartheta^{-1}$. □

Consider the basis $\{b_U\}$ of $\bigwedge W_{2n}$, with

$$b_U = (-1)^{\#U} e_{k_1} \cdot e_{k_2} \cdots e_{k_r} \in \bigwedge W_{2n}, \tag{4.5}$$

where $U = \{k_1 < k_2 < \cdots < k_r\}$ ranges over the subsets of $\{1, 2, \ldots, n\}$. The U of even size form a basis for $\bigwedge^+ W_{2n}$; and the U with odd size form a basis for $\bigwedge^- W_{2n}$. Order the b_U for U odd, and the b_U for U even in such a way that the ordering of $\{b_U\}_{|U|:\text{even}}$ corresponds to that of $\{b_U\}_{|U|:\text{odd}}$ via $b_U \mapsto \vartheta b_U/\sqrt{-1}$. Then these orderings of the b_U gives us two identifications

$$\mathrm{GL}\left(\bigwedge\nolimits^+ W_{2n}\right) \xrightarrow{\sim} \mathrm{GL}_{2^{n-1}} \quad \text{and} \quad \mathrm{GL}\left(\bigwedge\nolimits^- W_{2n}\right) \xrightarrow{\sim} \mathrm{GL}_{2^{n-1}}, \tag{4.6}$$

such that the following proposition holds.

Proposition 4.5. *The following diagram commutes*

$$\begin{tikzcd}
& \mathrm{GSpin}_{2n} \arrow[dd, "\theta"] \arrow[dr, "\mathrm{spin}^+"] & \\
\mathrm{GSpin}_{2n-1} \arrow[ur, "i_{\mathrm{std}}"] \arrow[dr, "i_{\mathrm{std}}"'] \arrow[rr, "\mathrm{spin}"] & & \mathrm{GL}_{2^{n-1}} \\
& \mathrm{GSpin}_{2n} \arrow[ur, "\mathrm{spin}^-"'] &
\end{tikzcd}$$

Proof. This follows from Equation (4.5), Lemmas 4.3, and (proof of) Lemma 4.4. □

5. Some special subgroups of GSpin_{2n}

In this section, the base field of all algebraic groups is an algebraically closed field of characteristic 0 such as $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$. We begin with principal morphisms for GSpin_{2n-1} and GSpin_{2n}. (See [64, Sect. 7] and [29, 73] for general discussions.) The following notation will be convenient for us. Denote by

$$j_{\mathrm{reg}} \colon \mathbb{G}_{\mathrm{m}} \times \mathrm{SL}_2 \to \mathrm{GSpin}_{2n-1}$$

the product of the central embedding $\mathbb{G}_m \hookrightarrow \mathrm{GSpin}_{2n-1}$ and a fixed principal SL_2-mapping. Note that j_{reg} has the following kernel[6]

$$\begin{cases} \langle(-1, \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right))\rangle, & \text{if } n(n-1)/2 \text{ is odd,} \\ \langle(1, \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right))\rangle, & \text{if } n(n-1)/2 \text{ is even.} \end{cases}$$

We write $G_{\mathrm{pri}} \subset \mathrm{GSpin}_{2n-1}$ for the image of j_{reg}. The group G_{pri} is isomorphic to GL_2 if $n(n+1)/2$ is odd, and to $\mathbb{G}_{\mathrm{m}} \times \mathrm{PGL}_2$ otherwise. Using i_{std} from (3.6), we define

$$i_{\mathrm{reg}} = i_{\mathrm{std}} \circ j_{\mathrm{reg}} \colon \mathbb{G}_{\mathrm{m}} \times \mathrm{SL}_2 \to \mathrm{GSpin}_{2n}.$$

The map $\mathrm{pr}^\circ \circ i_{\mathrm{reg}} : \mathbb{G}_{\mathrm{m}} \times \mathrm{SL}_2 \to \mathrm{SO}_{2n}$ factors through $\mathrm{PGL}_2 \to \mathrm{SO}_{2n}$, to be denoted i°_{reg}, via the natural projection from $\mathbb{G}_{\mathrm{m}} \times \mathrm{SL}_2 \to \mathrm{PGL}_2$ (trivial on the $\mathbb{G}_{\mathrm{m}}$-factor). We see that the preimage of $i^\circ_{\mathrm{reg}}(\mathrm{PGL}_2)$ in GSpin_{2n} is $i_{\mathrm{std}}(G_{\mathrm{pri}})$. Denote by $\overline{j_{\mathrm{reg}}} : \mathrm{PGL}_2 \to \mathrm{PSO}_{2n-1}$ the map induced by j_{reg} on the adjoint groups.[7] We also introduce the map

$$\overline{i_{\mathrm{reg}}} = \overline{i_{\mathrm{std}}} \circ \overline{j_{\mathrm{reg}}} \colon \mathrm{PGL}_2 \to \mathrm{PSO}_{2n}.$$

Recall that we have fixed earlier the group SO_8 in (2.1) (cf. below (2.3)). Let $T_{\mathrm{GSpin}_7} \subset \mathrm{GSpin}_7$ be as in [49, §Notation] and put $T_{\mathrm{Spin}_7} = T_{\mathrm{GSpin}_7} \cap \mathrm{Spin}_7$.

We will now fix a convenient basis for $X_*(T_{\mathrm{Spin}_7})$. We have $X_*(T_{\mathrm{GSpin}_7}) = X^*(T_{\mathrm{GSp}_6}) = \bigoplus_{i=0}^3 \mathbb{Z} e_i$, the center $Z_{\mathrm{GSp}_6} \subset T_{\mathrm{GSp}_6}$ equals $\{(t^2, t, t, t) | t \in \mathbb{G}_{\mathrm{m}}\}$ (use the roots α_i listed in [*loc. cit.*, p. 10]), and so $X^*(T_{\mathrm{GSp}_6}) \to X^*(Z_{\mathrm{GSp}_6})$ identifies with $\mathbb{Z}^4 \to \mathbb{Z}$, $(x_i) \mapsto 2a_0 + a_1 + a_2 + a_3$. Thus $X_*(T_{\mathrm{Spin}_7}) = \{(a_i) \in \mathbb{Z}^4 \,|\, 2a_0 + a_1 + a_2 + a_3 = 0\}$. By projecting $(a_i) \in \mathbb{Z}^4$ onto (a_1, a_2, a_3) we obtain

$$X_*(T_{\mathrm{Spin}_7}) = \{(a_1, a_2, a_3) \in \mathbb{Z}^3 : a_1 + a_2 + a_3 \equiv 0 \mod 2\}. \tag{5.1}$$

[6] To see this, one can use Proposition 6.1 of [30], where the SL_2-representations appearing in the composition $\mathrm{SL}_2 \overset{\mathrm{pri}}{\to} \mathrm{GSpin}_{2n-1} \overset{\mathrm{spin}}{\to} \mathrm{GL}_{2^{n-1}}$ are computed.

[7] When denoting the group standing alone, we prefer SO_{2n-1} to PSO_{2n-1}. When thinking of a projective representation or a subgroup of PSO_{2n} via $\overline{i_{std}}$, we usually write PSO_{2n-1}.

We write $T_{\mathrm{SO}_8} \subset \mathrm{SO}_8$ for the maximal torus corresponding to (2.2) (so with $t_0 = 1$). The spin representation of Spin_7 is orthogonal ([49, Lem. 0.1]), yielding an embedding $\mathrm{spin}^{\circ\prime} : \mathrm{Spin}_7 \hookrightarrow \mathrm{SO}(q)$, for some quadratic form q in 8 variables. We fix an isomorphism $u \colon \mathrm{SO}(q) \xrightarrow{\sim} \mathrm{SO}_8$, in such a way that the composition

$$\mathrm{spin}^\circ := u \circ \mathrm{spin}^{\circ\prime} \colon \mathrm{Spin}_7 \hookrightarrow \mathrm{SO}_8$$

maps T_{Spin_7} into T_{SO_8} and such that

$$\mathrm{spin}^\circ(a) = \left(\tfrac{1}{2}(\tau_1^j a_1 + \tau_2^j a_2 + \tau_3^j a_3)\right) \in X_*(T_{\mathrm{SO}_8}) \subset \mathbb{Z}^8, \tag{5.2}$$

for some choice of numbering $\tau^j = (\tau_1^j, \tau_2^j, \tau_3^j) \in \{\pm 1\}^3$ for $j = 1, ..., 8$, such that $\tau^j = -\tau^{j+4}$ for $j = 1, 2, 3, 4$. In (5.2) the embedding $X_*(T_{\mathrm{SO}_8}) \subset \mathbb{Z}^8$ comes from (2.2).

We write $\overline{\mathrm{spin}^\circ} : \mathrm{Spin}_7 \hookrightarrow \mathrm{PSO}_8$ for the projectivization of spin°. Fixing a non-isotropic line in the underlying 8-dimensional space, the stabilizer of the line in Spin_7 is isomorphic to a group of type G_2, cf. [31, p.169, Prop. 2.2(4)]. Thereby we obtain an embedding $j_{\mathrm{spin}} \colon G_2 \hookrightarrow \mathrm{Spin}_7$. Alternatively, an embedding $G_2 \hookrightarrow \mathrm{Spin}_7$ can be constructed using the octonion algebra [16, Sect. 2.5]. The conjugacy class of j_{spin} is unique (thus independent of choices) by [*loc. cit.*, Prop. 2.11]. Denote by

$$i_{\mathrm{spin}} \colon G_2 \hookrightarrow \mathrm{Spin}_8 \tag{5.3}$$

the composite $i_{\mathrm{std}} \circ j_{\mathrm{spin}}$. The restriction of $\mathrm{spin}^\varepsilon : \mathrm{Spin}_8 \to \mathrm{GL}_8$ via i_{spin} is isomorphic to $\mathbf{1} \oplus \mathrm{std}$, where $\mathbf{1}$ and std are the trivial and the unique irreducible 7-dimensional representation of G_2, respectively. (This is easy to see by dimension counting, as the other irreducible representations have dimension ≥ 14.)

Lemma 5.1. *The representation* $\mathrm{spin}^\circ : \mathrm{Spin}_7 \hookrightarrow \mathrm{SO}_8$ *is* O_8*-conjugate to* $\theta^\circ \mathrm{spin}^\circ$ *but not locally conjugate (thus not conjugate) as an* SO_8*-valued representation. In fact, there exists an open dense subset* $U \subset \mathrm{Spin}_7$ *such that* $\mathrm{spin}^\circ t$ *and* $\theta^\circ \mathrm{spin}^\circ t$ *are not conjugate for any* $t \in U$*. Moreover* $\mathrm{spin}^\circ(\mathrm{Spin}_7)$ *and* $\theta^\circ \mathrm{spin}^\circ(\mathrm{Spin}_7)$ *are not* SO_8*-conjugate. The analogous assertion holds for* $\overline{\mathrm{spin}} : \mathrm{SO}_7 \hookrightarrow \mathrm{PSO}_8$*.*

Proof. Evidently spin° and $\theta^\circ \mathrm{spin}^\circ$ are O_8-conjugate since $\theta^\circ = \mathrm{Int}(\vartheta^\circ)$ with $\vartheta^\circ \in \mathrm{O}_8$. Let $T_{\mathrm{GL}_8} \subset \mathrm{GL}_8$ be the diagonal torus. Let $\Omega_{\mathrm{Spin}_7}, \Omega_{\mathrm{SO}_8}, \Omega_{\mathrm{GL}_8}$ denote the Weyl groups corresponding to $T_{\mathrm{Spin}_7}, T_{\mathrm{SO}_8}, T_{\mathrm{GL}_8}$. In view of the weights of the spin representation [25, Prop. 20.20], we know that

$$\mathrm{std}(\mathrm{spin}^\circ(a_1, a_2, a_3)) \in \Omega_{\mathrm{GL}_8}((\varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_3 a_3)/2 : \varepsilon_i \in \{\pm 1\}).$$

(The Ω_{GL_8}-orbit of 8-tuples is simply an unordered 8-tuple.) When $\varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_3 a_3$ are all distinct, the right hand side breaks up into exactly two Ω_{SO_8}-orbits, which are permuted by θ°. Similarly, if U_T is the open dense subset of T_{Spin_7} consisting of $t = (t_1, t_2, t_3) \in T_{\mathrm{Spin}_7}$ with $t_1^{\varepsilon_1}, t_2^{\varepsilon_2}, t_3^{\varepsilon_3}$ all distinct, then $\mathrm{spin}^\circ(t)$ and $\theta^\circ(\mathrm{spin}^\circ(t))$ are not SO_8-conjugate. This implies the existence of U as in the lemma by taking U to be the set of regular semisimple elements whose conjugacy classes meet U_T.

Now assume that $\mathrm{spin}^\circ(\mathrm{Spin}_7) = g\theta^\circ \mathrm{spin}^\circ(\mathrm{Spin}_7) g^{-1}$ for some $g \in \mathrm{SO}_8$. Then the composition $c_g = \mathrm{spin}^{\circ,-1} \circ \mathrm{Int}_g \circ \theta^\circ \mathrm{spin}^\circ$ is an automorphism of Spin_7, which is hence inner and of the form $x \mapsto hxh^{-1}$ for some $h \in \mathrm{Spin}_7$. Thus spin° and $\theta^\circ \mathrm{spin}^\circ$ are conjugate by $g^{-1}\mathrm{spin}^\circ(h)$, a contradiction. Thus $\mathrm{spin}^\circ(\mathrm{Spin}_7)$ and $\theta^\circ \mathrm{spin}^\circ(\mathrm{Spin}_7)$ are not SO_8-conjugate. The projective analogue for $\overline{\mathrm{spin}} \colon \mathrm{SO}_7 \hookrightarrow \mathrm{PSO}_8$ also follows. □

Lemma 5.2. *Write* $H := \mathrm{pr}^{-1}(\mathrm{spin}^\circ(\mathrm{Spin}_7)) \subset \mathrm{Spin}_8$*. The restriction of* $\mathrm{spin}^\varepsilon$ *to* H *is irreducible if and only if* $\varepsilon = -$*. More precisely, we have* $\overline{\mathrm{spin}^+} \circ \mathrm{spin}^\circ \simeq \overline{\mathrm{std} \oplus \mathbf{1}}$ *and* $\overline{\mathrm{spin}^-} \circ \mathrm{spin}^\circ \simeq \overline{\mathrm{spin}^\circ}$*.*

Proof. We compute the composition $\overline{\mathrm{spin}^\varepsilon} \circ \mathrm{spin}^\circ \colon \mathrm{Spin}_7 \xrightarrow{\mathrm{spin}^\circ} \mathrm{SO}_8 \to \mathrm{PSO}_8 \xrightarrow{\overline{\mathrm{spin}^\varepsilon}} \mathrm{PGL}_8$ on T_{Spin_7}. For $a \in X_*(T_{\mathrm{Spin}_7})$, $\mathrm{spin}^\circ(a)$ is given by (5.2), and for $b \in X_*(T_{\mathrm{SO}_8}) = \mathbb{Z}^4$ we have $\overline{\mathrm{spin}^\varepsilon}(b) = (\frac{1}{2}(\tau_1 b_1 + \tau_2 b_2 + \tau_3 b_3 + \tau_4 b_4))_{\tau \in \{\pm 1\}^4, \prod_{i=1}^4 \tau_i = \varepsilon}$ (both up to the Weyl group actions). From this it follows that $\overline{\mathrm{spin}^\varepsilon} \circ \mathrm{spin}^\circ|_{T_{\mathrm{Spin}_7}}$ is conjugated to $\overline{\mathrm{std} \oplus \mathbf{1}}|_{T_{\mathrm{Spin}_7}}$ and $\overline{\mathrm{spin}^\circ}|_{T_{\mathrm{Spin}_7}}$ for $\varepsilon = +$ and $\varepsilon = -$, respectively. The lemma now follows from Lemma 1.2. □

Lemma 5.3. *Let $n \geq 3$ and $H \subsetneq \mathrm{SO}_{2n}$ be a proper connected reductive subgroup containing a regular unipotent element. Then H is isomorphic to a quotient of Spin_{2n-1}, SL_2 or G_2 (the last can occur only if $n = 4$).*

Proof. We begin with some preliminaries. When G is a reductive group, write $\Sigma(G)$ for the set of maximal proper connected reductive subgroups M of G that contain a regular unipotent element of G. From [70, Thms. A,B] we have the following[8]:

(a) Case $G \simeq \mathrm{SO}_{2n}$ ($n \geq 3$), then every $M \in \Sigma(G)$ is isomorphic to a quotient of Spin_{2n-1}.
(b) Case $G \simeq \mathrm{SO}_{2n-1}$ ($n \geq 3$, $n \neq 4$), then every $M \in \Sigma(G)$ is isomorphic to a quotient of SL_2.
(c) Case $G \simeq \mathrm{SO}_7$, then every $M \in \Sigma(G)$ is isomorphic to G_2.
(d) Case $G \simeq G_2$, then every $M \in \Sigma(G)$ is isomorphic to a quotient of SL_2.

We prove the following claim: If H is a connected reductive subgroup of some connected reductive group G (over $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$), such that H contains a regular unipotent element u of G, then u is also regular unipotent in H. To see this, write $B_u \subset G$ for the unique Borel subgroup that contains u. Now let $B_0 \ni u$ be a Borel subgroup of H that contains u. Then B_0 is a connected solvable subgroup of G, and hence is contained in a Borel subgroup B_1 of G. As $u \in B_1$, we must have $B_1 = B_u$. Hence $B_0 \subset B_u \cap H$. Since $(B_u \cap H)^0$ is connected solvable and contains B_0, we must have $(B_u \cap H)^0 = B_0$ by maximality of B_0. This shows that in H, the element u is contained in exactly one Borel subgroup. Therefore $u \in H$ is regular unipotent.

Now let H be as in the statement of the lemma. Let $u \in H$ be regular unipotent in SO_{2n}. Let $M \in \Sigma(\mathrm{SO}_{2n})$ such that $H \subset M$. If $H = M$, we are done by (a) above. So assume $H \neq M$. Again by (a), M is a quotient of Spin_{2n-1}, and hence $M_{\mathrm{ad}} \simeq \mathrm{SO}_{2n-1}$. Then H_{ad} maps to SO_{2n-1} (since the center of H commutes with u, it is contained in $Z_{\mathrm{SO}_{2n}}$ by [75, Thm. 4.11] thus also in Z_M), and by the claim we can find an $M' \in \Sigma(\mathrm{SO}_{2n-1})$ that contains the image of H_{ad} in SO_{2n-1}. If $H_{\mathrm{ad}} = M'$, we are done by (b) or (c) if $n = 4$. If $H_{\mathrm{ad}} \neq M'$, then again M' is either G_2 or a quotient of SL_2, and we can argue similarly. □

If H is an algebraic group, we write $\Upsilon(H)$ for the set of SO_{2n}-conjugacy classes of morphisms $H \to \mathrm{SO}_{2n}$ that have a regular unipotent element in their image. By abuse of notation, we often identify $\Upsilon(H)$ with a set of representatives for the conjugacy classes.

Lemma 5.4. *We have*

$$\begin{aligned}
\Upsilon(\mathrm{Spin}_{2n-1}) &= \{i^\circ_{\mathrm{std}} \colon \mathrm{Spin}_{2n-1} \to \mathrm{SO}_{2n}\} && (n \geq 3, n \neq 4)\\
\Upsilon(\mathrm{Spin}_7) &= \{i^\circ_{\mathrm{std}}, \mathrm{spin}^\circ, \theta^\circ \mathrm{spin}^\circ \colon \mathrm{Spin}_7 \to \mathrm{SO}_8\} && (n = 4)\\
\Upsilon(\mathrm{SL}_2) &= \{i_{\mathrm{reg}} \colon \mathrm{SL}_2 \to \mathrm{SO}_{2n}\} && (n \geq 3)\\
\Upsilon(G_2) &= \{i_{\mathrm{spin}} \colon G_2 \to \mathrm{SO}_8\} && (n = 4)
\end{aligned}$$

Proof. Let $f \colon H \to \mathrm{SO}_{2n}$ be a morphism with a regular unipotent element in its image, where H is one of the 4 groups in the lemma.

Case $H = \mathrm{Spin}_{2n-1}$. We have $\ker(f) \subset \{\pm 1\}$. Assume first $\ker(f) = \{\pm 1\}$. Then $r := \mathrm{std} \circ f$, $r' := \mathrm{std} \circ i^\circ_{\mathrm{std}}$ are two faithful representations of dimension $2n$. By Lemma 2.8, r and r' are isomorphic. By acceptability of O_{2n} we find a $g \in \mathrm{O}_{2n}$ that conjugates r to r' [49, Prop. B.1]. The element g might have negative determinant. In this case we can replace g by $g \cdot \vartheta^\circ$, as ϑ° centralizes $i^\circ_{\mathrm{std}}(\mathrm{SO}_{2n-1})$ by Lemma 3.6(ii).

Now assume $\ker(f) = 1$. Then $r = \mathrm{std} \circ f$ is a faithful $2n$-dimensional representation of Spin_{2n-1}. The smallest such representation by dimension is the spin representation, and therefore $2^{n-1} \leq 2n$, and $n \leq 4$. We distinguish in subcases $n = 3$ or $n = 4$:

(When $n = 3$.) We show that this subcase ($H = \mathrm{Spin}_{2n-1}, \ker(f) = 1, n = 3$) does not occur. Assume $f \colon \mathrm{Spin}_5 \to \mathrm{SO}_6$ is injective with a regular unipotent element in its image. Recall $\mathrm{SO}_6 \simeq \mathrm{SL}_4/\{\pm 1\}$. Write $\widetilde{H} \subset \mathrm{SL}_4$ for the pre-image of $f(\mathrm{Spin}_5)$ in SL_4. Let $M \subset \mathrm{SL}_4$ be a

[8]The statement of [70, Thms. A,B] are not entirely clear on whether the list describes H^0 or H. We interpret it as the former since that is what their proof shows. For instance, regarding (i)(a) of their theorem, a maximal reductive subgroup of type B_{n-1} in SO_{2n} is not $i^\circ_{\mathrm{std}}(\mathrm{SO}_{2n-1})$ but $Z(\mathrm{SO}_{2n}) \times i^\circ_{\mathrm{std}}(\mathrm{SO}_{2n-1})$, which is disconnected.

proper maximal connected reductive subgroup of SL_4 that contains H. Then M is isomorphic to Sp_4 by [70, Thm. B]. By dimension consideration we must have $\widetilde{H} = M$. In particular the image of f in $\mathrm{SL}_4/\{\pm 1\}$ must be isomorphic to PSp_4. Hence $\mathrm{Spin}_5 \xrightarrow{\sim} \mathrm{PSp}_4$, a contradiction.

(When $n = 4$.) We want to classify all conjugacy classes of injections $f\colon \mathrm{Spin}_7 \to \mathrm{SO}_8$ with a regular unipotent element in their image. In this case they do exist, as spin° is an example. The representations $\mathrm{std} \circ f$ and $\mathrm{std} \circ \mathrm{spin}^\circ$ are both faithful representations of Spin_7 of dimension 8. Hence they are isomorphic. By acceptability of O_8 there exists a $g \in \mathrm{O}_8$ that conjugates f to spin°. This implies that f is SO_8-conjugate to either spin° or $\theta^\circ\mathrm{spin}^\circ$. By Lemma 5.1, the representations spin° and $\theta^\circ\mathrm{spin}^\circ$ are not SO_8-conjugate. Observe finally that $\ker(i^\circ_{\mathrm{std}}) = \{\pm 1\}$ while spin° and $\theta^\circ\mathrm{spin}^\circ$ are injective. Hence i°_{std} is not conjugate to either spin° or $\theta^\circ\mathrm{spin}^\circ$. This verifies the description of $\Upsilon(\mathrm{Spin}_7)$ in the lemma.

Case $H = \mathrm{SL}_2$. We want to classify the morphisms $f\colon \mathrm{SL}_2 \to \mathrm{SO}_{2n}$ with a regular unipotent element in the image. In fact, such a morphism is called the "principal morphism" in the literature, and it is well-known that it is unique up to conjugacy. However, we could not find a precise reference, so we give some detail for a general connected reductive group G.

We first note that the natural map $\mathrm{Hom}(\mathrm{SL}_2, G) \to \mathrm{Hom}(\mathrm{Lie}\,(\mathrm{SL}_2), \mathrm{Lie}\,(G))$, equivariant for the adjoint action of G, is a bijection. To construct the inverse, let $g \in \mathrm{Hom}(\mathrm{Lie}\,(\mathrm{SL}_2), \mathrm{Lie}\,(G))$. The composition $\mathrm{Rep}(G) \to \mathrm{Rep}(\mathrm{Lie}\,(G)) \to \mathrm{Rep}(\mathrm{Lie}\,(\mathrm{SL}_2)) \xleftarrow{\sim} \mathrm{Rep}(\mathrm{SL}_2)$ is a $\otimes$-functor preserving the underlying vector spaces, where the last arrow is an equivalence (e.g., see [11, VIII.1.5]). Thus the composition arises from a morphism of groups $f\colon \mathrm{SL}_2 \to G$ by [21, Cor. 2.9], and one checks directly that $g \mapsto f$ and $f \mapsto \mathrm{Lie}\,(f)$ are inverse to each other. By the Jacobson–Morozov lemma, $\mathrm{Hom}(\mathrm{Lie}\,(\mathrm{SL}_2), \mathrm{Lie}\,(G))$ is in bijection with the set of nilpotent elements in $\mathrm{Lie}\,(G)$ via $g \mapsto g(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})$. Above we consider f such that $f(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ is regular unipotent, thus $g(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})$ is regular nilpotent in $\mathrm{Lie}\,(G)$, and hence unique up to conjugacy. The same statement follows for f then as well.

Case $H = G_2$ **and** $n = 4$. Let $f\colon G_2 \to \mathrm{SO}_8$ be a morphism with a regular unipotent element in its image. Recall $i_{\mathrm{spin}}\colon G_2 \to \mathrm{Spin}_8$ from (5.3), it induces a morphism $\overline{i_{\mathrm{spin}}}\colon G_2 \to \mathrm{SO}_8$. The representations $\mathrm{std} \circ f$ and $\mathrm{std} \circ \overline{i_{\mathrm{spin}}}$ are both faithful and of dimension 8, hence isomorphic (they are both isomorphic to $r_7 \oplus \mathbf{1}$, where r_7 is the unique representation of G_2 of dimension 7). By O_8-acceptability, we can find a $g \in O_8$ such that $f = g\overline{i_{\mathrm{spin}}}g^{-1}$. If $\det(g) = 1$ we are done. The element $\vartheta^\circ \in O_8$ centralizes the subgroup SO_{2n-1}. The map i_{spin} factors over the map $j_{\mathrm{spin}}\colon G_2 \to \mathrm{Spin}_7$ (see above (5.3)). In particular $\vartheta^\circ i_{\mathrm{spin}} \vartheta^{\circ,-1} = \overline{i_{\mathrm{spin}}}$, and we can replace g by $g\vartheta^\circ$. $\square$

Proposition 5.5. *Let $n \geq 3$. Let $\overline{H} \subset \mathrm{PSO}_{2n}$ be a (possibly disconnected) reductive subgroup (over $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$) containing a regular unipotent element. Up to conjugation by an element of PSO_{2n}, the following holds (in particular $\overline{H}$ is connected in all cases):*

- *(i) if $n \neq 4$, then $\overline{H} = \mathrm{PSO}_{2n}$, $\overline{H} = \overline{i_{\mathrm{std}}}(\mathrm{PSO}_{2n-1})$, or $\overline{H} = \overline{i_{\mathrm{reg}}}(\mathrm{PGL}_2)$;*
- *(ii) if $n = 4$, then $\overline{H}$ is either as in (1), $\overline{H} = \overline{\mathrm{spin}^\circ}(\mathrm{SO}_7)$, $\overline{H} = \overline{\theta^\circ\mathrm{spin}^\circ}(\mathrm{SO}_7)$, or $\overline{H} = \overline{i_{\mathrm{spin}}}(G_2)$.*

If $H \subset \mathrm{SO}_{2n}$ is a (possibly disconnected) reductive subgroup containing a regular unipotent element, then $H^0 \subset H \subset H^0 \cdot Z(\mathrm{SO}_{2n})$ and H^0 surjects onto $\overline{H} \subset \mathrm{PSO}_{2n}$ as in the list above. (See the proof for the list of possible H^0.)

Proof. We first focus on the classification of reductive subgroups $H \subset \mathrm{SO}_{2n}$ containing a regular unipotent element. If $H = \mathrm{SO}_{2n}$ there is nothing to do, so we assume H is proper. By Lemma 5.3 the group H^0 is isomorphic to a quotient of $\mathrm{Spin}_{2n-1}, G_2$ or SL_2. Using Lemma 5.4 we conjugate so that $f\colon H^0 \hookrightarrow \mathrm{SO}_8$ is one of the maps listed in that lemma. So

(a) When $n \neq 4$, f equals $\overline{i^\circ_{\mathrm{std}}}$ or i_{reg}.
(b) When $n = 4$, f equals $\overline{i^\circ_{\mathrm{std}}}$, i_{reg}, $\overline{i_{\mathrm{spin}}}$, spin° or $\theta^\circ\mathrm{spin}^\circ$.

From this list we see that if $\mathrm{std}(H^0)$ is reducible (so $f \neq \mathrm{spin}^\circ, \theta^\circ\mathrm{spin}^\circ$) then $\mathrm{std}(H)$ is contained in a parabolic subgroup of GL_{2n} with Levi component $\mathrm{GL}_{2n-1} \times \mathrm{GL}_1$. By reductivity $\mathrm{std}(H)$ is contained in $\mathrm{GL}_{2n-1} \times \mathrm{GL}_1$, and it is an irreducible subgroup. We see that $H^0 \subset H^+ :=$

$i^\circ_{\mathrm{std}}(\mathrm{SO}_{2n-1}) \times Z(\mathrm{SO}_{2n})$, and by Schur's lemma, the centralizer of H^0 in H^+ is $Z(\mathrm{SO}_{2n})$. Since H^0 has no nontrivial outer automorphism, the conjugation by each $h \in H$ on H^0 is an inner automorphism. Thus there exists $h' \in H^0$ such that $h'h^{-1}$ centralizes H^0. It follows that $H \subset H^0 \times Z(\mathrm{SO}_{2n})$. If $\mathrm{std}(H^0) \subset \mathrm{GL}_{2n}$ is irreducible, the centralizer of H^0 in SO_{2n} is $Z(\mathrm{SO}_{2n})$ again by Schur's lemma, with no nontrivial outer automorphism for H^0. As in the reducible case, we deduce $H^0 \subset H \subset H^0 \times Z(\mathrm{SO}_{2n})$.

Finally, if a reductive subgroup $\overline{H} \subset \mathrm{PSO}_{2n}$ contains a regular unipotent element, then so does its preimage H in SO_{2n}. By the previous argument we may conjugate so that H^0 is of type (a) or (b), and moreover we find $H^0 \subset H \subset H^0 \times Z(\mathrm{SO}_{2n})$. In particular $\overline{H}$ is connected and of the type listed in (i) and (ii). □

In the next lemma, and also in the later sections, the following group will play a role

$$H_{2n-1} := \mathrm{GSpin}_{2n-1} Z(\mathrm{GSpin}_{2n}) \subset \mathrm{GSpin}_{2n}. \tag{5.4}$$

Recall $z^\varepsilon \in Z(\mathrm{GSpin}_{2n})$ is such that $\langle z^\varepsilon \rangle = \ker(\mathrm{spin}^\varepsilon)$. We have $H_{2n-1} = \mathrm{GSpin}_{2n-1} \times \langle z^+ \rangle$. By projecting we obtain a quadratic character

$$\kappa \colon H_{2n-1} \to \langle z^+ \rangle \subset Z(\mathrm{GSpin}_{2n-1}) \tag{5.5}$$

such that the composition $\mathrm{spin}^\varepsilon \circ \kappa$ is trivial if $\varepsilon = +$ and otherwise equal to the composition

$$\kappa_0 \colon H_{2n-1} \xrightarrow{\kappa} \langle z^+ \rangle \simeq \{\pm 1\}. \tag{5.6}$$

In the definition of κ we could have also used z^-, in that case the convention would be slightly different. But notice that κ_0 does not depend on this choice as this character is simply the canonical map of H_{2n-1} onto its component group. Observe also that θ acts trivially on GSpin_{2n-1} and on z^+ via $z^+ \mapsto -z^+$. This gives the simple formula

$$\theta(g) = \kappa_0(g) g \qquad \text{for all } g \in H_{2n-1}. \tag{5.7}$$

Lemma 5.6. *Let $r \colon \Gamma \to \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$ be a semisimple representation containing a regular unipotent element in its image. Let $\chi \colon \Gamma \to \overline{\mathbb{Q}}_\ell^\times$ be a character and $\varepsilon \in \{+, -\}$.*

- *(i) If $\chi \otimes \mathrm{spin}^\varepsilon(r) \simeq \mathrm{spin}^\varepsilon(r)$ then $\chi = 1$.*
- *(ii) If $\chi \otimes \mathrm{spin}^+(r) \simeq \mathrm{spin}^-(r)$ then r has image in the group $H_{2n-1} \subset \mathrm{GSpin}_{2n}$ up to conjugation, and χ is equal to $\kappa_0 \circ r$.*
- *(iii) If $\chi^+ \mathrm{spin}^+(r) \oplus \chi^- \mathrm{spin}^-(r) \simeq \mathrm{spin}(r)$ for two characters $\chi^\pm \colon \Gamma \to \overline{\mathbb{Q}}_\ell^\times$. Then $\chi^\pm$ are both trivial, or r has image in H_{2n-1} up to conjugation and $\chi^+ = \chi^- = \kappa_0 \circ r$.*

Proof. (i) Write $\overline{r} \colon \Gamma \to \mathrm{PSO}_{2n}(\overline{\mathbb{Q}}_\ell)$ for the projectivization of r. By Proposition 5.5 we can distinguish between two cases for the Zariski closure of the image of $\overline{r}$ in $\mathrm{PSO}_{2n}(\overline{\mathbb{Q}}_\ell)$. If the Zariski closure of $\overline{r}$ is either PSO_{2n} or $\overline{i_{\mathrm{std}}}(\mathrm{PSO}_{2n-1})$ then $\mathrm{spin}^\varepsilon r$ is strongly irreducible, and the statement follows from [49, Lem. 4.8(i)]. In the remaining cases the Zariski closure of $\mathrm{Im}(\overline{r})$ is, $\overline{\mathrm{spin}^\circ}(\mathrm{SO}_7)$, $\overline{\theta^\circ \mathrm{spin}^\circ}(\mathrm{SO}_7)$, $\overline{i_{\mathrm{reg}}}(\mathrm{PGL}_2)$ or $i_{\mathrm{spin}}(G_2)$. In the last two of these, $\mathrm{Im}(\overline{r}) \subset \overline{i_{\mathrm{std}}}(\mathrm{SO}_{2n-1}(\overline{\mathbb{Q}}_\ell))$ so $\mathrm{Im}(r)$ is contained in $\overline{\mathbb{Q}}_\ell$-points of H_{2n-1} (which is the preimage of $i_{\mathrm{std}}(\mathrm{SO}_{2n-1})$ in GSpin_{2n}). Then we show $\chi = 1$ by the argument exactly as in Cases (i), (ii), (iv) in the proof of [49, Lem. 5.1, Prop. 5.2], noting that $\mathrm{spin}^\varepsilon$ restricts to spin on GSpin_{2n-1} by Proposition 4.5. Finally, if the Zariski closure of the image is $\overline{\mathrm{spin}^\circ}(\mathrm{SO}_7)$ (resp. $\overline{\theta^\circ \mathrm{spin}^\circ}(\mathrm{SO}_7)$), then $\overline{\mathrm{spin}^\varepsilon} \circ \overline{r}$ is irreducible if $\varepsilon = -$ (resp. $\varepsilon = +$) and isomorphic to $\mathrm{std} \oplus \mathbf{1}$ if $\varepsilon = +$ (resp. $\varepsilon = -$) by Lemma 5.2. In particular the representation $\mathrm{spin}^\varepsilon r$ satisfies the conditions of [49, Prop. 4.9], and so $\chi = 1$ in this case as well.

(ii) If the Zariski closure of the image of r contains Spin_{2n}, then (ii) cannot occur. Thus r has either image in H_{2n-1}, or it has image in $\mathrm{pr}^{-1}(\mathrm{spin}^\circ(\mathrm{Spin}_7))$. In the latter case, $\mathrm{spin}^- r$ is strongly irreducible while $\mathrm{spin}^+ r$ is not by Lemma 5.2, which is a contradiction. Thus $\mathrm{Im}(r) \subset H_{2n-1}$. For $g \in H_{2n-1}$ we have

$$\mathrm{spin}^+(g) = \mathrm{spin}^-(\theta g) = \mathrm{spin}^-(\kappa_0(g) \cdot g) = \kappa_0(g) \mathrm{spin}^-(g).$$

Put $t = \kappa_0 \circ r$. Then $t \otimes \mathrm{spin}^+ r \simeq \mathrm{spin}^- r$ and $t\chi^{-1} \otimes \mathrm{spin}^+ r \simeq \mathrm{spin}^+ r$, and $t = \chi$ by (i).

(*iii*) Write H for the Zariski closure of the image of r. By the proof of (i) we see that either $\mathrm{Spin}_{2n} \subset H$, or $H \subset H_{2n-1}$ or $H \subset \mathrm{pr}^{-1}(\mathrm{spin}^\circ(\mathrm{Spin}_7))$ up to conjugation. Assume that $H \not\subset H_{2n-1}$ (even after conjugation), so that $\mathrm{Spin}_{2n} \subset H$ or $H \subset \mathrm{pr}^{-1}(\mathrm{spin}^\circ(\mathrm{Spin}_7))$. In this case, we need to show that $\chi^+ = \chi^- = 1$. Suppose $\chi^- \neq 1$ to the contrary. Since $\mathrm{spin}^-(r)$ is strongly irreducible by assumption on H (cf. Lemma 5.2), we have

$$\mathrm{Hom}(\chi^- \otimes \mathrm{spin}^-(r), \mathrm{spin}^-(r)) = 0 \tag{5.8}$$

by [49, Lem. 4.8(i)]. In particular $\chi^+\mathrm{spin}^+(r) \oplus \chi^-\mathrm{spin}^-(r) \xrightarrow{\sim} \mathrm{spin}(r)$ induces an isomorphism $\chi^- \otimes \mathrm{spin}^-(r) \xrightarrow{\sim} \ker(\mathrm{spin}(r) \twoheadrightarrow \mathrm{spin}^-(r)) = \mathrm{spin}^+(r)$. As $H \not\subset H_{2n-1}$, this contradicts (ii). Therefore $\chi^- = 1$. From $\chi^+\mathrm{spin}^+ r \oplus \chi^-\mathrm{spin}^- r \cong \mathrm{spin}(r)$ we then obtain $\chi^+\mathrm{spin}^+ r \cong \mathrm{spin}^+ r$, which implies $\chi^+ = 1$ by item (i).

Now assume $H \subset H_{2n-1}$ up to conjugation. We obtain a character $t = \kappa_0 \circ r \colon \Gamma \to \{\pm 1\}$. Write $\Gamma_0 := \ker(t)$. We have $\chi^+\mathrm{spin}^+(r) \oplus \chi^-\mathrm{spin}^-(r) \simeq \mathrm{spin}^+(r) \oplus \mathrm{spin}^-(r)$. By Proposition 4.5 we have $\mathrm{spin}^+(r|_{\Gamma_0}) \simeq \mathrm{spin}_{2n-1}(r|_{\Gamma_0})$, where spin_{2n-1} denotes the spin representation of GSpin_{2n-1}. By the proof of [49, Prop. 5.1], $\mathrm{spin}_{2n-1}(r|_{\Gamma_0})$ decomposes as a direct sum $r_1^{e_1} \oplus \cdots \oplus r_k^{e_k}$, such that $k, e_i \in \mathbb{Z}_{\geq 1}$, r_i is irreducible, and $\dim(r_i) \neq \dim(r_j)$ for $i \neq j$. Moreover, the projective image of $\mathrm{spin}_{2n-1}(r|_{\Gamma_0})$ in $\mathrm{PGL}_{2^{n-1}}$ is Zariski connected. This implies that the r_i are strongly irreducible Γ_0-representations [Prop. 4.8(ii), *loc. cit.*].

Suppose $\chi^-|_{\Gamma_0} \neq 1$. We again claim that (5.8) holds. To see this, assume $f \colon \chi^- \otimes \mathrm{spin}^-(r) \to \mathrm{spin}^-(r)$ is a non-trivial Γ_0-morphism. Since the $\dim(r_i)$ are distinct, the morphism f induces an isomorphism from $\chi^- \otimes r_1$ to one of the copies of r_1 in $\mathrm{spin}^-(r)$. Since r_1 is strongly irreducible by [Prop. 4.8(i), *loc. cit.*], this implies $\chi^-|_{\Gamma_0} = 1$. Thus $\chi^- \in \{t, 1\}$.

Arguing with $+$ instead of $-$ we find similarly $\chi^+ \in \{t, 1\}$. If $\chi^+ = 1$, then we have

$$\mathrm{spin}^+(r) \oplus \chi^-\mathrm{spin}^-(r) \simeq \mathrm{spin}^+(r) \oplus \mathrm{spin}^-(r),$$

which implies $\chi^-\mathrm{spin}^-(r) \simeq \mathrm{spin}^-(r)$, and thus $\chi^- = 1$ by (i). By the same argument, if $\chi^- = 1$, then $\chi^+ = 1$. Thus $\chi^+ = \chi^-$. The statement follows. □

Lemma 5.7. *Let $\gamma, \gamma' \in \mathrm{GSpin}_{2n}$ be two semi-simple elements. Then γ, γ' are GPin_{2n}-conjugate if and only if they are conjugate in the representations $\mathcal{N}, \mathrm{std}$ and spin.*

Proof. This follows from [49, §1] and the fact that GPin_{2n} has $\{\mathrm{std}, \mathcal{N}, \mathrm{spin}\}$ as a fundamental set in the sense thereof, which follows from the fact that $\{\mathrm{std}, \mathcal{N}, \mathrm{spin}^+, \mathrm{spin}^-\}$ is a fundamental set for GSpin_{2n} as checked therein. □

Proposition 5.8. *Let E/F be a quadratic extension of characteristic zero fields. Let H be one of the following algebraic groups*

$$\mathrm{SO}_{2n}, \mathrm{GSpin}_{2n}, \mathrm{SO}_{2n} \rtimes \Gamma_{E/F}, \mathrm{GSpin}_{2n} \rtimes \Gamma_{E/F},$$

where $\Gamma_{E/F}$ acts through θ° or θ in the semi-direct products. We write H^0 for the neutral component of H. Let

$$r_1, r_2 : \Gamma_F \to H(\overline{\mathbb{Q}}_\ell)$$

be semisimple Galois representations such that

- *r_1 and r_2 are locally conjugate and*
- *the Zariski closure of $r_1(\Gamma)$ contains a regular unipotent element.*

Then r_1 and r_2 are H^0-conjugate.

Proof. For simplicity we abbreviate $H(\overline{\mathbb{Q}}_\ell)$ as H if there is no danger of confusion.

The case $H = \mathrm{SO}_{2n}$. Write $\overline{r}_1, \overline{r}_2 : \Gamma \to \mathrm{PSO}_{2n}$ for the projectivizations of r_1, r_2. Write I_i for the Zariski closure of $\overline{r}_i(\Gamma)$ in PSO_{2n}, for $i = 1, 2$. Since O_{2n} is acceptable [49, Prop. B.1], r_1 and r_2 are conjugate by an element of $w \in \mathrm{O}_{2n}$, i.e., $r_2 = w r_1 w^{-1}$. In particular the Zariski closure of $r_2(\Gamma)$ also contains a regular unipotent element. We are done if $w \in \mathrm{SO}_{2n}$, so we may assume that $w \notin \mathrm{SO}_{2n}$ henceforth.

There are now three cases by Proposition 5.5: either (A) $I_1 = \mathrm{PSO}_{2n}$, (B) I_1 is SO_{2n-1}, PGL_2, or G_2 (the last case when $n = 4$) or (C) $n = 4$ and I_1 is $\overline{\mathrm{spin}}(\mathrm{SO}_7)$ or $\theta^\circ\overline{\mathrm{spin}}(\mathrm{SO}_7)$.

Case (A). Since r_1 has Zariski dense image in SO_{2n}, there exists $\mathfrak{q}$ such that $r_1(\mathrm{Frob}_{\mathfrak{q}})$ and $wr_1(\mathrm{Frob}_{\mathfrak{q}})w^{-1}$ are not outer conjugate by Lemma 1.1. This contradicts $r_2(\mathrm{Frob}_{\mathfrak{q}}) = wr_1(\mathrm{Frob}_{\mathfrak{q}})w^{-1}$ since $w \in \mathrm{O}_{2n}\backslash\mathrm{SO}_{2n}$.

Case (B). The image of r_1 is contained in $i^\circ_{\mathrm{std}}(\mathrm{SO}_{2n-1}) \times Z_{\mathrm{SO}_{2n}}$, which is centralized by $\vartheta^\circ \in \mathrm{O}_{2n}\backslash\mathrm{SO}_{2n}$. Since ϑ° and w belong to the same SO_{2n}-coset, it follows that r_1 and r_2 are SO_{2n}-conjugate.

Case (C). Without loss of generality, we may assume $I_1 = \overline{\mathrm{spin}}(\mathrm{SO}_7)$. Since $r_1 = \mathrm{Int}(w) \circ r_2$ for $w \in \mathrm{O}_8\backslash\mathrm{SO}_8$, we see that I_2 is PSO_8-conjugate to $\theta^\circ\overline{\mathrm{spin}}(\mathrm{SO}_7)$.

We claim that this case does not arise. By assumption r_1 and r_2 are locally conjugate representations with values in SO_8.

We may assume, after replacing r_2 with a suitable SO_8-conjugate, that $r_1(\gamma) = \theta^\circ r_2(\gamma)$ for all $\gamma \in \Gamma$. There exists a representation $\widetilde{r}\colon \Gamma \to \mathrm{Spin}_7$ such that $r_1(\gamma) = \mathrm{spin}(\widetilde{r}(\gamma))$, $r_2(\gamma) = \theta^\circ\mathrm{spin}(\widetilde{r}(\gamma))$. On the other hand, by Lemma 1.1 there exists an F-place $\mathfrak{q}$ where $\widetilde{r}$ is unramified and $\widetilde{r}(\mathrm{Frob}_{\mathfrak{q}}) \in U$ with U is as in Lemma 5.1. Then $\mathrm{spin}(\widetilde{r}(\mathrm{Frob}_{\mathfrak{q}}))$ and $\theta^\circ\mathrm{spin}(\widetilde{r}(\mathrm{Frob}_{\mathfrak{q}}))$ are not SO_8-conjugate. This contradicts the assumption that r_1 and r_2 are locally conjugate. The claim is proved.

The case $H = \mathbf{SO}_{2n} \rtimes \Gamma_{E/F}$. By the preceding case, we may assume that $r_1|_{\Gamma_E} = r_2|_{\Gamma_E}$. (Strictly speaking, we proved the SO_{2n}-case for $\Gamma = \Gamma_F$, but the proof goes through without change for Γ_E.) Since $\mathrm{SO}_{2n} \rtimes \Gamma_{E/F} \simeq \mathrm{O}_{2n}$ via $g \rtimes c \mapsto g\vartheta^\circ$, we identify the two groups. In particular H is acceptable, so there exists $w \in \mathrm{O}_{2n}$ such that $r_2 = wr_1w^{-1}$. We are done if $w \in \mathrm{SO}_{2n}$, so assume that $w \notin \mathrm{SO}_{2n}$. Depending on the projective image of $r_1|_{\Gamma_E}$, we have Cases (A), (B), (C) as above. The arguments there tell us that Cases (A) and (C) are impossible when $w \notin \mathrm{SO}_{2n}$. In Case (B), we know $r_1(\Gamma_E)$ is contained in $i^\circ_{\mathrm{std}}(\mathrm{SO}_{2n-1}) \times Z_{\mathrm{SO}_{2n}}$. The normalizer of the latter in O_{2n} is $\mathrm{O}_{2n-1} \times \mathrm{O}_1$ (embedded in O_{2n} via i°_{std}), which is centralized by ϑ°. Hence if we write $w = w_0\vartheta^\circ$ with $w_0 \in \mathrm{SO}_{2n}$, then $r_2 = w_0r_1w_0^{-1}$. Namely r_1 and r_2 are H^0-conjugate.

The $\mathbf{GSpin}_{2n}$-case. Write r_1°, r_2° for the composition of r_1, r_2 with $\mathrm{pr}^\circ : \mathrm{GSpin}_{2n} \to \mathrm{SO}_{2n}$. Then r_1° and r_2° are conjugate by the SO_{2n}-case treated above. Hence we may assume that $r_2 = \chi r_1$ with a continuous character $\chi : \Gamma \to \overline{\mathbb{Q}}_\ell^\times$, where $\overline{\mathbb{Q}}_\ell^\times = \ker(\mathrm{GSpin}_{2n} \to \mathrm{SO}_{2n})$ via Lemma 3.1 (ii). Since r_1 and $\chi \otimes r_1$ are locally conjugate by the initial assumption, we have

$$\mathrm{spin}^\varepsilon(r_1) \simeq \mathrm{spin}^\varepsilon(\chi \otimes r_1) \simeq \chi \otimes \mathrm{spin}^\varepsilon(r_1), \qquad \varepsilon \in \{\pm 1\}.$$

It follows from Lemma 5.6 that $\chi = 1$.

The $\mathbf{GSpin}_{2n} \rtimes \Gamma_{E/F}$-case. By the GSpin_{2n}-case above, we may assume that $r_1|_{\Gamma_E} = r_2|_{\Gamma_E}$. Writing $r_i^\circ := \mathrm{pr}^\circ \circ r_i$ for $i = 1, 2$, we have $r_1^\circ|_{\Gamma_E} = r_2^\circ|_{\Gamma_E}$. By the preceding argument, we deduce that $r_1^\circ = r_2^\circ$. On the other hand, $r_1|_{\Gamma_E} = r_2|_{\Gamma_E}$ implies that $r_1 = r_2$ or $r_1 = r_2 \otimes \chi$ by Example A.6, with χ as in that example. If $r_1 = r_2$ then we are done so suppose $r_1 = r_2 \otimes \chi$. Then $r_1^\circ = r_2^\circ \otimes \chi_{E/F}$ for $\chi_{E/F} : \Gamma_F \twoheadrightarrow \Gamma_{E/F} = \{\pm 1\}$. Set $R_i := \mathrm{std} \circ r_i^\circ$ for $i = 1, 2$, so that $R_1 = R_2 \otimes \chi_{E/F}$. Since r_1 and r_2 are locally conjugate, the GL_{2n}-valued representations R_1 and R_2 are locally conjugate and thus conjugate. So $R_1 \simeq R_1 \otimes \chi_{E/F}$. By [49, Lem. 4.8], R_1 is not strongly irreducible. Considering the projective image of $r_1|_{\Gamma_E}$ as in the SO_{2n}-case above, we see that Case (A) is excluded and only Case (B) or (C) occurs. In either case, again because R_1 is not strongly irreducible, the only possibility is that $R_1|_{\Gamma_E}$ decomposes into two strongly irreducible representations of dimensions $(2n-1)$ and 1. Then it is easy to see that $R_1 = R_1' \oplus R_1''$ already on Γ_F, with strongly irreducible R_1' and R_1'' of dimensions $(2n-1)$ and 1. It follows from $R_1 \simeq R_1 \otimes \chi_{E/F}$ that $R_1' \simeq R_1' \otimes \chi_{E/F}$ (and similarly for R_1''), but this contradicts strong irreducibility of R_1' [49, Lem. 4.8]. □

6. On SO_{2n}-valued Galois representations

In this section we construct Galois representations associated with automorphic representations of even orthogonal groups over a totally real field F. More precisely, we will derive a weaker version of Conjecture 1 for such groups from the literature. Let either

- $E = F$, or

- E be a CM quadratic extension of F.

In the latter case write c for the nontrivial element of $\Gamma_{E/F} := \mathrm{Gal}(E/F)$. Write $\mathrm{SO}_{2n}^{E/F}$ for the split group SO_{2n} if $E = F$, and the quasi-split outer form of SO_{2n} over F relative to E/F otherwise. To be precise, in the latter case,

$$\mathrm{O}_{2n}^{E/F}(R) := \{g \in \mathrm{GL}_{2n}(E \otimes_F R) \mid c(g) = \vartheta^\circ g \vartheta^\circ, g^t \left(\begin{smallmatrix} 0 & 1_n \\ 1_n & 0 \end{smallmatrix}\right) g = \left(\begin{smallmatrix} 0 & 1_n \\ 1_n & 0 \end{smallmatrix}\right)\} \tag{6.1}$$

for F-algebras R, and $\mathrm{SO}_{2n}^{E/F}$ is the connected component where $\det(g) = 1$. We can extend the standard embedding $\mathrm{std} : \mathrm{SO}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$ to a map (still denoted std)

$$\mathrm{std} : {}^L(\mathrm{SO}_{2n}^{E/F}) = \mathrm{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_{E/F} \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C}), \tag{6.2}$$

whose image is $\mathrm{SO}_{2n}(\mathbb{C})$ if $E = F$ and $\mathrm{O}_{2n}(\mathbb{C})$ if $E \neq F$. More precisely, when $E \neq F$, we fix the extended map std by requiring $c \mapsto \vartheta^\circ$. (We defined O_{2n} explicitly in the last section, and ϑ° was given in (2.4).)

Let $\pi^\flat$ be a cuspidal automorphic representation of $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$. The following will be key assumptions on $\pi^\flat$. (Recall from §1 that $\mathrm{St}_{\mathrm{SO},\mathfrak{q}_{\mathrm{St}}}$ denotes the Steinberg representation.)

(coh°) $\pi^\flat_\infty$ is cohomological for an irreducible algebraic representation $\xi^\flat = \otimes_{y \in \mathcal{V}_\infty} \xi^\flat_y$ of $\mathrm{SO}_{2n,F\otimes\mathbb{C}}^{E/F}$.

(St°) There exists a prime $\mathfrak{q}_{\mathrm{St}}$ of F such that $\pi^\flat_{\mathfrak{q}_{\mathrm{St}}} \simeq \mathrm{St}_{\mathrm{SO},\mathfrak{q}_{\mathrm{St}}}$ up to a character twist.

Condition (coh°) implies that the infinitesimal character of $\xi^\flat_y$ is given by $\rho_{\mathrm{SO}} + \lambda(\xi^\flat_y)$ at each $y \in \mathcal{V}_\infty$; see [8, Thm. I.5.3]. In particular $\pi^\flat$ is C-algebraic in the sense of Buzzard–Gee [12, Lem. 7.2.2], thus also L-algebraic as the half sum of positive (co)roots is integral for $\mathrm{SO}_{2n}^{E/F}$. In (St°), characters of $\mathrm{SO}_{2n}^{E/F}(F_{\mathfrak{q}_{\mathrm{St}}})$ are exactly the characters factoring through the cokernel of $\mathrm{Spin}_{2n}^{E/F}(F_{\mathfrak{q}_{\mathrm{St}}}) \to \mathrm{SO}_{2n}^{E/F}(F_{\mathfrak{q}_{\mathrm{St}}})$. (This is a special case of the general fact [50, Cor. 2.3.3].) Such characters are in a natural bijection with characters of $F^\times_{\mathfrak{q}_{\mathrm{St}}}/(F^\times_{\mathfrak{q}_{\mathrm{St}}})^2 \simeq H^1(F_{\mathfrak{q}_{\mathrm{St}}}, \{\pm 1\})$.

Write $\mathrm{T}_{\mathrm{SO}} := \mathrm{T}_{\mathrm{GSO}} \cap \mathrm{SO}_{2n}$ over $\mathbb{C}$ and choose the Borel subgroup containing T_{SO} in $\mathrm{SO}_{2n}^{E/F}$ as in the preceding section. For each $y \in \mathcal{V}_\infty$, the highest weight of $\xi^\flat_y$ gives rise to a dominant cocharacter $\lambda(\xi^\flat_y) \in X_*(\mathrm{T}_{\mathrm{SO}})$. Let $\phi_{\pi^\flat_y} : W_{F_y} \to {}^L\mathrm{SO}_{2n}^{E/F}$ denote the L-parameter of $\pi^\flat_y$ assigned by [53]. Recall $\mathrm{std} : \mathrm{SO}_{2n} \hookrightarrow \mathrm{GL}_{2n}$ denotes the standard embedding. We also consider the following conditions:

(std-reg°) $\mathrm{std} \circ \phi_{\pi^\flat_y}|_{W_{\overline{F}_y}}$ is regular (i.e., the centralizer group in $\mathrm{GL}_{2n}(\mathbb{C})$ is a torus) for every $y \in \mathcal{V}_\infty$.

(disc-∞) If n is odd then $[E : F] = 2$. If n is even then $E = F$.

Since E is either F or a CM quadratic extension of F, condition (disc-∞) is equivalent to requiring $\mathrm{SO}_{2n}^{E/F}(F_y)$ to admit discrete series at all infinite places y of F (or equivalently, to admit compact maximal tori). Condition (std-reg°), when (coh°) is imposed, amounts to requiring that $\mathrm{std} \circ (\rho_{\mathrm{SO}} + \lambda(\xi^\flat_y))$ should be a regular cocharacter of GL_{2n} for every $y \in \mathcal{V}_\infty$, since $\phi_{\pi^\flat_y}|_{W_{\overline{F}_y}}$ encodes the infinitesimal character of π_y according to [78, Prop. 7.4].

When v is a prime of F, write $\phi_{\pi^\flat_v} : W_{F_\mathfrak{q}} \to {}^L\mathrm{SO}_{2n}^{E/F}$ for the L-parameter of $\pi^\flat_v$ as given by [1, Thm 1.5.1]. (By the Langlands quotient theorem, $\pi^\flat_v$ is the unique quotient of an induced representation from a character twist of a tempered representation on a Levi subgroup. Apply Arthur's theorem to this tempered representation.) Note that $\phi_{\pi^\flat_v}$ is well-defined only up to $\mathrm{O}_{2n}(\mathbb{C})$-conjugacy in *loc. cit.* (This does not matter for the statement of (SO-i) in Theorem 6.5 below.)

Let $\mathrm{Unr}(\pi^\flat)$ denote the set of finite primes $\mathfrak{q}$ of F such that $\mathfrak{q}$ is unramified in E and $\pi^\flat_\mathfrak{q}$ is unramified. In this case, the unramified L-parameter $\phi_{\pi^\flat_\mathfrak{q}}$ is determined (up to $\mathrm{SO}_{2n}(\mathbb{C})$-conjugacy, not just up to outer automorphism) by the Satake isomorphism.

Thanks to Arthur, we can lift $\pi^\flat$ to an automorphic representation of GL_{2n} as follows. The Hecke character $F^\times \backslash \mathbb{A}_F^\times \to \{\pm 1\}$ corresponding to the Galois character $\chi_{E/F} : \Gamma_F \twoheadrightarrow \Gamma_{E/F} = \{\pm 1\}$ is still denoted $\chi_{E/F}$. Let $\chi_{E/F,\mathfrak{q}}$ denote its local component at $\mathfrak{q}$.

Proposition 6.1 (Arthur)**.** *Assume that $\pi^\flat$ satisfies (St°). Then there exists a self-dual automorphic representation $\pi^\sharp$ of $\mathrm{GL}_{2n}(\mathbb{A}_F)$, which is either cuspidal or the isobaric sum of two cuspidal self-dual representations of $\mathrm{GL}_{2n-1}(\mathbb{A}_F)$ and $\mathrm{GL}_1(\mathbb{A}_F)$, such that*

(Ar1) $\pi^\sharp_\mathfrak{q}$ is unramified at every $\mathfrak{q} \in \mathrm{Unr}(\pi^\flat)$.
(Ar2) $\pi^\sharp_{\mathfrak{q}_{\mathrm{St}}} \simeq \mathrm{St}_{2n-1} \boxplus \chi_{E/F,\mathfrak{q}_{\mathrm{St}}}$ up to a quadratic character of $\mathrm{GL}_{2n}(\mathbb{A}_F)$.
(Ar3) $\phi_{\pi^\sharp_v} \simeq \mathrm{std} \circ \phi_{\pi^\flat_v}$ at every F-place v.

If $\pi^\flat$ satisfies both (St°) and (coh°) then we furthermore have

(Ar4) $\pi^\sharp_y$ and $\pi^\flat_y$ are tempered for all infinite F-places y.

If $\pi^\flat$ has properties (coh°), (St°), and (std-reg°), then the following strengthening holds:

(Ar4)+ $\pi^\sharp_v$ and $\pi^\flat_v$ are tempered for all F-places v.

Remark 6.2. In fact (Ar1) is implied by (Ar3) since $\phi_{\pi^\flat_\mathfrak{q}}$ is an unramified parameter at every $\mathfrak{q} \in \mathrm{Unr}(\pi^\flat)$, but we state (Ar1) to make (SO-ii) below more transparent.

Proof. Consider $\pi^\flat$ satisfying (St°). For notational convenience, we assume $\pi^\flat_{\mathfrak{q}_{\mathrm{St}}} \simeq \mathrm{St}_{\mathrm{SO},\mathfrak{q}_{\mathrm{St}}}$ (not just up to a quadratic character twist) as the general case works in the same way. By [1, Thm. 1.5.2] (using the notation there),[9] we have a formal global parameter ψ (as in [1, 1.4]) such that $\pi^\flat_v$ appears as a subquotient of a member of the packet $\tilde{\Pi}(\psi_v)$ at every place v of F, where ψ_v denotes the localization of ψ at v as in *loc. cit.* (A priori, members of $\tilde{\Pi}(\psi_v)$ may be reducible due to possible failure of the generalized Ramanujan conjecture. Only from the argument below it follows that ψ is a generic parameter, i.e., its SU(2)-part is trivial. Then $\tilde{\Pi}(\psi_v)$ consists of irreducible representations by [81, Appendix A].)

Since $\mathrm{St}_{\mathrm{SO},\mathfrak{q}_{\mathrm{St}}}$ appears as a subquotient of a member of $\tilde{\Pi}(\psi_{\mathfrak{q}_{\mathrm{St}}})$, Proposition B.1 implies that $\psi_{\mathfrak{q}_{\mathrm{St}}} \simeq \psi_{\mathrm{St},\mathfrak{q}_{\mathrm{St}}}$, where ψ_{St} is defined above that proposition. Thus

$$\psi_{\mathrm{St},\mathfrak{q}_{\mathrm{St}}} \simeq \psi_{\mathrm{St}_{2n-1},\mathfrak{q}_{\mathrm{St}}} \oplus \chi_{E/F,\mathfrak{q}_{\mathrm{St}}}, \tag{6.3}$$

where $\psi_{\mathrm{St}_{2n-1},\mathfrak{q}_{\mathrm{St}}}$ (resp. $\psi_{\mathbf{1},\mathfrak{q}_{\mathrm{St}}}$) denotes the A-parameter for the Steinberg (resp. trivial) representation St_{2n-1} of $\mathrm{GL}_{2n-1}(F_{\mathfrak{q}_{\mathrm{St}}})$ (resp. $\mathrm{GL}_1(F_{\mathfrak{q}_{\mathrm{St}}})$). It follows that either $\psi = \pi^\#$ or $\psi = \pi^\#_1 \boxplus \pi^\#_2$, where $\pi^\#$, $\pi^\#_1$, and $\pi^\#_2$ are cuspidal self-dual automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A}_F)$, $\mathrm{GL}_{2n-1}(\mathbb{A}_F)$, and $\mathrm{GL}_1(\mathbb{A}_F)$, respectively. In the second case, we take $\pi^\#$ to be the isobaric sum of $\pi^\#_1$ and $\pi^\#_2$. Now (Ar2) follows from (6.3). We define $\phi_\mathfrak{q} \in \tilde{\Phi}(\mathrm{SO}^{E/F}_{2n,F_\mathfrak{q}})$ as the restriction of $\psi_\mathfrak{q} \in \tilde{\Psi}(\mathrm{SO}^{E/F}_{2n,F_\mathfrak{q}})$ from $\mathcal{L}_{F_\mathfrak{q}} \times \mathrm{SU}(2)$ to $\mathcal{L}_{F_\mathfrak{q}}$. Then Properties (Ar1) and (Ar3) with $\phi_\mathfrak{q}$ in place of $\phi_{\pi^\flat_\mathfrak{q}}$ are part of Arthur's result already cited.

To complete the proof of (Ar1) and (Ar3), it suffices to verify that $\phi_v = \phi_{\pi^\flat_v}$ in $\tilde{\Phi}(G_{F_v})$. In the notation of [1] (between Theorems 1.5.1 and 1.5.2), ϕ_v gives rise to

- a F_v-rational parabolic subgroup $P_v \subset G_{F_v}$ with a Levi factor M_v,
- a bounded parameter $\phi_{M_v} \in \tilde{\Phi}(M_v)$,
- a point λ in the open chamber for P_v in $X_*(M_v)_{F_v} \otimes_\mathbb{Z} \mathbb{R}$,

such that ϕ_v comes from the λ-twist $\phi_{M_v,\lambda}$ of ϕ_{M_v}. (This is the counterpart of the Langlands quotient construction for L-parameters.) The statement of [1, Thm. 1.5.2] tells us that $\pi^\flat_v$ is a subrepresentation of the normalized induction $\mathrm{Ind}^{G(F_v)}_{P_v(F_v)}(\sigma_{v,\lambda})$ for some $\sigma_v \in \tilde{\Pi}(M_v)$, where $\sigma_{v,\lambda}$ denotes the λ-twist of σ_v, since $\pi^\flat_v$ appears in the packet of ψ_v in *loc. cit.* According to the same theorem, $\mathrm{Ind}^{G(F_v)}_{P_v(F_v)}(\sigma_{v,\lambda})$ must be completely reducible since it appears in the L^2-discrete spectrum. This means that $\pi^\flat_v$ is irreducible and the Langlands quotient of $\mathrm{Ind}^{G(F_v)}_{P_v(F_v)}(\sigma_{v,\lambda})$ (thus

[9] E.g., $\tilde{\Phi}(\mathrm{SO}^{E/F}_{2n,F_\mathfrak{q}})$ means the set of isomorphism classes of L-parameters for $G = \mathrm{SO}^{E/F}_{2n,F_\mathfrak{q}}$ *modulo the action of the outer automorphism group* $\widetilde{\mathrm{Out}}_{2n}(G)$ as defined in [1, 1.2]. Similarly the packet $\tilde{\Pi}(\psi_\mathfrak{q})$ of [1, 1.5] consists of finitely many $\widetilde{\mathrm{Out}}_{2n}(G)$-orbits of isomorphism classes of representations of $G(F_\mathfrak{q})$. By abuse of terminology, a representation will often mean the outer automorphism orbit of representations in this proof.

$\pi_v^\flat$ is isomorphic to the latter). Since the formation of Langlands parametrization is compatible with the Langlands quotient, it follows that ϕ_v is the L-parameter of $\pi_v^\flat$, namely that $\phi_v = \phi_{\pi_v^\flat}$.

It remains to check (Ar4) and (Ar4)+. Assume (coh°) in addition to (St°). Thanks to (Ar3), $\pi^\#$ is L-algebraic since L-algebraicity is preserved by std. Applying [18, Lem. 4.9] to $\pi^\# \otimes |\det|^{1/2}$ if $\pi^\#$ is cuspidal, and $\pi_1^\#$ and $\pi_2^\#$ otherwise, to deduce that $\pi_v^\#$ is essentially tempered at all $y|\infty$. Since $\pi^\#$ is self-dual, $\pi_y^\#$ are a fortiori tempered. Now suppose furthermore that $\pi_y^\flat$ has property (std-reg°). Then $\pi^\#$ is regular L-algebraic. Arguing as above but applying [13, Thm. 1.2] to $\pi^\#$ at finite places, in place of [18, Lem. 4.9] at infinite places, we deduce (Ar4)+. Finally, whenever $\pi_v^\#$ is tempered (for finite or infinite v), this implies that ψ_v is bounded, hence that $\pi_v^\flat$ is tempered by [1, Thm. 1.5.1]. □

Corollary 6.3. *Assume (disc-∞). If $\pi^\flat$ satisfies (St°) and (coh°) then $\pi_y^\flat$ is a discrete series representation for every infinite place y.*

Proof. The condition (disc-∞) guarantees that $\mathrm{SO}_{2n}^{E/F}(F_y)$ contains an elliptic maximal torus at infinite places y, so that it admits discrete series. In this case, a tempered ξ-cohomological representation is a discrete series representation by [8, Thm. III.5.1]. Thus the corollary follows from (Ar4) of the preceding proposition. □

Under the assumptions of the corollary, let us describe $\phi_{\pi_y^\flat}|_{W_{\overline{F}_y}}$ explicitly. Fix an $\mathbb{R}$-isomorphism $\overline{F}_y \simeq \mathbb{C}$ once and for all, so that we can identify $W_{\overline{F}_y} = \mathbb{C}^\times$. We noted that the infinitesimal character of $\xi_y^\flat$ is $\rho_{\mathrm{SO}} + \lambda(\xi_y^\flat)$. The half sum of positive coroots $\rho_{\mathrm{SO}} \in X_*(\mathrm{T}_{\mathrm{SO}})$ is equal to $(n-1)e_1 + (n-2)e_2 + \cdots + e_{n-1}$. It follows from the construction of discrete series L-packets in [53, p.134] that possibly after $\mathrm{SO}_{2n}^{E/F}(\mathbb{C})$-conjugation, we have

$$\phi_{\pi_y^\flat}(z) = (z/\overline{z})^{\rho_{\mathrm{SO}}+\lambda(\xi_y^\flat)}, \quad z \in W_{\overline{F}_y}. \tag{6.4}$$

Continue to assume (St°) and (coh°) for $\pi^\flat$ as well as (disc-∞). We noted that $\pi^\flat$ is L-algebraic thanks to (coh°). Then Conjecture 1 predicts the existence of an ${}^L\mathrm{SO}_{2n}^{E/F}$-valued Galois representation attached to $\pi^\flat$. When (std-reg°) is also assumed, Theorem 6.5 below proves the conjecture modulo outer automorphisms in that (SO-i) is weaker than what is predicted. (This is to be upgraded by (SO-i+) in §13; also see Remark 13.2.) The proof is carried out by reducing to the known results for $\pi^\#$ on GL_{2n}. We will get to the theorem after observing that (disc-∞) is automatically satisfied under the additional hypothesis (std-reg°); this observation is related to (SO-v) of the theorem.

Lemma 6.4. *Suppose there exists a cuspidal automorphic representation $\pi^\flat$ of $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ such that (St°),(coh°), and (std-reg°) hold. Then (disc-∞) is satisfied.*[10]

Proof. For each $y \in \mathcal{V}_\infty$, Proposition 6.1 tells us that $\phi_{\pi_y^\flat}$ is tempered. From this and (coh°), we obtain a decomposition of the form

$$\mathrm{std} \circ \phi_{\pi_y^\flat} = \bigoplus_{i \in I} \mathrm{Ind}_{W_{\overline{F}_y}}^{W_{F_y}} \chi_{a_i} \oplus \bigoplus_{i' \in I'} \omega_{i'}, \quad a_i \in \mathbb{Z}_{>0}$$

where $\chi_{a_i} : \mathbb{C}^\times \to \mathbb{C}^\times$ is given by $z \mapsto (z/\overline{z})^{a_i}$ using the identification $W_{\overline{F}_y} = \mathbb{C}^\times$ above, and $\omega_{i'}$ is a quadratic character of W_{F_y}. (In fact, a_i are mutually distinct.) By the dimension reason $|I'|$ is even. On the other hand, (std-reg°) implies that $|I'| \le 1$. Hence I' is empty and $|I| = n$. Now the image of $j \in W_{F_y}$ in $\mathrm{GL}_{2n}(\mathbb{C})$ under $\mathrm{std} \circ \phi_{\pi_y^\flat}$ has determinant $(-1)^n$ since the determinant of j is -1 in each induced representation. In view of (6.2), we deduce that $E = F$ if n is even and $E \neq F$ otherwise. That is, (disc-∞) holds true. □

[10] We heartily thank the referee for pointing out this lemma and explaining its proof.

Theorem 6.5. *Let $\pi^\flat$ be a cuspidal automorphic representation of $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ satisfying (coh°), (St°), and (std-reg°). Then there exists a semisimple Galois representation (depending on ι)*

$$\rho_{\pi^\flat} = \rho_{\pi^\flat,\iota} \colon \Gamma_F \to \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F},$$

whose restriction to $\Gamma_{F_\mathfrak{q}}$ at every F-place $\mathfrak{q}|\ell$ is potentially semistable, such that the following hold. Here $\overset{\circ}{\sim}$ means $\mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$-conjugacy.

(SO-i) For every finite F-place $\mathfrak{q}$ (including $\mathfrak{q}|\ell$), in the convention of §1, we have

$$\iota\phi_{\pi^\flat_\mathfrak{q}} \overset{\circ}{\sim} \mathrm{WD}(\rho_{\pi^\flat}|_{\Gamma_{F_\mathfrak{q}}})^{\text{F-ss}}.$$

(SO-ii) Let $\mathfrak{q} \in \mathrm{Unr}(\pi^\flat)$. If $\mathfrak{q} \nmid \ell$ then $\rho_{\pi^\flat,\mathfrak{q}}$ is unramified at $\mathfrak{q}$, and for all eigenvalues α of $\mathrm{std}(\rho_{\pi^\flat}(\mathrm{Frob}_\mathfrak{q}))_{\mathrm{ss}}$ and all embeddings $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$ we have $|\alpha| = 1$.

(SO-iii) For each $\mathfrak{q}|\ell$, and for each $y\colon F \hookrightarrow \mathbb{C}$ such that ιy induces $\mathfrak{q}$, we have $\mu_{\mathrm{HT}}(\rho_{\pi^\flat,\mathfrak{q}}, \iota y) \overset{\circ}{\sim} \iota\mu_{\mathrm{Hodge}}(\xi^\flat, y)$.

(SO-iv) If $\pi^\flat_\mathfrak{q}$ is unramified at $\mathfrak{q}|\ell$, then $\rho_{\pi^\flat,\mathfrak{q}}$ is crystalline. If $\pi^\flat_\mathfrak{q}$ has a non-zero Iwahori fixed vector at $\mathfrak{q}|\ell$, then $\rho_{\pi^\flat,\mathfrak{q}}$ is semistable.

(SO-v) $\rho_{\pi^\flat}$ is totally odd. More explicitly, for each real place y of F and the corresponding complex conjugation $c_y \in \Gamma_F$ (well-defined up to conjugacy),

$$\rho_{\pi^\flat}(c_y) \sim \begin{cases} \mathrm{diag}(\underbrace{1,\dots,1}_{n/2}, \underbrace{-1,\dots,-1}_{n/2}, \underbrace{1,\dots,1}_{n/2}, \underbrace{-1,\dots,-1}_{n/2}), & n : \textit{even}, \\ \mathrm{diag}(\underbrace{1,\dots,1}_{(n-1)/2}, \underbrace{-1,\dots,-1}_{(n-1)/2}, 1, \underbrace{1,\dots,1}_{(n-1)/2}, \underbrace{-1,\dots,-1}_{(n-1)/2}, 1) \rtimes c, & n : \textit{odd}. \end{cases}$$

Condition (SO-i) characterizes $\rho_{\pi^\flat}$ uniquely up to $\mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$-conjugation.

Remark 6.6. Since $\pi^\flat_\infty$ is a discrete series representation, the conjugation by $\phi_{\pi^\flat_y}(j)$ on T_{SO} is the inverse map, where j denotes the usual element of the real Weil group. Thus (SO-v) and (6.4) imply Buzzard–Gee's prediction on the image of complex conjugation in [12, Conj. 3.2.1, 3.2.2]. When n is odd, we also observe that (SO-v) is equivalent to

$$\rho_{\pi^\flat}(c_y) \sim \mathrm{diag}(\underbrace{1,\dots,1}_{(n-1)/2}, \underbrace{-1,\dots,-1}_{(n-1)/2}, a, \underbrace{1,\dots,1}_{(n-1)/2}, \underbrace{-1,\dots,-1}_{(n-1)/2}, a^{-1}) \rtimes c, \quad \forall a \in \overline{\mathbb{Q}}_\ell^\times.$$

Remark 6.7. Without (St°), an analogous theorem can be proved only under (coh°) and (std-reg), but in a weaker and less precise form. The strategy is similar: transfer $\pi^\flat$ to a regular algebraic automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$, which is an isobaric sum of cuspidal self-dual automorphic representations, and apply the known results on associating Galois representations.

Proof of Theorem 6.5. Let $\pi^\#$ be as in Proposition 6.1 so that

Case 1: $\pi^\#$ is cuspidal, or

Case 2: $\pi^\# = \pi_1^\# \boxplus \pi_2^\#$, with $\pi_1^\#$ (resp. $\pi_2^\#$) a cuspidal automorphic representation of $\mathrm{GL}_{2n-1}(\mathbb{A}_F)$ (resp. $\mathrm{GL}_1(\mathbb{A}_F)$).

As in the proof there, we know that $\pi^\#$ is L-algebraic.

In Case 1, consider the C-algebraic twist $\Pi := \pi^\sharp \otimes |\det|^{(1-2n)/2}$, which is regular by (std-reg), and essentially self-dual ("essentially" means up to a character twist). Applying the well-known construction of Galois representations (see [3, Thm. 2.1.1] for a summary and further references) to Π, we obtain a semisimple Galois representation (recall $\Gamma = \Gamma_F$ by convention)

$$\rho_\Pi \colon \Gamma \to \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_\ell),$$

satisfying the obvious analogues of properties (SO-i) through (SO-iv) for GL_{2n}, with ρ_Π and GL_{2n} in place of $\rho_{\pi^\flat}$ and O_{2n}; call these analogues (GL-i), ..., (GL-iv). By 'obvious', we mean for instance that (GL-ii) is about the eigenvalues of $\rho_\Pi(\mathrm{Frob}_\mathfrak{q})$ having absolute value 1. We also spell out (GL-i), which states that

$$\iota\phi_{\Pi_\mathfrak{q}\otimes|\det|_\mathfrak{q}^{(1-2n)/2}} \sim \mathrm{WD}(\rho_\Pi|_{\Gamma_{F_\mathfrak{q}}})^{\text{F-ss}}, \quad \mathfrak{q} \nmid \ell. \tag{6.5}$$

In particular, for all $\mathfrak{q} \in \mathrm{Unr}(\pi^\flat)$, since $\Pi_\mathfrak{q}$ is unramified by (Ar1), we see that ρ_Π is unramified at $\mathfrak{q}$ as well and that

$$\rho_\Pi(\mathrm{Frob}_\mathfrak{q})_{\mathrm{ss}} \sim \iota\phi_{\Pi_\mathfrak{q} \otimes |\det|_\mathfrak{q}^{(1-2n)/2}}(\mathrm{Frob}_\mathfrak{q}) \sim \iota\phi_{\pi^\#_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{q}) \sim \iota\mathrm{std}(\phi_{\pi^\flat_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{q})). \tag{6.6}$$

Since each $\pi^\#_\mathfrak{q}$ is self-dual, we see that ρ_Π is self-dual. By (Ar2) and (6.5) at $\mathfrak{q} = \mathfrak{q}_{\mathrm{St}}$ as well as semisimplicity of ρ_Π, we see that either

- ρ_Π is irreducible, or
- $\rho_\Pi = \rho_1 \oplus \rho_2$ for self-dual irreducible subrepresentations ρ_1 and ρ_2 with $\dim \rho_1 = n-1$ and $\dim \rho_2 = 1$.

Either way, it follows from [6, Cor. 1.3] that every irreducible constituent of ρ_Π is orthogonal in the sense of *loc. cit.* (As we are in Case 1, apply their corollary with $\eta = |\cdot|^{2n-1}$, in which case $\eta_\lambda(c) = -1$ in their notation.)

Now we turn to Case 2. Take $\Pi_1 := \pi^\#_1 |\det|^{1-n}$ and $\Pi_2 := \pi^\#_2$. Each of Π_1 and Π_2 is cuspidal, regular C-algebraic, and essentially self-dual, so the same construction yields ρ_{Π_1} and ρ_{Π_2}, which are $2n-1$ and 1-dimensional, respectively. Then put $\rho_\Pi := \rho_{\Pi_1} \oplus \rho_{\Pi_2}$. As before, (GL-i), ..., (GL-iv) hold true for ρ_Π. Moreover an argument as in Case 1 shows that ρ_{Π_1} and ρ_{Π_2} are self-dual and orthogonal. It follows from (Ar2) and (6.5) at $v = \mathfrak{q}_{\mathrm{St}}$ that ρ_{Π_1} and ρ_{Π_2} are irreducible.

From here on, we treat the two cases together. Since ρ_Π is self-dual and orthogonal, after conjugating ρ_Π by an element of $\mathrm{GL}_{2n}(\overline{\mathbb{Q}}_\ell)$, we can ensure that $\rho_\Pi(\Gamma) \subset \mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$. Write

$$\rho_{\pi^\flat} : \Gamma \to \mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$$

for the $\mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$-valued representation that ρ_Π factors through. (In case ρ_Π is reducible, we even have $\rho_\Pi(\Gamma) \subset (\mathrm{O}_{2n-1} \times \mathrm{O}_1)(\overline{\mathbb{Q}}_\ell)$.) Let us check that this is the desired Galois representation and deduce properties (SO-i) through (SO-v) from (GL-i) through (GL-iv).

We start with the case $E = F$. Then $\phi_{\pi^\flat_v}(\mathrm{Frob}_\mathfrak{q}) \in \mathrm{SO}_{2n}(\mathbb{C})$ in (6.6), so we deduce via the Chebotarev density theorem that $\rho_{\pi^\flat}$ has image in $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$. Note that (GL-ii) is the same statement as (SO-ii). The Hodge-theoretic properties at ℓ in (SO-iii) and (SO-iv) may be checked after composing with a faithful representation, so these properties hold. One sees from [49, Appendix B] (for O_{2n}) that (GL-i) implies (SO-i). (Alternatively, one can appeal to [26, Thm. 8.1].) The assertion on the cocharacters in (SO-iii) also follows (GL-iii) that the two cocharacters become conjugate in GL_{2n}.

We now prove (SO-v), namely that $\rho_{\pi^\flat}$ is totally odd. The following claim

(E) The element $\mathrm{std}(\rho_{\pi^\flat}(c_y)) \in \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_\ell)$ has eigenvalues 1 and -1 with multiplicity n each, for every $y \in \mathcal{V}_\infty$.

follows from [14] (in fact it can also be deduced from Taïbi's theorem [76, Thm. 6.3.4] when $\pi^\sharp$ is cuspidal, and from Taylor [77, Prop. A] when $\pi^\sharp$ is not cuspidal).

As $\rho_{\pi^\flat}(c_y) \in \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ has order 2, we have

$$\rho_{\pi^\flat}(c_y) \sim \mathrm{diag}(\underbrace{1,\dots,1}_{a_y}, \underbrace{-1,\dots,-1}_{b_y}, \underbrace{1,\dots,1}_{a_y}, \underbrace{-1,\dots,-1}_{b_y}), \quad a_y + b_y = n, \quad a_y, b_y \in \mathbb{Z}_{\geq 0}.$$

So (E) implies that $a_y = b_y$; this is possible as n is even, which follows from Lemma 6.4 and the running assumption that $E = F$. Now one computes the adjoint action of $\rho_{\pi^\flat}(c_y)$ on $\mathrm{Lie}\,\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ to be $-n$. (A similar computation is done in the proof of [49, Lem. 1.9] for GSp_{2n}.) Thus $\rho_{\pi^\flat}$ is totally odd.

It remains to treat the case $E \neq F$. In this case, the standard embedding $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F} \hookrightarrow \mathrm{GL}_{2n}$ identifies $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F} \xrightarrow{\sim} \mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$. The composition of $\rho_{\pi^\flat}$ with this isomorphism is still to be denoted by $\rho_{\pi^\flat}$. Since $\phi_{\pi^\flat_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{q}) \in \mathrm{O}_{2n}(\mathbb{C}) \backslash \mathrm{SO}_{2n}(\mathbb{C})$ (resp. $\phi_{\pi^\flat_y}(\mathrm{Frob}_\mathfrak{q}) \in \mathrm{SO}_{2n}(\mathbb{C})$) in (6.6) when $\mathfrak{q}$ is inert (resp. split) in E by the unramified Langlands correspondence, we see that

$$\rho_{\pi^\flat} : \Gamma \to \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$$

commutes with the natural projections onto $\Gamma_{E/F}$. (By continuity it suffices to check the commutativity on Frobenius conjugacy classes.) Thus $\rho_{\pi^\flat}$ is a Galois representation valued in ${}^L(\mathrm{SO}_{2n}^{E/F})$. Properties (SO-i) through (SO-iv) follow from (GL-i) through (GL-iv) in the same way as for the $E = F$ case.

We now prove (SO-v). The argument for claim (E) still applies, and since n is odd by Lemma 6.4, we have

$$\begin{aligned}
(6.7)\ \mathrm{std}\rho_{\pi^\flat}(c_y) &\sim \mathrm{diag}(\underbrace{1,\dots,1}_{n},\underbrace{-1,\dots,-1}_{n}) \\
(6.8)\qquad &\sim \mathrm{diag}(\underbrace{1,\dots,1}_{\frac{n-1}{2}},\underbrace{-1,\dots,-1}_{\frac{n+1}{2}},\underbrace{1,\dots,1}_{\frac{n-1}{2}},\underbrace{-1,\dots,-1}_{\frac{n+1}{2}})\cdot\mathrm{std}(c) \quad \text{in } \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_\ell).
\end{aligned}$$

(Recall that $\mathrm{std}(c) = \vartheta^\circ$ is the $2n \times 2n$ permutation matrix switching n and $2n$.) Therefore

$$\rho_{\pi^\flat}(c_y) \sim \mathrm{diag}(\underbrace{1,\dots,1}_{\frac{n-1}{2}},\underbrace{-1,\dots,-1}_{\frac{n+1}{2}},\underbrace{1,\dots,1}_{\frac{n-1}{2}},\underbrace{-1,\dots,-1}_{\frac{n+1}{2}}) \rtimes c \quad \text{in } {}^L\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell).$$

From this, it follows that the adjoint action of $\rho_{\pi^\flat}(c_y)$ on $\mathrm{Lie}\,\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ has trace equal to $-n$. Hence $\rho_{\pi^\flat}$ is totally odd. □

The following corollary allows us to apply Proposition 5.5 to identify the Zariski closure of the image of $\rho_{\pi^\flat}$.

Corollary 6.8. *In the setup of Theorem 6.5, the image of $\rho_{\pi^\flat}$ (thus also $\rho_{\pi^\flat}(\Gamma_E)$ contains a regular unipotent element of $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$.*

Proof. Suppose that $\mathfrak{q}_{\mathrm{St}} \nmid \ell$. Then $\iota\phi_{\pi^\flat_{\mathfrak{q}_{\mathrm{St}}}} \overset{\circ}{\sim} \mathrm{WD}(\rho_{\pi^\flat}|_{\Gamma_{F_\mathfrak{q}}})^{\text{F-ss}}$ by (SO-i), where the two sides are compared through [28, Prop. 2.2] by the convention of §1. Since $\phi_{\pi^\flat_{\mathfrak{q}_{\mathrm{St}}}}$ contains a regular unipotent element in the image, so does $\mathrm{WD}(\rho_{\pi^\flat}|_{\Gamma_{F_\mathfrak{q}}})$. Therefore $\rho_{\pi^\flat}|_{\Gamma_{F_\mathfrak{q}}}$ has a regular unipotent in the image. If $\mathfrak{q}_{\mathrm{St}}|\ell$ then the same is shown following the argument of [49, Lem. 3.2]. □

The next corollary is solely about automorphic representations, but proved by means of Galois representations. Interestingly we do not know how to derive it within the theory of automorphic forms. The corollary is not needed in this paper as (disc-∞) will be imposed in the main case of interest.

Corollary 6.9. *Let $\pi^\flat$ be a cuspidal automorphic representation of $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ satisfying (coh°), (St°), and (std-reg°). If (disc-∞) is false (i.e., n is odd and $E = F$, or n is even and $[E : F] = 2$), then $\pi^\#$ in Proposition 6.1 (the functorial lift of $\pi^\flat$ to GL_{2n}) is the isobaric sum of cuspidal self-dual automorphic representations of $\mathrm{GL}_{2n-1}(\mathbb{A}_F)$ and $\mathrm{GL}_1(\mathbb{A}_F)$.*

Proof. Fix a real place y of F. Up to conjugation, we may assume that

$$\rho_{\pi^\flat}(c_y) = \mathrm{diag}(t_1,\dots,t_n,t_1^{-1},\dots,t_n^{-1}) \rtimes c_y,$$

where the latter c_y means its image in $\Gamma_{E/F}$; so $\mathrm{std}(c_y) = 1$ if $E = F$ and $\mathrm{std}(c_y) = \vartheta^\circ$ if $[E : F] = 2$. The proof of Theorem 6.5 shows that $\mathrm{std}(\rho_{\pi^\flat}(c_y)) \in \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_\ell)$ is odd for every real place y. That is, $\mathrm{std}(\rho_{\pi^\flat}(c_y))$ has each of the eigenvalues 1 and -1 with multiplicity n. It is elementary to see that this is impossible when (disc-∞) is false. Indeed, if n is odd and $E = F$, then the number of 1's on the diagonal of $\rho_{\pi^\flat}(c_y)$ is obviously even (so cannot equal n). If n is even and $[E : F] = 2$, this is elementary linear algebra. □

Remark 6.10. The corollary suggests that in that setup, $\pi^\flat$ should come from an automorphic representation on $\mathrm{Sp}_{2n-2}(\mathbb{A}_F)$, where Sp_{2n-2} is viewed as a twisted endoscopic group for $\mathrm{SO}_{2n}^{E/F}$ (see the paragraph containing (1.2.5) in [1]).

If we assume (coh°) and (St°) but not (std-reg°), then some expected properties to be needed in our arguments are not known. We formulate them as a hypothesis so that our results become unconditional once the hypothesis is verified. (In the preceding arguments in this section, (std-reg°) allowed us to apply the results on the Ramanujan conjecture and construction of automorphic Galois representations for *regular* algebraic cuspidal automorphic representations of GL_n which are self-dual.)

Hypothesis 6.11. *Assume (disc-∞). When $\pi^\flat$ satisfies (coh°) and (St°) but not (std-reg°), the following hold true.*

(1) $\pi^\flat_{\mathfrak{q}}$ is tempered at every finite prime $\mathfrak{q}$ where $\pi^\flat_{\mathfrak{q}}$ is unramified.

(2) There exists a semisimple Galois representation $\rho_{\pi^\flat} : \Gamma_F \to \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$ satisfying (SO-i) at every $\mathfrak{q}$ where $\pi^\flat_{\mathfrak{q}}$ is unramified as well as (SO-iii), (SO-iv), and (SO-v). Moreover $\rho_{\pi^\flat}(\Gamma_F)$ contains a regular unipotent element.

The hypothesis readily implies (SO-ii) for $\rho_{\pi^\flat}$. We expect that this hypothesis is accessible via suitable orthogonal Shimura varieties. If one is only interested in constructing the GSpin_{2n}-valued representation ρ_π without proving its ℓ-adic Hodge-theoretic properties, then (SO-iii) and (SO-iv) may be dropped from the hypothesis.

Remark 6.12. Corollary 6.8 (or the above hypothesis, if (std-reg°) fails) tells us that the Zariski closure of $\rho_{\pi^\flat}(\Gamma_F)$ belongs to the list of subgroups of SO_{2n} in Proposition 5.5. In the list, the PGL_2, G_2, and PSO_{2n-1} cases can only occur when (std-reg°) is not satisfied. Since PGL_2 and G_2 are contained in PSO_{2n-1} (up to conjugation), we only need to observe this for PSO_{2n-1}. In this case, $\mu_{\mathrm{HT}}(\rho_{\pi^\flat,\mathfrak{q}}, \iota y)$ of Theorem 6.5 must factor through $i^\circ_{std} : \mathrm{SO}_{2n-1} \hookrightarrow \mathrm{SO}_{2n}$, thus cannot be regular as a cocharacter of GL_{2n}. By (SO-iii) of the theorem, $\mathrm{std}(\mu_{\mathrm{Hodge}}(\xi^\flat, y))$ is not regular either, contradicting (std-reg°).

7. Extension and restriction

In this section we study how the local conditions (St), (coh) on a cuspidal automorphic representation of $\mathrm{GSO}^{E/F}_{2n}(\mathbb{A}_F)$ (introduced in the introduction and §10 respectively) compare to conditions (St°), (Coh°) on an irreducible $\mathrm{SO}^{E/F}_{2n}(\mathbb{A}_F)$-subrepresentation (given in §6).

Lemma 7.1. *Let $\mathfrak{q}$ be a finite place of F. Let π be an irreducible admissible representation of $\mathrm{GSO}^{E/F}_{2n}(F_{\mathfrak{q}})$, and let $\pi^\flat \subset \pi$ be an irreducible $\mathrm{SO}^{E/F}_{2n}(F_{\mathfrak{q}})$-subrepresentation. Then π is a character twist of the Steinberg representation of $\mathrm{GSO}^{E/F}_{2n}(F_{\mathfrak{q}})$ if and only if $\pi^\flat$ is a character twist of the Steinberg representation of $\mathrm{SO}^{E/F}_{2n}(F_{\mathfrak{q}})$.*

Proof. Write $G = \mathrm{GSO}^{E/F}_{2n}(F_{\mathfrak{q}})$ and $G_0 = \mathrm{SO}^{E/F}_{2n}(F_{\mathfrak{q}})$. To lighten notation, when H is an algebraic group over $F_{\mathfrak{q}}$, we still write H for $H(F_{\mathfrak{q}})$ in this proof when there is no danger of confusion.

(⇒) Let $P \subset G$ be a parabolic subgroup, and write C_P for the space of smooth functions on $P\backslash G$. Fixing a Borel subgroup B and taking $P \supset B$, we may view $C_P \subset C_B$ as those functions on G/B that are P-invariant. These spaces C_P define a (non-linear) filtration on C_B, and the Steinberg representation St_G is the quotient of C_B generated by all subrepresentations C_P with $P \subset G$ proper [8, X.4.6]. There is a natural bijection between the parabolic subgroups of G with those of G_0 by $P \mapsto P_0 := P \cap G_0$. Applying *loc. cit.* now to G_0, we take $B_0 := G_0 \cap B$ and consider the spaces $C_{B_0} \supset C_{P_0}$ for $B_0 \subset P_0 \subsetneq G_0$ and St_{G_0} as before. The inclusion $G_0 \hookrightarrow G$ induces an isomorphism $P_0\backslash G_0 \xrightarrow{\sim} P\backslash G$ for each parabolic P (injectivity is clear; surjectivity can be seen by using the Bruhat decomposition, for instance). Thereby we have a G_0-equivariant filtration-preserving isomorphism $C_B \xrightarrow{\sim} C_{B_0}$ restricting to $C_P \xrightarrow{\sim} C_{P_0}$ for each P. Therefore $\mathrm{St}_G|_{G_0} \simeq \mathrm{St}_{G_0}$.

(⇐) Write $G' = \mathrm{GSpin}^{E/F}_{2n}(F_{\mathfrak{q}})$ and $G'_0 = \mathrm{Spin}^{E/F}_{2n}(F_{\mathfrak{q}})$. By abuse of notation, write $G_0/G'_0 :=$ coker(pr : $G'_0 \to G_0$) and likewise for G/G'. These are finite abelian groups. We claim that every smooth character $G_0 \to \mathbb{C}^\times$ extends to a smooth character $G \to \mathbb{C}^\times$. Since such characters factor through G_0/G'_0 and G/G', respectively (see e.g., [50, Cor. 2.6]) the claim would follow once we verify that $G_0/G'_0 \to G/G'$ is injective. So let $g_0 \in G_0$ and suppose that $g_0 = \mathrm{pr}(g)$ for $g \in G'$. Then $1 = \mathrm{sim}(g_0) = \mathrm{sim}(\mathrm{pr}(g)) = \mathcal{N}(g)^2$ by Lemma 3.1 (iii). If $\mathcal{N}(g) = -1$ then we replace g with zg using $z \in Z_{\mathrm{GSpin}}$ such that $\mathcal{N}(z) = -1$ and $\mathrm{pr}(z) = 1$ (in the coordinates of Lemma 2.5, choose $z = (1, -1)$ if n is odd, and $z = (\zeta_4, -1)$ if n is even); so we may assume that $\mathcal{N}(g) = 1$. But this means that g_0 is trivial in G_0/G_0'. The claim has been proved.

Thanks to the claim, we may assume $\pi^\flat = \mathrm{St}_{G_0}$ after twisting by a character. Since $\pi|_{G_0}$ contains St_{G_0}, we can twist π such that the central character of π is trivial. (The central character is a character χ of $F^\times/\{\pm 1\}$, so there exists a smooth character $\chi^{1/2} : F^\times \to \mathbb{C}^\times$ whose square is χ. Then we twist by $\mathrm{sim} \circ \chi^{-1/2}$.) By assumption

$$0 \neq \mathrm{Hom}_{G_0}(\mathrm{St}_{G_0}, \pi) = \mathrm{Hom}_{G_0}(\mathrm{St}_G, \pi) = \mathrm{Hom}_{Z(G)G_0}(\mathrm{St}_G, \pi),$$

where the first equality is from the implication ($\Rightarrow$), and the second from the triviality of central characters. By Frobenius reciprocity, this realizes π as a constituent of $\mathrm{Ind}_{Z(G)G_0}^G \mathrm{St}_G$, which is the direct sum of twists of St_G by characters of the finite abelian group $G/Z(G)G_0$. □

Let y be a real place of F so that $E_y/F_y = \mathbb{C}/\mathbb{R}$ if n is odd and $E_y = F_y = \mathbb{R}$ if n is even.

Lemma 7.2. *Let π be an irreducible admissible representation of $\mathrm{GSO}_{2n}^{E_y/F_y}(F_y)$ with central character ω_π. Let $\pi^\flat$ be an irreducible $\mathrm{SO}_{2n}^{E_y/F_y}(F_y)$-subrepresentation. Let ξ be an irreducible algebraic representation of $\mathrm{GSO}_{2n}^{E_y/F_y}$, and $\xi^\flat$ its pullback to $\mathrm{SO}_{2n}^{E_y/F_y}$. Then:*

(1) The representation π is essentially unitary if and only if $\pi^\flat$ is unitary.

(2) The representation π is a discrete series representation if and only if $\pi^\flat$ is a discrete series representation.

(3) Assume π is essentially unitary. Then π is ξ-cohomological if and only if $\pi^\flat$ is $\xi^\flat$-cohomological and $\omega_\pi = \omega_\xi^{-1}$, where ω_ξ is the central character of ξ on $Z(\mathrm{GSO}_{2n}^{E_y/F_y})(F_y)$

Proof. Write $G = \mathrm{GSO}_{2n}^{E_y/F_y}(F_y)$, $G_0 = \mathrm{SO}_{2n}^{E_y/F_y}(F_y)$, and $F_y^\times \subset G$ for the image of $\mathbb{G}_m(F_y)$.

(1) The "only if" direction is obvious. For the "if" direction, assume $\pi^\flat$ is unitary. We may assume $\omega_\pi = 1$. Choose a Hermitian form $h(\cdot, \cdot)$ on π, extending the G_0-equivariant one on $\pi^\flat$. Choose representatives $\{g_1, \dots, g_r\}$ for the quotient $G/F_y^\times G_0$ and define $h'(\cdot, \cdot) = \sum_{i=1}^r h(g_i \cdot, g_i \cdot)$. Then $h'(\cdot, \cdot)$ is a G-equivariant Hermitian form on π.

(2) This follows directly from the characterization of discrete series representations through square-integrability (modulo center) of their matrix coefficients.

(3) This is implied by the fact that a unitary representation is cohomological if and only if its central character and infinitesimal character coincide with those of an algebraic representation. The "only if" direction is true without the unitarity condition by [8, Thm. I.5.3.(ii)]. We explain the "if" direction in the case of interest. (This argument adapts to the general case.) For G_0, this follows from [69, Thm. 1.8], which applies to connected semisimple real Lie groups. The case of G follows from that of G_0, by applying [8, Cor. I.6.6] to $\pi \otimes \xi$ by taking H there to be $Z(G)$, and similarly to $(\pi|_{G_0}) \otimes \xi^\flat$ with $H = Z(G_0)$. □

8. Certain forms of GSO_{2n} and outer automorphisms

In this section we introduce a certain form of the split group GSO_{2n} over a totally real field F, to be used to construct Shimura varieties. We start by considering real groups. Let $\mathrm{GO}_{2n}^{\mathrm{cpt}}, \mathrm{O}_{2n}^{\mathrm{cpt}}, \mathrm{SO}_{2n}^{\mathrm{cpt}}, \mathrm{PSO}_{2n}^{\mathrm{cpt}}$, and $\mathrm{GSO}_{2n}^{\mathrm{cpt}}$ be the various versions of the orthogonal group defined by the quadratic form $x_1^2 + x_2^2 + \dots + x_{2n}^2$ on $\mathbb{R}^{2n}$. Consider the matrix $J = \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right) \in \mathrm{SO}_{2n}^{\mathrm{cpt}}(\mathbb{R})$. We define the group GSO_{2n}^J over $\mathbb{R}$ to be the inner form of $\mathrm{GSO}_{2n,\mathbb{R}}^{\mathrm{cpt}}$ obtained by conjugating the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action by J (using that J^2 is central). Namely, for all $\mathbb{R}$-algebras R we have

$$\mathrm{GSO}_{2n}^J(R) = \{g \in \mathrm{GSO}_{2n}^{\mathrm{cpt}}(\mathbb{C} \otimes_{\mathbb{R}} R) \mid J\overline{g}J^{-1} = g\}. \tag{8.1}$$

For $g \in \mathrm{GSO}_{2n}^{\mathrm{cpt}}(\mathbb{C} \otimes_{\mathbb{R}} R)$ we have $g^{\mathrm{t}} J \overline{g} = \mathrm{sim}(g) J$ if and only if $J\overline{g}J^{-1} = g$, and thus $\mathrm{GSO}_{2n}^J(\mathbb{R})$ is the group of matrices $g \in \mathrm{GL}_{2n}(\mathbb{C})$ preserving the forms

$$\begin{cases} x_1^2 + x_2^2 + \dots + x_{2n}^2 \\ -x_1\overline{x}_{n+1} + x_{n+1}\overline{x}_1 - x_2\overline{x}_{n+2} + x_{n+2}\overline{x}_2 - \dots - x_n\overline{x}_{2n} + x_{2n}\overline{x}_n \end{cases} \tag{8.2}$$

up to the scalar $\mathrm{sim}(g) \in \mathbb{R}^\times$ (the scalar is required to be the same for both forms), and such that g satisfies the condition $\det(g) = \mathrm{sim}(g)^n$.

Similarly we define the inner forms $\mathrm{GO}^J_{2n}, \mathrm{SO}^J_{2n}, \mathrm{O}^J_{2n}, \mathrm{PSO}^J_{2n}$ of $\mathrm{GO}^{\mathrm{cpt}}_{2n}, \mathrm{SO}^{\mathrm{cpt}}_{2n}, \mathrm{O}^{\mathrm{cpt}}_{2n}, \mathrm{PSO}^{\mathrm{cpt}}_{2n}$. Then $\mathrm{SO}^J_{2n}(\mathbb{R})$ is the real Lie group which is often denoted $\mathrm{SO}^*(2n)$ in the literature (e.g., [36, Sect. X.2, p.445]). Note that $\mathrm{SO}^J_{2n}(\mathbb{R})$ is not isomorphic to any of the classical groups $\mathrm{SO}(p, q)$, where $2n = p + q$ (see [44, thm 6.105(c)]). The group $\mathrm{SO}(p, q)$ with $2n = p + q$ is quasi-split if and only if $|n - p| \leq 1$, giving rise to two classes of inner twists (recall that $\mathrm{SO}(p, q)$ and $\mathrm{SO}(p', q')$ lie in the same class if and only if $p \equiv p' \mod 2$). The group SO^J_{2n}, and hence the group GSO^J_{2n}, is not quasi-split since SO^J_{2n} is not isomorphic to any group of the form $\mathrm{SO}(p, q)$.

We pin down the isomorphisms

$$C_X\colon \mathrm{GSO}^{\mathrm{cpt}}_{2n}(\mathbb{C}) \xrightarrow{\sim} \mathrm{GSO}_{2n}(\mathbb{C}), \quad g \mapsto X^{-1}gX, \quad X = \left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right),$$

$$\mathrm{GSO}^{\mathrm{cpt}}_{2n}(\mathbb{C}) \xrightarrow{\sim} \mathrm{GSO}^J_{2n}(\mathbb{C}), \quad g \mapsto (g, J^{-1}gJ) \in \mathrm{GSO}^{\mathrm{cmpt}}_{2n}(\mathbb{C})^2 = \mathrm{GSO}^{\mathrm{cmpt}}_{2n}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}). \tag{8.3}$$

Lemma 8.1. *(i) The group GSO^J_{2n} is an inner form of the split group $\mathrm{GSO}(n, n)$ over $\mathbb{R}$ if n is even, and an outer form otherwise.*

(ii) Explicitly,

$$\mathrm{GO}^J_{2n}(\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \in \mathrm{GL}_{2n}(\mathbb{C}) \;\middle|\; \begin{array}{c} A, B \in \mathrm{M}_n(\mathbb{C}) \text{ such that} \\ A^tA + \overline{B}^t\overline{B} = \lambda \cdot 1_n \text{ (where } \lambda = \mathrm{sim}(g) \in \mathbb{R}^\times) \\ A^tB = \overline{B}^t\overline{A} \end{array} \right\}.$$

(iii) The following are true:

(a) The groups $\mathrm{SO}^J_{2n}(\mathbb{R})$ and $\mathrm{O}^J_{2n}(\mathbb{R})$ are connected and equal to each other.

(b) The map $\mathrm{sim}\colon \mathrm{GO}^J_{2n}(\mathbb{R}) \to \mathbb{R}^\times$ is surjective; $\mathrm{sim}\colon \mathrm{GSO}^J_{2n}(\mathbb{R}) \to \mathbb{R}^\times$ is surjective if and only if n is even.

(c) We have $\mathrm{GSO}^J_{2n}(\mathbb{R}) = \mathrm{GO}^J_{2n}(\mathbb{R})$ if and only if n is even.

(d) If n is even (resp. odd) then $|\pi_0(\mathrm{GSO}^J_{2n}(\mathbb{R}))|$ equals 2 (resp. 1).

(iv) The mapping

$$\theta^J\colon \mathrm{GSO}^J_{2n}(\mathbb{R}) \to \mathrm{GSO}^J_{2n}(\mathbb{R}), \quad g = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \mapsto TgT^{-1} = \begin{pmatrix} \overline{A} & -\overline{B} \\ B & A \end{pmatrix}$$

for $T = i \cdot \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{GO}^J_{2n}(\mathbb{R})$ is an automorphism of GSO^J_{2n} over $\mathbb{R}$. It is outer if and only if n is odd.

(v) The group SO^J_{2n} (resp. GSO^J_{2n}) has a nontrivial outer automorphism defined over $\mathbb{R}$ that acts trivially on the center if and only if n is odd.

(vi) The groups $\mathrm{SO}^{\mathrm{cpt}}_{2n}(\mathbb{R})$ and $\mathrm{GSO}^{\mathrm{cpt}}_{2n}(\mathbb{R})$ are connected.

Proof. (i). The group SO^J_{2n} is an inner form of $\mathrm{SO}^{\mathrm{cpt}}_{2n,\mathbb{R}}$ and the compact form lies in the split inner class if and only if n is even.

(ii). Let $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GL}_{2n}(\mathbb{C})$. Write $\lambda = \mathrm{sim}(g)$. We compute

$$J\overline{g} = gJ \Leftrightarrow \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\begin{pmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{pmatrix} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \Leftrightarrow \begin{pmatrix} -\overline{C} & -\overline{D} \\ \overline{A} & \overline{B} \end{pmatrix} = \left(\begin{smallmatrix} B & -A \\ D & -C \end{smallmatrix}\right) \Leftrightarrow g = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}^t \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} = \lambda\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \Leftrightarrow \lambda\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \begin{pmatrix} A^t & -\overline{B}^t \\ B^t & \overline{A}^t \end{pmatrix}\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} = \begin{pmatrix} A^tA+\overline{B}^t\overline{B} & A^tB-\overline{B}^t\overline{A} \\ B^tA-\overline{A}^t\overline{B} & B^tB+\overline{A}^t\overline{A} \end{pmatrix}.$$

These identities are equivalent to the stated conditions on g.

$(iii.a)$ By [83, Cor. 6.3], $\det\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \geq 0$ for all $A, B \in \mathrm{M}_n(\mathbb{C})$. By Lemma 8.1(ii) any $g \in \mathrm{O}^J_{2n}(\mathbb{R})$ has $\det(g) \geq 0$ and thus $\det(g) = 1$. Thus $\mathrm{O}^J_{2n}(\mathbb{R}) = \mathrm{SO}^J_{2n}(\mathbb{R})$. By [44, prop I.1.145] the group $\mathrm{SO}^J_{2n}(\mathbb{R})$ (and hence $\mathrm{O}^J_{2n}(\mathbb{R})$) is connected.

$(iii.b)$ By restricting to the center we see that the image of the similitudes factor contains $\mathbb{R}^\times_{>0}$ in all stated cases. The element $g = \left(\begin{smallmatrix} A & 0 \\ 0 & \overline{A} \end{smallmatrix}\right)$ with $A = i1_n$ lies in $\mathrm{GO}^J_{2n}(\mathbb{R})$ and has $\mathrm{sim}(g) = -1$, proving the first part. Since $\det(g) = 1$ we have $g \in \mathrm{GSO}^J_{2n}(\mathbb{R})$ if n is even, proving the second part in that case. Assume for a contradiction that $\mathrm{GSO}^J_{2n}(\mathbb{R}) \to \mathbb{R}^\times$ is surjective when n is odd. Take some $g' \in \mathrm{GSO}^J_{2n}(\mathbb{R})$ with $\mathrm{sim}(g') = -1$. Then $\mathrm{sim}(gg') = 1$ thus $gg' \in \mathrm{O}^J_{2n}(\mathbb{R}) = \mathrm{SO}^J_{2n}(\mathbb{R})$ and hence $g \in \mathrm{GSO}^J_{2n}(\mathbb{R})$: Contradiction.

(*iii.c*) For n odd the element $g = \left(\begin{smallmatrix} A & 0 \\ 0 & \overline{A} \end{smallmatrix}\right)$ from (*iii.b*) shows $\mathrm{GSO}^J_{2n}(\mathbb{R}) \neq \mathrm{GO}^J_{2n}(\mathbb{R})$. Assume n even. If $h \in \mathrm{GO}^J_{2n}(\mathbb{R})$, choose $g \in \mathrm{GSO}^J_{2n}(\mathbb{R})$ with $\mathrm{sim}(g) = \mathrm{sim}(h)^{-1}$ using (*iii.b*). Then $hg \in \mathrm{O}^J_{2n}(\mathbb{R}) = \mathrm{SO}^J_{2n}(\mathbb{R})$, hence also $h \in \mathrm{GSO}^J_{2n}(\mathbb{R})$. Thus $\mathrm{GO}^J_{2n}(\mathbb{R}) = \mathrm{GSO}^J_{2n}(\mathbb{R})$.

(*iii.d*) Write $c := \#\pi_0(\mathrm{GSO}^J_{2n}(\mathbb{R}))$. As $\mathrm{H}^1(\mathbb{R}, \mu_2)$ has 2 elements, we have $c \leq 2$. If n is even, then sim is surjective, hence $c \geq 2$ and $c = 2$. If n is odd, we have $\mathbb{R}^\times_{>0} \times \mathrm{SO}^J_{2n}(\mathbb{R}) \xrightarrow{\sim} \mathrm{GSO}^J_{2n}(\mathbb{R})$, hence $c = 1$.

(*iv*) We have $T^tT = -1$ and $J\overline{T}J^{-1} = J$, so indeed $T \in \mathrm{GO}^J_{2n}(\mathbb{R})$. As $\mathrm{sim}(T) = -1$ and $\det(T) = i^{2n}(-1)^n = 1$, we have $\mathrm{sim}(T)^n \neq \det(T)$ if and only if n is odd.

(*v*) By the example in (*iv*) we may assume n even. Any $\mathbb{R}$-automorphism $\theta \in \mathrm{Aut}(\mathrm{GSO}^J_{2n})$ acting trivially on the center is given by $\theta\colon g \mapsto YgY^{-1}$ for some $Y \in \mathrm{GO}^J_{2n}(\mathbb{C})$. Replacing Y with tY for some $t \in \mathbb{C}^\times$ we may assume that $\mathrm{sim}(Y) = 1$ (as θ does not change, it is still defined over $\mathbb{R}$). Write $\sigma\colon \mathrm{GO}^{\mathrm{cpt}}_{2n}(\mathbb{C}) \to \mathrm{GO}^{\mathrm{cpt}}_{2n}(\mathbb{C})$ for the automorphism $g \mapsto J\overline{g}J^{-1}$, so that $\mathrm{GO}^J_{2n}(\mathbb{R}) = \mathrm{GO}^{\mathrm{cpt}}_{2n}(\mathbb{C})^{\sigma=\mathrm{id}}$. As θ is defined over $\mathbb{R}$,

$$\theta(\sigma g) = \sigma\theta(g) \qquad \forall g \in \mathrm{GO}^J_{2n}(\mathbb{C}),$$

and therefore $YJ \cdot \overline{g} \cdot J^{-1}Y^{-1} = J\overline{Y} \cdot \overline{g} \cdot \overline{Y}^{-1}J^{-1}$, so $\overline{Y}^{-1}J^{-1}YJ \cdot \overline{g} = \overline{g} \cdot \overline{Y}^{-1}J^{-1}YJ$. Thus

$$\lambda \cdot YJ = J\overline{Y} \qquad \text{for some } \lambda \in Z(\mathrm{GSO}^J_{2n}(\mathbb{C})) = \mathbb{C}^\times.$$

We have $Y^tY = 1$, so we compute as follows using $J^tJ = 1$:

$$1 = \overline{Y}^t\overline{Y} = (\lambda J^{-1}YJ)^t(\lambda J^{-1}YJ) = \lambda^2(J^tY^tYJ) = \lambda^2.$$

Therefore $\lambda \in \{\pm 1\}$. If $\lambda = 1$ then $Y \in \mathrm{O}^J_{2n}(\mathbb{R}) = \mathrm{SO}^J_{2n}(\mathbb{R})$, and θ is inner. If $\lambda = -1$ then $\sigma(Y) = -Y$ so $iY \in \mathrm{GO}^J_{2n}(\mathbb{R}) = \mathrm{GSO}^J_{2n}(\mathbb{R})$ (n is even). Thus $\theta = (g \mapsto (iy)g(iY)^{-1})$ is inner.

(*vi*) It is standard that $\mathrm{SO}^{\mathrm{cpt}}_{2n}(\mathbb{R})$ is connected. Let us show that $\mathrm{GSO}^{\mathrm{cpt}}_{2n}(\mathbb{R})$ is connected from this. The multiplication map $\mathrm{SO}^{\mathrm{cpt}}_{2n}(\mathbb{R}) \times \mathbb{R}^\times \to \mathrm{GSO}^{\mathrm{cpt}}_{2n}(\mathbb{R})$ has connected image since $\mathrm{SO}^{\mathrm{cpt}}_{2n}(\mathbb{R})$ meets both connected components of $\mathbb{R}^\times$. So we will be done if we check the surjectivity. This is equivalent to the injectivity of $H^1(\mathbb{R}, \{\pm 1\}) \to H^1(\mathbb{R}, \mathrm{SO}^{\mathrm{cpt}}_{2n} \times \mathrm{GL}_1)$, which follows from the fact that there is no $g \in \mathrm{SO}^{\mathrm{cpt}}_{2n}(\mathbb{C})$ with $g^{-1}\overline{g} = -1$. (Via $h = \sqrt{-1}g$, the latter is equivalent to non-existence of $h \in \mathrm{GL}_{2n}(\mathbb{R})$ with $h^th = -1$, which is clear.) $\square$

Now we turn to the global setup. Let n and E/F be as in §6 and impose condition (disc-∞) from now on. In analogy with the SO_{2n}-case, we introduce a quasi-split form G^* of GSO_{2n} over F. If n is even, we have $E = F$ and take the split form $G^* := \mathrm{GSO}_{2n,F}$ (or simply written as GSO_{2n}). If n is odd then E/F is a totally imaginary quadratic extension. In this case, let G^* be the quasi-split form $\mathrm{GSO}^{E/F}_{2n,F}$ of $\mathrm{GSO}_{2n,F}$ (up to F-automorphism) given by the 1-cocycle $\mathrm{Gal}(E/F) \to \mathrm{Aut}(\mathrm{GSO}_{2n,E})$ sending the nontrivial element to $\theta^\circ = \mathrm{Int}(\vartheta^\circ)$. Since $\vartheta^\circ \in \mathrm{O}_{2n}(E)$, this cocycle comes from the $\mathrm{Aut}(\mathrm{SO}_{2n,E})$-valued cocycle determining $\mathrm{SO}^{E/F}_{2n}$ as an outer form of SO_{2n}, thus we have $\mathrm{SO}^{E/F}_{2n} \hookrightarrow \mathrm{GSO}^{E/F}_{2n}$. Concretely, in analogy with (6.1),

(8.4)
$$\mathrm{GSO}^{E/F}_{2n}(R) = \left\{ g \in \mathrm{GL}_{2n}(E \otimes_F R) \;\middle|\; \begin{array}{c} \text{there exists } \lambda \in R^\times \text{ such that} \\ c(g) = \vartheta^\circ g \vartheta^\circ,\ g^t\left(\begin{smallmatrix} 0 & 1_n \\ 1_n & 0 \end{smallmatrix}\right)g = \lambda\left(\begin{smallmatrix} 0 & 1_n \\ 1_n & 0 \end{smallmatrix}\right),\ \det(g) = \lambda^n \end{array} \right\},$$

and $\mathrm{GO}^{E/F}_{2n}(R)$ is defined by removing the condition $\det(g) = \lambda^n$.

We write $G^* = \mathrm{GSO}^{E/F}_{2n}$ for both parities of n, understanding that $E = F$ if n is even, for a streamlined exposition. In both cases, we have an exact sequence

$$1 \to \mathrm{SO}^{E/F}_{2n} \to \mathrm{GSO}^{E/F}_{2n} \to \mathbb{G}_m \to 1, \tag{8.5}$$

where the similitude map $\mathrm{GSO}^{E/F}_{2n} \to \mathbb{G}_m$ is the usual one if $E = F$, and $g \mapsto \lambda$ in (8.4) if $E \neq F$. Note that $\widehat{G^*_{\mathrm{ad}}}$ is isomorphic to $\mathrm{Spin}_{2n}(\mathbb{C})$, on which Γ acts trivially (resp. non-trivially via $\mathrm{Gal}(E/F)$ as $\{1, \theta\}$) if n is even (resp. odd).

Write $(\cdot)^D$ for the Pontryagin dual of a locally compact abelian group. Let v be a place of F. By [46, Thm 1.2] we have a map[11]

$$\alpha_v\colon \mathrm{H}^1(F_v, G^*_{\mathrm{ad}}) \to \pi_0(Z(\widehat{G^*_{\mathrm{ad}}})^{\Gamma_v})^D,$$

which is an isomorphism if v is a finite place (but not if v is infinite).

Lemma 8.2. *We have*

$$Z(\widehat{G^*_{\mathrm{ad}}})^{\Gamma_v} \simeq \begin{cases} (\mu_2)^2, & n \text{ is even}, \\ \mu_2, & n \text{ is odd, } v \text{ is non-split in } E/F, \\ \mu_4, & n \text{ is odd, } v \text{ is split in } E/F. \end{cases}$$

Proof. This follows from Lemma 3.7. □

By [46, Prop. 2.6] and the Hasse principle from [68, Thm. 6.22] we have an exact sequence of pointed sets

$$1 \to \mathrm{H}^1(F, G^*_{\mathrm{ad}}) \to \bigoplus_v \mathrm{H}^1(F_v, G^*_{\mathrm{ad}}) \xrightarrow{\Sigma_v \alpha_v} \pi_0(Z(\widehat{G^*_{\mathrm{ad}}})^{\Gamma})^D \to 1. \tag{8.6}$$

Since $Z(\widehat{G^*_{\mathrm{ad}}})$ is finite, we may forget $\pi_0(\cdot)$ in (8.6) and the proof of the lemma below. From now until the end of §9, we fix a finite place $\mathfrak{q}_{\mathrm{St}}$ and an infinite place y_∞ of F.

Lemma 8.3. *Let $\mathfrak{q}_{\mathrm{St}}$ (resp. y_∞) be a fixed finite (resp. infinite) place of F. There exists an inner twist G of G^* such that for all F-places $v \neq \mathfrak{q}_{\mathrm{St}}$, we have*

$$G_v \simeq \begin{cases} \mathrm{GSO}^J_{2n,F_v} & v = y_\infty \\ \mathrm{GSO}^{\mathrm{cpt}}_{2n,F_v} & v \in \mathcal{V}_\infty \backslash \{y_\infty\} \\ G^*_{F_v} & v \notin \mathcal{V}_\infty \cup \{\mathfrak{q}_{\mathrm{St}}\}. \end{cases} \tag{8.7}$$

This inner twist G is unique up to isomorphism if either n is even or $\mathfrak{q}_{\mathrm{St}}$ is non-split in E/F; otherwise there are two choices for G. (Recall the notion of inner twist from §1.)

Proof. Put

$$a_{\mathfrak{q}_{\mathrm{St}}} := -\alpha_{y_\infty}(\mathrm{GSO}^J_{2n,F_{y_\infty}}) - \sum_{v \neq y_\infty} \alpha_v(\mathrm{GSO}^{\mathrm{cpt}}_{2n,F_v}) \in (Z(\widehat{G^*_{\mathrm{ad}}})^{\Gamma})^D. \tag{8.8}$$

By duality, the inclusion $Z(\widehat{G^*_{\mathrm{ad}}})^{\Gamma} \subset Z(\widehat{G^*_{\mathrm{ad}}})^{\Gamma_v}$ induces a surjection $(Z(\widehat{G^*_{\mathrm{ad}}})^{\Gamma_v})^D \twoheadrightarrow (Z(\widehat{G^*_{\mathrm{ad}}})^{\Gamma})^D$. Hence we can choose some invariant $\widetilde{a_{\mathfrak{q}_{\mathrm{St}}}} \in (Z(\widehat{G^*_{\mathrm{ad}}})^{\Gamma_v})^D$ mapping to the expression on the right hand side of (8.8). Let $G_{\mathfrak{q}_{\mathrm{St}}}$ be the inner twist of G^* over $F_{\mathfrak{q}_{\mathrm{St}}}$ corresponding to $\widetilde{a_{\mathfrak{q}_{\mathrm{St}}}}$. Then, by (8.6) the collection of local inner twists $\{G_v\}_{\text{places } v}$ comes from a global inner twist G/F, unique up to isomorphism. Conversely, any G as in the lemma satisfies $\alpha_{\mathfrak{q}_{\mathrm{St}}}(G) = a_{\mathfrak{q}_{\mathrm{St}}}$ by (8.6). Therefore the number of choices for G equals the number of choices for $\widetilde{a_{\mathfrak{q}_{\mathrm{St}}}}$, which can be computed using Lemma 8.2. □

Remark 8.4. The group $G_{\mathfrak{q}_{\mathrm{St}}}$ in the lemma is never quasi-split, regardless of the parity of $[F : \mathbb{Q}]$. It is always a unitary group for a Hermitian form over a quaternion algebra. This corresponds to the "$d = 2$ case" in [1, §9.1]. In this case the rank of $G_{\mathfrak{q}_{\mathrm{St}}}$ is roughly $n/2$ (see [1] for precise information).

9. Shimura varieties of type D corresponding to spin$^\pm$

We continue in the same global setup, with an inner form G of a quasi-split form G^* of GSO_{2n} over a totally real field F, depending on the fixed places $\mathfrak{q}_{\mathrm{St}}$ and y_∞ of F. We are going to construct Shimura data associated with $\mathrm{Res}_{F/\mathbb{Q}} G$ by giving an $\mathbb{R}$-morphism $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to$

[11] This map has been computed explicitly by Arthur [1, Section 9.1] for all inner forms of classical groups of type B, C, and D.

$(\mathrm{Res}_{F/\mathbb{Q}}G) \otimes_{\mathbb{Q}} \mathbb{R}$. Our running assumption (disc-∞) is clearly a necessary condition for the existence of such Shimura data. We define

$$h_{(-1)^n} \colon \mathbb{S} \to \mathrm{GSO}^J_{2n}, \quad x+yi \mapsto \begin{pmatrix} x1_n & y1_n \\ -y1_n & x1_n \end{pmatrix}$$
$$h_{(-1)^{n+1}} \colon \mathbb{S} \to \mathrm{GSO}^J_{2n}, \quad x+yi \mapsto \begin{pmatrix} x1_n & \mathrm{diag}(y1_{n-1}, -y) \\ \mathrm{diag}(-y1_{n-1}, y) & x1_n \end{pmatrix}.$$

We will often omit 1_n in the cases similar to the above if a matrix is clearly $2n \times 2n$ in the context.

By slight abuse of notation, we write Ad for either the natural map from $\mathrm{GSO}^J_{2n} \to \mathrm{GSO}^J_{2n,\mathrm{ad}}$ or the adjoint representation of GSO^J_{2n} on $\mathrm{Lie}\,\mathrm{GSO}^J_{2n}$.

Lemma 9.1. *Let $\varepsilon \in \{+,-\}$ and put $K_\varepsilon := \mathrm{Cent}_{\mathrm{GSO}^J_{2n}(\mathbb{R})}(h_\varepsilon)$. The following hold.*

- *(i) In the representation of $\mathbb{C}^\times$ on $\mathrm{Lie}\,\mathrm{GSO}^J_{2n}(\mathbb{C})$ via $\mathrm{Ad} \circ h_\varepsilon$, only the characters $z \mapsto z^{-1}\overline{z}$, $z \mapsto 1$, and $z \mapsto z\overline{z}^{-1}$ appear.*
- *(ii) The involution on $\mathrm{GSO}^J_{2n,\mathrm{ad}}$ given by $\mathrm{Ad}\ h_\varepsilon(i)$ is a Cartan involution.*
- *(iii) K_+ and K_- are $\mathrm{GSO}^J_{2n}(\mathbb{R})$-conjugate.*

Proof. For (i) and (ii), we only treat the case of $\varepsilon = (-1)^n$ as the argument for $-\varepsilon$ is the same. Let $z = x + yi \in \mathbb{C}^\times$ and consider the left-multiplication action of the matrix $h_\varepsilon(x+iy) = \left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)$ on $\mathrm{M}_{2n}(\mathbb{C})$. The matrix $\left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)$ is conjugate to $\left(\begin{smallmatrix} x+yi & \\ & x-yi \end{smallmatrix}\right)$ via $\left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right)$. Hence only the characters $z\overline{z}^{-1}$, $\overline{z}z^{-1}$ and 1 appear in the representation of $\mathbb{S}$ on $\mathrm{M}_{2n}(\mathbb{C})$ via conjugation by $h_\varepsilon(x+iy)$. Since $\mathrm{Lie}\,\mathrm{GSO}^J_{2n}(\mathbb{R})$ is contained in $\mathrm{M}_{2n}(\mathbb{C})$ via the standard representation, (i) is true for $h_\varepsilon(z)$. Since $J^{-1} = h_\varepsilon(i)$, the inner form of GSO^J_{2n} defined by $h_\varepsilon(i)$ is the compact-modulo-center form $\mathrm{GSO}^{\mathrm{cpt}}_{2n,\mathbb{R}}$, so part (ii) follows.

Let us prove (iii). Write $\overline{h}_\varepsilon := \mathrm{Ad} \circ h_\varepsilon$. Clearly $\mathrm{Ad}(K_\varepsilon) \subset \mathrm{Cent}_{\mathrm{GSO}^J_{2n,\mathrm{ad}}(\mathbb{R})}(\overline{h}_\varepsilon)$. The Lie algebra $\mathrm{Lie}(K_\varepsilon)$ (resp. the Lie algebra of $\mathrm{Cent}_{\mathrm{GSO}^J_{2n,\mathrm{ad}}(\mathbb{R})}(\overline{h}_\varepsilon)$) is the $(0,0)$ part of $\mathrm{Lie}(\mathrm{GSO}^J_{2n})$ (resp. $\mathrm{Lie}(\mathrm{GSO}^J_{2n,\mathrm{ad}})$) via h_ε, in the sense of [20]. In particular

$$\mathrm{ad}\colon \mathrm{Lie}(K_\varepsilon) \to \mathrm{Lie}\left(\mathrm{Cent}_{\mathrm{GSO}^J_{2n,\mathrm{ad}}(\mathbb{R})}(\overline{h}_\varepsilon)\right)$$

is surjective. Therefore $\mathrm{Ad}(K_\varepsilon) \supset \mathrm{Cent}_{\mathrm{GSO}^J_{2n,\mathrm{ad}}(\mathbb{R})}(\overline{h}_\varepsilon)^0$. Since $\mathrm{Cent}_{\mathrm{GSO}^J_{2n,\mathrm{ad}}(\mathbb{R})}(\overline{h}_\varepsilon)$ is connected by [20, proof of Prop. 1.2.7], we have $\mathrm{Ad}(K_\varepsilon) = \mathrm{Cent}_{\mathrm{GSO}^J_{2n,\mathrm{ad}}(\mathbb{R})}(\overline{h}_\varepsilon)$. The latter is the identity component of a maximal compact subgroup of $\mathrm{GSO}^J_{2n,\mathrm{ad}}(\mathbb{R})$ by *loc. cit.* so $\mathrm{Ad}(K_-)$ and $\mathrm{Ad}(K_+)$ are conjugate in $\mathrm{GSO}^J_{2n,\mathrm{ad}}(\mathbb{R})$. Since $K_\varepsilon = \mathrm{Ad}^{-1}(\mathrm{Ad}(K_\varepsilon))$ and since $\mathrm{Ad}\colon \mathrm{GSO}^J_{2n} \to \mathrm{GSO}^J_{2n,\mathrm{ad}}$ is surjective on real points by Hilbert 90, we lift a conjugating element to see that K_+ and K_- are conjugate in $\mathrm{GSO}^J_{2n}(\mathbb{R})$. □

Recall the cocharacters μ_+, μ_- from (2.9), which are outer conjugate as $\mu_+ = \vartheta^\circ \mu_- (\vartheta^\circ)^{-1}$ (but not inner, cf. (2.8)).

Lemma 9.2. *Let $\varepsilon \in \{+,-\}$.*

- *(i) Consider the inclusion of $\mathbb{C}^\times$ in $(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times = (\mathbb{C}^\times)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$ indexed by the identity $\mathrm{id}_{\mathbb{C}/\mathbb{R}} \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Then $C_X h_{\varepsilon,\mathbb{C}}|_{\mathbb{C}^\times} = \mu_\varepsilon$.*
- *(ii) The complex conjugate morphism $z \mapsto h_\varepsilon(\overline{z})$ is $\mathrm{GSO}^J_{2n}(\mathbb{R})$-conjugate to $h_{(-1)^n\varepsilon}$.*

Proof. In the proof, put $\varepsilon = (-1)^n$. *(i)*. Recall C_X from (8.3), which induces $\mathrm{GSO}^J_{2n}(\mathbb{R}) \hookrightarrow \mathrm{GSO}^{\mathrm{cpt}}_{2n}(\mathbb{C}) \overset{C_X}{\simeq} \mathrm{GSO}_{2n}(\mathbb{C})$. The morphism $C_X h_\varepsilon$ equals $x+yi \mapsto \left(\begin{smallmatrix} x+yi & 0 \\ 0 & x-yi \end{smallmatrix}\right)$. The holomorphic part of this morphism is $z \mapsto \left(\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right)$, which is μ_ε. Then $h_\varepsilon = \vartheta^\circ h_{-\varepsilon} \vartheta^\circ$, where ϑ° is as in (2.4). Write $\vartheta^c = \left(\begin{smallmatrix} -1_{2n-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Note that $C_X(\vartheta^c) = \vartheta^\circ$, so ϑ^c-conjugation becomes ϑ°-conjugation under C_X. As ϑ° swaps μ_ε and $\mu_{-\varepsilon}$, we obtain $C_X h_{-\varepsilon}|_{\mathbb{C}^\times} = \mu_{-\varepsilon}$.

(ii). Write $z = x + yi \in \mathbb{C}$. We compute

$$h_\varepsilon(\overline{z}) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = T h_\varepsilon(z) T^{-1},$$

where $T = i\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{GO}^J_{2n}(\mathbb{R})$. By Lemma 8.1(iv) $T \in \mathrm{GSO}^J_{2n}(\mathbb{R})$ if n is even, which proves (ii) in that case. For n odd the above identity shows $z \mapsto h_\varepsilon(\overline{z})$ and h_ε are conjugate under an outer automorphism. Thus, by (i), the cocharacter attached to $h_\varepsilon(\overline{z})$ is conjugate to $\mu_{-\varepsilon}$.

By Lemma 9.1 the conditions of [20, Prop. 1.2.2] on $(\mathrm{PSO}^J_{2n}, h_\pm(z))$ and $(\mathrm{PSO}^J_{2n}, h_\pm(\overline{z}))$ are satisfied. By this proposition there exists a $g \in \mathrm{PSO}^J_{2n}(\mathbb{C})$ that conjugates $(\mathrm{PSO}^J_{2n,\mathbb{R}}, h_\varepsilon(\overline{z}))$ to $(\mathrm{PSO}^J_{2n,\mathbb{R}}, h_{-\varepsilon}(z))$. This implies that for all $x \in \mathrm{PSO}^J_{2n}(\mathbb{C})$ we have $g\overline{x}g^{-1} = \overline{gxg^{-1}}$. Thus also $\overline{g}^{-1}g\overline{x} = \overline{x}\overline{g}^{-1}g$ and $\overline{g}^{-1}g \in Z(\mathrm{PSO}^J_{2n}(\mathbb{C})) = \{1\}$, and thus $\overline{g} = g$, which means $g \in \mathrm{PSO}^J_{2n}(\mathbb{R})$. By Hilbert 90 we may lift g to an element $\widetilde{g} \in \mathrm{GSO}^J_{2n}(\mathbb{R})$. Then the map $\mathbb{C}^\times \ni z \mapsto \chi(z) := h_\varepsilon^{-1}(\overline{z})\widetilde{g}h_{-\varepsilon}(z)\widetilde{g}^{-1}$ is a continuous homomorphism to $Z(\mathrm{GSO}^J_{2n}(\mathbb{R})) = \mathbb{R}^\times$. It suffices to show that χ is trivial. The subgroup $\chi(U(1)) \subset \mathbb{R}^\times$ is connected and compact, hence trivial. Since h_ε and $h_{-\varepsilon}$ agree and are central on $\mathbb{R}^\times \subset \mathbb{C}^\times$, we have $\chi(\mathbb{R}^\times) = \{1\}$ as well. So χ is trivial as desired. The proof for $h_{-\varepsilon}$ is similar. $\square$

Let G be as in Lemma 8.3. Let X^ε be the $(\mathrm{Res}_{F/\mathbb{Q}}G)(\mathbb{R})$-conjugacy class of the morphism

$$h^\varepsilon \colon \mathbb{S} \to (\mathrm{Res}_{F/\mathbb{Q}}G)_\mathbb{R}, \quad z \mapsto (h_\varepsilon(z), 1, \dots, 1) \in \prod_{y \in \mathcal{V}_\infty} G_{F_y}, \tag{9.1}$$

where the non-trivial component corresponds to the place y_∞. Then $\mu^\varepsilon = (\mu_\varepsilon, 1, \dots, 1) \in X_*((\mathrm{Res}_{F/\mathbb{Q}}G)_\mathbb{C}) = X_*(\mathrm{GSO}_{2n,\mathbb{C}})^{\mathcal{V}_\infty}$ is the cocharacter attached to h_ε, in the same way as in Lemma 9.2 (i). The reflex field of $(\mathrm{Res}_{F/\mathbb{Q}}G, X^\varepsilon)$ means the field of definition for the conjugacy class of μ^ε, as a subfield of $\mathbb{C}$.

Lemma 9.3. *Let $\varepsilon \in \{\pm 1\}$. Then*

(i) The pair $(\mathrm{Res}_{F/\mathbb{Q}}G, X^\varepsilon)$ is a Shimura datum of abelian type.
(ii) The Shimura data $(\mathrm{Res}_{F/\mathbb{Q}}G, X^+)$ and $(\mathrm{Res}_{F/\mathbb{Q}}G, X^-)$ are isomorphic only if n is odd.
(iii) If $F \neq \mathbb{Q}$, the Shimura varieties attached to $(\mathrm{Res}_{F/\mathbb{Q}}G, X^\varepsilon)$ are projective.
(iv) The reflex field of the datum $(\mathrm{Res}_{F/\mathbb{Q}}G, X^\varepsilon)$ is equal to E, equipped with an embedding $x_\infty : E \hookrightarrow \mathbb{C}$ extending $y_\infty : F \hookrightarrow \mathbb{C}$.

Remark 9.4. About (i): When $F = \mathbb{Q}$, the Shimura datum (G, X^ε) can be shown to be of Hodge type but we do not need this fact. About (ii): If n is odd and $\mathfrak{q}_{\mathrm{St}}$ is inert in E/F, then one can show that $(\mathrm{Res}_{F/\mathbb{Q}}G, X^+) \simeq (\mathrm{Res}_{F/\mathbb{Q}}G, X^-)$.

Proof. (i) Clearly, $(\mathrm{Res}_{F/\mathbb{Q}}G)_{\mathrm{ad}}$ has no compact factor defined over $\mathbb{Q}$, which is one of Deligne's axioms of Shimura datum [20, 2.1]. The remaining two axioms follow from Lemma 9.1, and hence $(\mathrm{Res}_{F/\mathbb{Q}}G, X^\varepsilon)$ is a Shimura datum. In the terminology of *loc. cit.*, $(\mathrm{Res}_{F/\mathbb{Q}}G, X^\varepsilon)$ is of type $D^\mathbb{H}$. By [20, Prop. 2.3.10], a datum (G', X') of type $D^\mathbb{H}$ is of abelian type if the derived group of $G'_\mathbb{C}$ is (a product of) $\mathrm{SO}_{2n,\mathbb{C}}$. (Not all Shimura data of type $D^\mathbb{H}$ are of abelian type.)

(ii) If n is even then every automorphism of $(G_{F_{y_\infty}})_{\mathrm{ad}}$ (isomorphic to $\mathrm{GSO}^J_{2n,\mathrm{ad}}$) is inner by Lemma 8.1 (v). On the other hand, it follows from Lemma 9.2 (i) that no inner automorphism of $\mathrm{GSO}^J_{2n,\mathrm{ad}}$ takes $\mathrm{Ad}\circ h_+$ to $\mathrm{Ad}\circ h_-$, since $\mathrm{Ad}\circ\mu_+$ to $\mathrm{Ad}\circ\mu_-$ are not conjugate by $\mathrm{GSO}_{2n,\mathrm{ad}}(\mathbb{C})$. Hence no automorphism of $(\mathrm{Res}_{F/\mathbb{Q}}G)_\mathbb{R}$ (thus also of $\mathrm{Res}_{F/\mathbb{Q}}G$) carries X^+ onto X^-.

(iii) If $F \neq \mathbb{Q}$ there exists some real place $y'_\infty \in \mathcal{V}_\infty$ of F different from y_∞. Since $G_{y'_\infty}$ is compact modulo center, $\mathrm{Res}_{F/\mathbb{Q}}G$ is anisotropic modulo center over $\mathbb{Q}$. Hence the associated Shimura varieties are projective by Bailey-Borel [2, Thm. 1].

(iv) Assume that n is odd (thus $[E : F] = 2$). Suppose that $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ stabilizes the conjugacy class of μ^ε. Since $\sigma(\mu^\varepsilon) \sim \mu^\varepsilon$ we have $\sigma(y_\infty) = y_\infty$, so $\sigma \in \mathrm{Aut}(\mathbb{C}/F)$ with respect to $y_\infty : F \hookrightarrow \mathbb{C}$. If σ has non-trivial image in $\mathrm{Gal}(E/F)$, then Lemma 9.2 (ii) tells us that $\sigma(\mu^\varepsilon) \sim (\mu_{-\varepsilon}, 1, \dots, 1)$, which is not $\mathrm{GSO}_{2n}(\mathbb{C})$-conjugate to μ^ε. Thus σ is trivial on E (embedded in $\mathbb{C}$ extending y_∞). Conversely, if $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$, then $\sigma(\mu^\varepsilon) = \mu^\varepsilon$. Hence the reflex field is E. When n is even (thus $E = F$), the preceding argument shows that the reflex field is F. $\square$

We introduce the following notation. Let $\varepsilon \in \{+, -\}$.

- Taking an algebraic closure of E in $\mathbb{C}$ via $x_\infty : E \hookrightarrow \mathbb{C}$, we fix $\overline{F} = \overline{E} \hookrightarrow \mathbb{C}$.
- We fix an isomorphism $G \otimes_F \mathbb{A}_F^{\infty,\mathfrak{q}_{\mathrm{St}}} \simeq G^* \otimes_F \mathbb{A}_F^{\infty,\mathfrak{q}_{\mathrm{St}}}$.
- Z is the center of G.
- $\xi = \otimes_{y|\infty} \xi_y$ is an irreducible algebraic representation of $(\mathrm{Res}_{F/\mathbb{Q}} G) \times_{\mathbb{Q}} \mathbb{C} = \prod_{y|\infty} G \times_{F,y} \mathbb{C}$.
- $\Pi_\xi^{G(F_\infty)}$ is the set of isomorphism classes of (irreducible) discrete series representations of $G(F_\infty)$ which have the same infinitesimal and central characters as $\xi^\vee$.
- K_∞^ε is the centralizer of h^ε in $(\mathrm{Res}_{F/\mathbb{Q}} G)(\mathbb{R}) = G(F_\infty)$.
- For irreducible admissible representations τ_∞ of $G(F_\infty)$, put

$$\mathrm{ep}^\varepsilon(\tau_\infty \otimes \xi) := \sum_{i=1}^{n(n-1)} (-1)^i \dim \mathrm{H}^i(\mathrm{Lie}\, G(F_\infty), K_\infty^\varepsilon; \tau_\infty \otimes \xi) \tag{9.2}$$

Let $\pi^\natural$ be a cuspidal automorphic representation of $G(\mathbb{A}_F)$ such that

- $\pi^\natural_{\mathfrak{q}_{\mathrm{St}}}$ is a Steinberg representation up to a character twist,
- $\pi^\natural$ is ξ-cohomological.

The latter condition that implies via (the proof of) [49, Lem. 7.1] the following condition:

(cent) There exists an integer $w \in \mathbb{Z}$, called the *central weight* of ξ, such that for every infinite F-place $y|\infty$ the central character of ξ_y is of the form $x \mapsto x^w$.

We also make the folowing assumption:

(temp) $\pi^\natural_{\mathfrak{q}}$ is essentially tempered at every finite F-place $\mathfrak{q}$ where $\pi_{\mathfrak{q}}$ is unramified.

This may seem strong, but (temp) will be satisfied in practice; see the paragraph above (10.4). Let $A(\pi^\natural)$ be the set of (isomorphism classes of) cuspidal automorphic representations τ of $G(\mathbb{A}_F)$ such that

(i) $\tau_{\mathfrak{q}_{\mathrm{St}}} \simeq \pi^\natural_{\mathfrak{q}_{\mathrm{St}}} \otimes \delta$ for an unramified character δ of the group $G(F_{\mathfrak{q}_{\mathrm{St}}})$,
(ii) $\tau^{\infty,\mathfrak{q}_{\mathrm{St}}} \simeq \pi^{\natural,\infty,\mathfrak{q}_{\mathrm{St}}}$, and
(iii) τ_∞ is ξ-cohomological.

By (temp), $\tau_{\mathfrak{q}}$ is essentially tempered at every $\mathfrak{q}$ where $\pi_{\mathfrak{q}}$ is unramified. Define

$$a^\varepsilon(\pi^\natural) := (-1)^{n(n-1)/2} N_\infty^{-1} \sum_{\tau \in A(\pi^\natural)} m(\tau) \cdot \mathrm{ep}^\varepsilon(\tau_\infty \otimes \xi) \in \mathbb{Q}, \tag{9.3}$$

where $m(\tau)$ is the multiplicity of τ in the discrete automorphic spectrum of G, and

$$N_\infty := |\Pi_\xi^{G(F_\infty)}| \cdot |\pi_0(G(F_\infty)/Z(F_\infty))| = \begin{cases} 2^{n-1} \cdot 2, & \text{if } n \text{ is even,} \\ 2^{n-1}, & \text{if } n \text{ is odd.} \end{cases} \tag{9.4}$$

Here $|\pi_0(G(F_\infty)/Z(F_\infty))| \in \{1, 2\}$ depending on the parity of n from Lemma 8.1 (iii), (vi).

Lemma 9.5. *The groups K_∞^+ and K_∞^- are $G(F_\infty)$-conjugate. In particular $a^-(\pi^\natural) = a^+(\pi^\natural)$.*

Henceforth we will write $a(\pi^\natural) \in \mathbb{Q}$ for the common value of $a^\varepsilon(\pi^\natural)$.

Proof. The y_∞-components of K_∞^ε is K_ε, which are conjugate to each other by Lemma 9.1. The components of K_∞^ε at the other real places y equal $G(F_y) \simeq \mathrm{GSO}_{2n}^{\mathrm{cpt}}(\mathbb{R})$, which is connected. Therefore K_∞^+ and K_∞^- are connected and $G(F_\infty)$-conjugate. It then follows that $\mathrm{ep}^+(\tau_\infty \otimes \xi) = \mathrm{ep}^-(\tau_\infty \otimes \xi)$ for all τ_∞. Thus $a^+(\pi^\natural) = a^-(\pi^\natural)$. $\square$

Since condition (cent) holds, we can attach to ξ a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_{\iota\xi}$ on $\mathrm{Sh}_K^\varepsilon$ as in [49, below Lem. 7.1] and [15, Sect. 2.1, 2.1.4]. We have a canonical model $\mathrm{Sh}_K^\varepsilon$ over E for each neat open compact subgroup $K \subset G(\mathbb{A}_F^\infty)$ (see [66, §0] for the definition of neat subgroups) and a distinguished embedding $E \subset \overline{F}$ (compatible with $E \subset \mathbb{C}$ and the fixed embedding $\overline{F} \hookrightarrow \mathbb{C}$). We take the limit over K of the étale cohomology of with compact support

$$\mathrm{H}^i_{\mathrm{c}}(\mathrm{Sh}^\varepsilon, \mathcal{L}_{\iota\xi}) := \varinjlim_K \mathrm{H}^i_{\mathrm{c}}(\mathrm{Sh}^\varepsilon_K \times_E \overline{F}, \mathcal{L}_{\iota\xi}),$$

equipped with commuting linear actions of $\Gamma_E = \mathrm{Gal}(\overline{F}/E)$ and $G(\mathbb{A}^\infty_F)$. The two groups act continuously and admissibly, respectively. Write $\mathrm{H}^i_{\mathrm{c}}(\mathrm{Sh}^\varepsilon_K, \mathcal{L}_{\iota\xi})^{\mathrm{ss}}$ for the semisimplification as a $\Gamma_E \times G(\mathbb{A}^\infty_F)$-module. (No semisimplification is necessary for the $G(\mathbb{A}^\infty_F)$-action if $F \neq \mathbb{Q}$, in which case $\mathrm{Sh}^\varepsilon_K$ is projective. This can be seen from the semisimplicity of the discrete L^2-automorphic spectrum via Matsushima's formula.)

We construct Galois representations of Γ_E by taking the $\iota\tau^\infty$-isotypic part in the cohomology as follows. We consider $\tau_1, \tau_2 \in A(\pi^\natural)$ are equivalent and write $\tau_1 \sim \tau_2$ if $\tau_1^\infty \simeq \tau_2^\infty$. Let $A(\pi^\natural)/{\sim}$ denote the set of (representatives for) equivalence classes. Let $\tau \in A(\pi^\natural)$. Define

$$\mathrm{H}^i_c(\mathrm{Sh}^\varepsilon, \mathcal{L}_\xi)[\iota\tau^\infty] := \mathrm{Hom}_{G(\mathbb{A}^\infty_F)}\big(\iota\tau^\infty, \mathrm{H}^i_{\mathrm{c}}(\mathrm{Sh}^\varepsilon, \mathcal{L}_{\iota\xi})^{\mathrm{ss}}\big), \tag{9.5}$$

$$\rho^{\mathrm{Sh},\varepsilon}_{\pi^\natural} := (-1)^{n(n-1)/2} \sum_{\tau \in A(\pi^\natural)/\sim} \sum_{i=0}^{n(n-1)} (-1)^i \mathrm{H}^i_c(\mathrm{Sh}^\varepsilon, \mathcal{L}_\xi)[\iota\tau^\infty]. \tag{9.6}$$

A priori $\rho^{\mathrm{Sh},\varepsilon}_{\pi^\natural}$ is an alternating sum of semisimple representations of Γ_E, thus a virtual representation (but see Theorem 9.6 below). Fix a neat open compact subgroup

$$K = \prod_{\mathfrak{q}\nmid\infty} K_{\mathfrak{q}} \subset G(\mathbb{A}^\infty_F) \quad \text{such that} \quad (\pi^{\natural,\infty})^K \neq 0,$$

and also such that $K_{\mathfrak{q}}$ is hyperspecial whenever $\pi^\natural_{\mathfrak{q}}$ (or equivalently $\pi_{\mathfrak{q}}$) is unramified. Let S_{bad} be the set of rational primes p for which either

- $p = 2$,
- $\mathrm{Res}_{F/\mathbb{Q}} G$ is ramified over $\mathbb{Q}_p$, or
- $K_p = \prod_{\mathfrak{q}|p} K_{\mathfrak{q}}$ is not hyperspecial.

We write S^F_{bad} (resp. S^E_{bad}) for the F-places (resp. E-places) above S_{bad}. We apply the Langlands–Kottwitz method at level K to compute the image of Frobenius elements under $\rho^{\mathrm{Sh},\varepsilon}_{\pi^\natural}$ at almost all primes.

Theorem 9.6. *Consider $\pi^\natural$ satisfying condition (temp); see below* (9.2)*. There exists a finite set of rational primes S containing S_{bad}, such that for all $\mathfrak{p}$ not above S and all sufficiently large integers j (with the lower bound for j depending on $\mathfrak{p}$), writing $\mathfrak{q} := \mathfrak{p} \cap F$, we have*

$$\mathrm{Tr}\, \rho^{\mathrm{Sh},\varepsilon}_{\pi^\natural}(\mathrm{Frob}^j_{\mathfrak{p}}) = \iota a(\pi^\natural) q_{\mathfrak{p}}^{jn(n-1)/4} \cdot \mathrm{Tr}\,(\mathrm{spin}^{\varepsilon,\vee}(\phi_{\pi^\natural_{\mathfrak{q}}}))(\mathrm{Frob}^j_{\mathfrak{p}}), \qquad \varepsilon \in \{+,-\}. \tag{9.7}$$

Moreover the summand of (9.6) *is nonzero only if $i = n(n-1)/2$. In particular the virtual representation $\rho^{\mathrm{Sh},\varepsilon}_{\pi^\natural}$ is a true semisimple representation.*

Proof. We mimic the proof of [49, Prop. 8.2] closely. Note that our $\rho^{\mathrm{Sh},\varepsilon}_{\pi^\natural}$ corresponds to ρ^{shim}_2 there. Another difference is that we use S to denote a set of primes of $\mathbb{Q}$ (not F or E). It is enough to find S as in the theorem for each ε separately, as we can take the union of the set for each of $+$ and $-$ (and take the maximum of lower bounds for j). We suppose that $F \neq \mathbb{Q}$ so that our Shimura varieties are proper. The case $F = \mathbb{Q}$ will be addressed at the end of proof.

Let $f_\infty = N_\infty^{-1} f_\xi$, where f_ξ is the Euler-Poincaré (a.k.a. Lefschetz) function for ξ on $G(F_\infty)$ as recalled in [49, Appendix A]. Then

$$\mathrm{Tr}\, \tau_\infty(f_\infty) = N_\infty^{-1} \mathrm{ep}^\varepsilon(\tau_\infty \otimes \xi) = N_\infty^{-1} \sum_{i=0}^{\infty} (-1)^i \dim \mathrm{H}^i(\mathfrak{g}, K^\varepsilon_\infty; \tau_\infty \otimes \xi).$$

Choose a decomposable Hecke operator $f^{\infty,\mathfrak{q}_{\mathrm{St}}} = \prod_{\mathfrak{q}\neq\mathfrak{q}_{\mathrm{St}}} f_{\mathfrak{q}} \in \mathcal{H}(G(\mathbb{A}_F^{\infty,\mathfrak{q}_{\mathrm{St}}}) \mathbin{/\!/} K^{\mathfrak{q}_{\mathrm{St}}})$ such that for all automorphic representations τ of $G(\mathbb{A}_F)$ with $\tau^{\infty,K}\neq 0$ and $\mathrm{Tr}\,\tau_\infty(f_\infty)\neq 0$ we have

$$\mathrm{Tr}\,\tau^{\infty,\mathfrak{q}_{\mathrm{St}}}(f^{\infty,\mathfrak{q}_{\mathrm{St}}}) = \begin{cases} 1 & \text{if } \tau^{\infty,\mathfrak{q}_{\mathrm{St}}} \simeq \pi^{\natural,\infty,\mathfrak{q}_{\mathrm{St}}} \\ 0 & \text{otherwise.}\end{cases}$$

This is possible since there are only finitely many such τ (one of which is $\pi^\natural$). Let $f_{\mathfrak{q}_{\mathrm{St}}}$ be a Lefschetz function from [49, Eq. (A.4)]. There exists a finite set of primes $\Sigma \supset S_{\mathrm{bad}}\cup\{\ell\}$ such that $f_{\mathfrak{p}'}$ is the characteristic function of $K_{\mathfrak{p}'}$ (which is hyperspecial) for every $\mathfrak{p}'$ not above Σ. We fix Σ and $f^\infty = \prod_{v\nmid\infty} f_v$ as above.

In the rest of the proof we fix an E-prime $\mathfrak{p}$ not above $\Sigma\cup\{\ell\}$. Write $\mathfrak{q} := \mathfrak{p}\cap F$, and p for the rational prime below $\mathfrak{p}$. To apply the Langlands–Kottwitz method, we need an integral model for $\mathrm{Sh}^\varepsilon_K$ over $\mathcal{O}_{E_\mathfrak{p}}$. Thus we choose an isomorphism $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ such that the valuation on $\overline{\mathbb{Q}}_p$ restricts to the $\mathfrak{p}$-adic valuation via $\iota_p x_\infty : E \hookrightarrow \overline{\mathbb{Q}}_p$. (Recall x_∞ from Lemma 9.3 (iv).) The $(\mathrm{Res}_{F/\mathbb{Q}}G)(\overline{\mathbb{Q}}_p)$-conjugacy class of $\iota_p\mu : \mathbb{G}_m \to (\mathrm{Res}_{F/\mathbb{Q}}G)_{\overline{\mathbb{Q}}_p}$ is defined over $E_\mathfrak{p}$.

For $j\in\mathbb{Z}_{\geq 1}$, let $f_p^{(j)}$ denote the function in the unramified Hecke algebra of $G(F_p)$ constructed in [47, §7] for the endoscopic group $H = G^*$, which is isomorphic to G over $F_p = F\otimes_{\mathbb{Q}}\mathbb{Q}_p$. (This is the function h_p in *loc. cit.* We take s and t_i's on p.179 there to be trivial, so that h_p is the image of ϕ_j under the standard base change map on p.180.) The L-group for $(\mathrm{Res}_{F/\mathbb{Q}}G)_{E_\mathfrak{p}}$ (with coefficients in $\mathbb{C}$) can be identified as

$${}^L(\mathrm{Res}_{F/\mathbb{Q}}G)_{E_\mathfrak{p}} = \Big(\prod_{\sigma\in\mathrm{Hom}(F,\overline{\mathbb{Q}}_p)} \widehat{G}\Big) \rtimes \Gamma_{E_\mathfrak{p}},$$

where $\Gamma_{E_\mathfrak{p}}$ acts trivially on the factor for $\sigma = \iota_p y_\infty$. (The Galois action may permute the other factors via its natural action on $\mathrm{Hom}(F,\overline{\mathbb{Q}}_p)$ but this does not matter to us.) The representation of ${}^L(\mathrm{Res}_{F/\mathbb{Q}}G)_{E_\mathfrak{p}}$ of highest weight $\iota_p\mu$ is the representation $(\mathrm{spin}^\varepsilon, 1,\dots,1)$. Here $\mathrm{spin}^\varepsilon$ is on the factor for $\sigma = \iota_p y_\infty$, where we identify

$$G\times_{F,\sigma}\overline{\mathbb{Q}}_p = \mathrm{GSO}^{E/F}_{2n}\times_{F,\sigma}\overline{\mathbb{Q}}_p \overset{\text{via } \iota_p x_\infty}{=\!=\!=} \mathrm{GSO}_{2n,\overline{\mathbb{Q}}_p},$$

(in the ambient group $\mathrm{GL}_{2n}(E\otimes_F\overline{\mathbb{Q}}_p) \simeq \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_p)\times\mathrm{GL}_{2n}(\overline{\mathbb{Q}}_p)$ of the left hand side, we project onto the $\iota_p x_\infty$-component) thus identify $\widehat{G} = \mathrm{GSpin}_{2n}$ on the $\iota_p y_\infty$-component. Now let $\tau_p = \prod_{\mathfrak{q}'|p}\tau_{\mathfrak{q}'}$ be an unramified representation of $G(F_p) = (\mathrm{Res}_{F/\mathbb{Q}}G)(\mathbb{Q}_p) = \prod_{\mathfrak{q}'|p} G(F_{\mathfrak{q}'})$, and denote by $\phi_{\tau_p} : W_{\mathbb{Q}_p} \to {}^L(\mathrm{Res}_{F/\mathbb{Q}}G)_{\mathbb{Q}_p}$ its L-parameter. Then the $\iota_p y_\infty$-component of $\phi_{\tau_p}|_{W_{E_\mathfrak{p}}}$ is given by $\phi_{\tau_\mathfrak{q}}|_{W_{E_\mathfrak{p}}}$. All in all, we can explicate [45, (2.2.1)] in our setup as[12]

$$\mathrm{Tr}\,\tau_p(f_p^{(j)}) = q_\mathfrak{p}^{jn(n-1)/4}\mathrm{Tr}\,(\mathrm{spin}^{\varepsilon,\vee}(\phi_{\tau_\mathfrak{q}})(\mathrm{Frob}^j_\mathfrak{p})). \tag{9.8}$$

As in the proof of [49, Prop. 8.2] (where our $f_p^{(j)}$ is denoted by $h_p^{G^*}$), the Lefschetz functions f_∞ and $f_{\mathfrak{q}_{\mathrm{St}}}$ allow us to simplify the stabilized Langlands-Kottwitz formula [43, Thm. 8.3.11] (recalled in [49, Thm. 7.3]) and obtain a simple stabilization of the trace formula for G; the outcomes are formulas (8.8) and (8.9) of [49]. Combining them, we obtain

$$\iota^{-1}\mathrm{Tr}\,(\iota f^{\infty,p}f_p\times\mathrm{Frob}^j_\mathfrak{p},\,\mathrm{H}_c(\mathrm{Sh}^\varepsilon,\mathcal{L}_{\iota\xi})) = T^G_{\mathrm{cusp},\chi}(f^{\infty,p}f_p^{(j)}f_\infty), \qquad j\gg 1. \tag{9.9}$$

[12] A word on the sign convention is appropriate here. The sign of [45, (2.2.1)] was flipped on [47, p.193], meaning that the highest weight $-\iota_p\mu_\varepsilon$ (up to the Weyl group action) should be used in (9.8). This was caused by the arithmetic vs geometric convention for Frobenius, and explains why $\mathrm{spin}^\varepsilon$ is dualized, cf. the paragraph above Lemma 4.2. (It may appear that the sign has to be changed once again when going from [47] to [48], since the latter paper asserts that (G,h^{-1}) in its notation, not (G,h), corresponds to the canonical model of [20]. However we think the sign change is unnecessary; it should be (G,h) as long as we fix the sign errors in [20] as pointed out at the end of §12 in [60].)

Note that f_p is the characteristic function of the hyperspecial subgroup $K_p = \prod_{\mathfrak{p}|p} K_{\mathfrak{p}}$. Following the argument from Equation (8.10) to (8.13) in [49], we compute

$$\iota^{-1}\mathrm{Tr}\,\rho^{\mathrm{Sh},\varepsilon}_{\pi^\natural}(\mathrm{Frob}^j_{\mathfrak{p}}) = a(\pi^\natural)\mathrm{Tr}\,\pi^\natural_p(f^{(j)}_p) \overset{(9.8)}{=\!=} a(\pi^\natural) q_{\mathfrak{p}}^{jn(n-1)/4}\cdot \mathrm{Tr}\,(\mathrm{spin}^{\varepsilon,\vee}(\phi_{\pi^\natural_{\mathfrak{q}}}))(\mathrm{Frob}^j_{\mathfrak{p}}). \tag{9.10}$$

Let us show that $\rho^{\mathrm{Sh},\varepsilon}_{\pi^\natural}$ is a true representation by showing that only the middle degree cohomology contributes to $\rho^{\mathrm{Sh},\varepsilon}_{\pi^\natural}$. Since the canonical smooth integral model of $\mathrm{Sh}^\varepsilon_K$ constructed by Kisin is proper as shown in [82, Thm. 2.1.29] (extending the analogous result for Hodge-type Shimura varieties by Madapusi Pera [58, Cor. 4.1.7]), the action of $\mathrm{Frob}_{\mathfrak{p}}$ on $\mathrm{H}^i_c(\mathrm{Sh}_K, \mathcal{L}_\xi)$ is pure of weight $-w+i$ by [22, Cor. 3.3.6] since $\mathcal{L}_\xi$ is pure of weight $-w$ [67, §5.4, Prop. 5.6.2]. (To obtain purity from Pink's result, we enlarge the set S if necessary; cf. [63, 1.3], especially the proof of (7) in Proposition 1.3.4 there.) The argument for Part (2) of [49, Lem. 8.1] (replacing Lemma 2.7 in the proof therein with our condition (temp)) implies that $\tau_{\mathfrak{q}}|\mathrm{sim}|^{w/2} = \pi^\natural_{\mathfrak{q}}|\mathrm{sim}|^{w/2}$ is tempered and unitary. Combining with (9.10) we conclude that $\mathrm{H}^i_c(\mathrm{Sh}^\varepsilon, \mathcal{L}_\xi)[\iota\tau^\infty] = 0$ unless $i = n(n-1)/2$.

Finally, the case $F = \mathbb{Q}$ is handled via intersection cohomology as in the proof of [49, Prop. 8.2]. Thus we content ourselves with giving a sketch. For each $\tau \in A(\pi^\natural)$, one observes as in [49, Lem. 8.1] that $\mathrm{H}^i_c(\mathrm{Sh}^\varepsilon, \mathcal{L}_\xi)[\iota\tau^\infty]$ is isomorphic to the $\iota\tau^\infty$-isotypic part of the intersection cohomology as Γ_E-representations. The point is that τ^∞ does not appear in any parabolic induction of an automorphic representation on a proper Levi subgroup of $G(\mathbb{A})$. (If it does appear, then restricting τ from $G(\mathbb{A})$ to its derived subgroup $G^{\mathrm{der}}(\mathbb{A})$ and transferring to the quasi-split inner form $\mathrm{SO}^{E/F}_{2n}(\mathbb{A})$ via [49, Prop. 6.3], we would have a cohomological automorphic representation $\tau^\flat$ of $\mathrm{SO}^{E/F}_{2n}(\mathbb{A})$ with a Steinberg component up to a twist that appears as a constituent in a parabolically induced representation. Then the Arthur parameter for $\tau^\flat$ cannot have the shape described in Proposition 6.1, leading to a contradiction.) The rest of the proof of [49, Prop. 8.2] carries over, via the analogue of part 2 of [49, Lem. 8.1] (the latter is justified using condition (temp) in our case), bearing in mind that the middle degree is $n(n-1)/2$ for us (which was $n(n+1)/2$ for the group GSp_{2n}). □

Corollary 9.7. *Let $\pi^\natural$ be as above. If $\tau \in A(\pi^\natural)$ then*

(1) τ_∞ belongs to the discrete series L-packet $\Pi^{G(F_\infty)}_\xi$,

(2) $\tau^\infty\tau'_\infty \in A(\pi^\natural)$ and $m(\tau) = m(\tau^\infty\tau'_\infty)$ for all $\tau'_\infty \in \Pi^{G(F_\infty)}_\xi$.

Moreover $a(\pi^\natural) = \sum_{\tau\in A(\pi^\natural)/\sim} m(\tau) \in \mathbb{Z}_{>0}$.

Proof. This is the exact analogue of [49, Cor. 8.4, Cor. 8.5] and the same proof applies. (Since $a(\pi^\natural) = a^+(\pi^\natural) = a^-(\pi^\natural)$, we adapt the argument there to either $\varepsilon \in \{+,-\}$ to compute.) □

Proposition 9.8. *Assume that $F \neq \mathbb{Q}$. Let $x_\infty : E \hookrightarrow \mathbb{C}$ and $y_\infty : F \hookrightarrow \mathbb{C}$ be as in Lemma 9.3. Then*

$$\mu_{\mathrm{HT}}(\rho^{\mathrm{Sh},\varepsilon}_{\pi^\natural}, \iota x_\infty) \sim i_{a(\pi^\natural)} \circ \mathrm{spin}^{\varepsilon,\vee} \circ \left(\mu_{\mathrm{Hodge}}(\xi_{y_\infty}) - \tfrac{n(n-1)}{4}\mathrm{sim}\right), \quad \varepsilon \in \{\pm 1\}.$$

Proof. We start by setting up some notation. Let $\mathfrak{p}$ be a prime of E above ℓ, and $\sigma : E \hookrightarrow \overline{\mathbb{Q}}_\ell$ an embedding inducing the $\mathfrak{p}$-adic valuation on E. Let r be a de Rham Galois representation of Γ_E on a $\overline{\mathbb{Q}}_\ell$-vector space. Write $D_{\mathrm{dR},\sigma}(r)$ for the filtered $\overline{\mathbb{Q}}_\ell$-vector space associated with $r|_{\Gamma_{E_{\mathfrak{p}}}}$ with respect to σ (as on p.99 of [33]). Define $\mathrm{HT}_\sigma(r)$ to be the multi-set containing each $j \in \mathbb{Z}$ with multiplicity $\dim \mathrm{gr}^j(D_{\mathrm{dR},\sigma}(r))$. (So the cardinality of $\mathrm{HT}_\sigma(r)$ equals $\dim r$.) When $a \in \mathbb{Z}_{>0}$ and A is a multi-set, we write $A^{\oplus a}$ to denote the multi-set such that the multiplicity of each element in $A^{\oplus a}$ is a times that in A.

Write $\lambda(\xi) = \{\lambda(\xi_y)\}_{y|\infty}$ for the highest weight of $\xi = \otimes_{y|\infty}\xi_y$. In the basis of §2 for $X^*(\mathrm{T}_{\mathrm{GSO}}) = X_*(\mathrm{T}_{\mathrm{GSpin}}) = \mathbb{Z}^{n+1}$, we write ξ_{y_∞} and the half sum of positive roots ρ for GSO_{2n} as

$$\begin{aligned} \lambda(\xi_{y_\infty}) &= (a_0, a_1, \ldots, a_n), \qquad a_1 \geq a_2 \geq \cdots \geq |a_n| \geq 0, \\ \rho &= (-n(n-1)/4, n-1, n-2, \ldots, 1, 0). \end{aligned}$$

Let $\mathscr{P}^\varepsilon(n)$ denote the collection of subsets of $\{1,2,\dots,n\}$ whose cardinality has the same parity as n if $\varepsilon=(-1)^n$ and different parity if $\varepsilon=(-1)^{n+1}$. Put

$$\begin{aligned}(b_0,b_1,\dots,b_n) &:= (a_0-\tfrac{n(n-1)}{2}, a_1+n-1, a_2+n-2,\dots,a_{n-1}+1,a_n)\\ &= \lambda(\xi_{y_\infty})+\rho-(n(n-1)/4,0,0,\dots,0),\end{aligned}$$

which equals $\mu_{\mathrm{Hodge}}(\xi_{x_\infty})-\frac{n(n-1)}{4}\mathrm{sim}$. Via the description of weights in the representation $\mathrm{spin}^\varepsilon$ in (2.10) (which gives the weights in $\mathrm{spin}^{\varepsilon,\vee}$), the proposition amounts to the assertion that (9.11)

$$\mathrm{HT}_{\iota x_\infty}(\rho^{\mathrm{Sh},\varepsilon}_{\pi^\natural})=\Big\{-b_0-\sum_{i\in I}b_i\,\Big|\,I\in\mathscr{P}^\varepsilon(n)\Big\}^{\oplus a(\pi^\natural)}=\Big\{-a_0-\sum_{i\in I}a_i+\sum_{i\notin I}(n-i)\,\Big|\,I\in\mathscr{P}^\varepsilon(n)\Big\}^{\oplus a(\pi^\natural)}.$$

We prove this assertion using a result from [52]. Let us introduce some more notation. Write $\mathrm{Sh}^\varepsilon_K(\mathbb{C})$ for the complex manifold obtained from $\mathrm{Sh}^\varepsilon_K$ by base change along $x_\infty: E\hookrightarrow\mathbb{C}$, and $\mathcal{L}^{\mathrm{top}}_\xi$ for the topological local system on $\mathrm{Sh}^\varepsilon_K(\mathbb{C})$ coming from ξ. Writing K^ε (Lemma 9.1) as $K^\varepsilon=\prod_y K^\varepsilon_y$, we have $K^\varepsilon_{y_\infty}=K_\varepsilon$ and $K^\varepsilon_y=G(F_y)\simeq \mathrm{GSO}^{\mathrm{cpt}}_{2n}(F_y)$ for $y\neq y_\infty$. Restricting $h^\varepsilon_{\mathbb{C}}$ to the first factor of $\mathbb{S}_{\mathbb{C}}=\mathbb{G}_{m,\mathbb{C}}\times\mathbb{G}_{m,\mathbb{C}}$ (labeled by the identity $\mathbb{C}\to\mathbb{C}$, not the complex conjugation), we obtain a cocharacter $\mathbb{G}_{m,\mathbb{C}}\to K^\varepsilon_{\mathbb{C}}$, which we denote by μ^ε; this is consistent with the definition of μ^ε below (9.1). We also have a parabolic subgroup $Q\subset(\mathrm{Res}_{F/\mathbb{Q}}G)_{\mathbb{C}}$ with Levi component $K^\varepsilon_{\mathbb{C}}$ as [23, p.57] (such that the Borel embedding goes into $(\mathrm{Res}_{F/\mathbb{Q}}G)(\mathbb{C})/Q$). Fix an elliptic maximal torus $T_\infty\subset K^\varepsilon$ and a Borel subgroup $B\subset(\mathrm{Res}_{F/\mathbb{Q}}G)_{\mathbb{C}}$ contained in Q such that B contains T_∞. Let R^+ denote the set of positive roots of T_∞ in B. By R^- we denote the set of roots of T_∞ in the opposite Borel subgroup. Write Ω for the Weyl group of $T_{\infty,\mathbb{C}}$ in $(\mathrm{Res}_{F/\mathbb{Q}}G)_{\mathbb{C}}$, and Ω_{nc} for the subset of $\omega\in\Omega$ such that $\omega\lambda$ is $B\cap K^\varepsilon_{\mathbb{C}}$-dominant whenever $\lambda\in X^*(T_\infty)$ is B-dominant. Let Ω_{c} denote the Weyl group of $T_{\infty,\mathbb{C}}$ in $K^\varepsilon_{\mathbb{C}}$. The inclusion $\Omega_{\mathrm{nc}}\subset\Omega$ induces a canonical bijection $\Omega_{\mathrm{nc}}\simeq\Omega/\Omega_{\mathrm{c}}$. We parametrize members of the discrete series L-packet $\Pi^{G(F_\infty)}_\xi$ as $\{\pi(\omega)|\omega\in\Omega_{\mathrm{nc}}\}$ following [34, 3.3]. (Our $\pi(\omega)$ is $\pi(\omega\lambda,\omega R^+)$ in their notation.) Even though Q,T_∞,B,Ω_c depend on ε (since they do on K^ε), we suppress it from the notation for simplicity.

Write $\rho_G\in X^*(T)$ for the half sum of all roots in R^+, and define $\omega\star\lambda_0:=\omega(\lambda_0+\rho_G)-\rho_G$ for $\lambda_0\in X^*(T_\infty)$. Every irreducible representation V_{λ_0} of $K^\varepsilon_{\mathbb{C}}$ of highest weight $\lambda_0\in X^*(T_\infty)$ gives rise to an automorphic vector bundle, to be denoted by $\mathcal{E}_{\lambda_0}$. Write $\lambda=\lambda(\xi)\in X^*(T_\infty)$ for the B-dominant highest weight of ξ.

For a finite multi-set A, write $\mathrm{mult}(a|A)\in\mathbb{Z}_{\geq 0}$ for the multiplicity of a in A. For each $j\in\mathbb{Z}$, define $\Omega_{\mathrm{nc}}(j)$ to be the set of $\omega\in\Omega_{\mathrm{nc}}$ such that the composition $\mathbb{G}_m\overset{(\mu^\varepsilon)^{-1}}{\to}T_{\infty,\mathbb{C}}\overset{\omega\star\lambda}{\to}\mathbb{G}_m$ equals $z\mapsto z^j$. Then

$$\begin{aligned}&\mathrm{mult}\big(j\,|\,\mathrm{HT}_{\iota x_\infty}(\rho^{\mathrm{Sh},\varepsilon}_{\pi^\natural})\big)\\ =&\sum_{\tau\in A(\pi^\natural)/\sim}\mathrm{mult}\big(j\,|\,\mathrm{HT}_{\iota x_\infty}(H^{n(n-1)/2}(\mathrm{Sh}^\varepsilon,\mathcal{L}_\xi)[\iota\tau^\infty]\big)\quad\text{by Thm. 9.6 and (9.6)},\\ =&\sum_{\tau\in A(\pi^\natural)/\sim}\sum_{\omega\in\Omega_{\mathrm{nc}}(j)}\dim H^{\frac{n(n-1)}{2}-l(\omega)}(\mathrm{Sh}^\epsilon(\mathbb{C}),\mathcal{E}_{\omega\star\lambda})[\tau^\infty],\quad\text{by [52, Thm. 6.2.9]}.\end{aligned}$$

From [34, §3] we have an isomorphism of $G(\mathbb{A}^\infty_F)$-modules:

$$H^k(\mathrm{Sh}(\mathbb{C}),\mathcal{E}_{\omega\star\lambda})\simeq\bigoplus_\tau m(\tau)\tau^\infty\otimes H^k(\mathrm{Lie}\,Q,K^\varepsilon,\tau_\infty\otimes V_{\omega\star\lambda}).$$

For each $\tau\in A(\pi^\natural)$, we pass to the τ^∞-isotypic parts (with notation as in (9.5)) to obtain

$$H^k(\mathrm{Sh}(\mathbb{C}),\mathcal{E}_{\omega\star\lambda})[\tau^\infty]\simeq\bigoplus_{\tau'_\infty}m(\tau^\infty\otimes\tau'_\infty)H^k(\mathrm{Lie}\,Q,K^\varepsilon,\tau'_\infty\otimes V_{\omega\star\lambda}),$$

where the sum runs over irreducible unitary representations of $G(F_\infty)$. By [34, Prop. 4.4.12], if the cohomology on the right hand side is nonzero then τ'_∞ is ξ-cohomological, so $\tau^\infty\otimes\tau'_\infty\in$

$A(\pi^\infty)$. It follows from Corollary 9.7 that $\tau'_\infty \in \Pi_\infty^{G(F_\infty)}$ and that $m(\tau^\infty \otimes \tau'_\infty) = m(\tau)$. Moreover, [34, Thm. 3.4] implies that $H^k(\operatorname{Lie} Q, K^\varepsilon, \tau'_\infty \otimes V_{\omega\star\lambda})$ is nonzero for a unique τ'_∞, in which case the cohomology is one-dimensional. We use this to resume the computation of $\operatorname{mult}(j \,|\, \mathrm{HT}_{\iota x_\infty}(\rho^{\mathrm{Sh},\varepsilon}_{\pi^\natural}))$ and obtain

$$\operatorname{mult}\big(j \,|\, \mathrm{HT}_{\iota x_\infty}(\rho^{\mathrm{Sh},\varepsilon}_{\pi^\natural})\big) = \sum_{\tau \in A(\pi^\natural)/\sim} \sum_{\omega \in \Omega_{\mathrm{nc}}(j)} m(\tau) = a(\pi^\natural) \cdot |\Omega_{\mathrm{nc}}(j)|.$$

To conclude (9.11), it remains to prove the following claim: that $|\Omega_{\mathrm{nc}}(j)|$ is precisely the number of ways j can be written as $-a_0 - \sum_{i\in I} a_i + \sum_{i \notin I}(n-i)$ with $I \in \mathscr{P}^\varepsilon(n)$.

As a preparation, we fix an isomorphism between the pairs $(T_{\infty,\mathbb{C}}, B)$ and $(\mathrm{T}_{\mathrm{GSO}}, B_{\mathrm{GSO}})$ induced by an inner twist $(\operatorname{Res}_{F/\mathbb{Q}} G)_{\mathbb{C}} \simeq (\operatorname{Res}_{F/\mathbb{Q}} G^*)_{\mathbb{C}}$. So the Weyl action of Ω is identified with the W_{GSO}-action in (2.8), while Ω_{c} is identified with $\mathfrak{S}_n$ therein if $\varepsilon = (-1)^n$. (If $\varepsilon = (-1)^{n+1}$ then Ω_{c} is the θ°-conjugate of $\mathfrak{S}_n$.) For a subset $I \subset \{1, ..., n\}$, let ω'_I denote the action on $(t_0, t_1, \dots, t_n) \in X^*(\mathrm{T}_{\mathrm{GSO}})$ by $t_i \mapsto t_i$ for $i \in I$, $t_i \mapsto -t_i$ for $i \notin I$, and $t_0 \mapsto t_0 + \sum_{i \in I} t_i$. Then $\omega'_I \in \Omega$ if and only if $n - |I|$ is even.

Let us prove the claim, starting with the case $\varepsilon = (-1)^n$. Then $n - |I|$ is even for each $I \in \mathscr{P}^\varepsilon(n)$. Write $\omega_I \in \Omega_{\mathrm{nc}}$ for the unique intersection of the Ω_{c}-orbit of ω_I with Ω_{nc}. We have bijections

$$\mathscr{P}^\varepsilon(n) \xrightarrow{\sim} \Omega/\Omega_{\mathrm{c}} \xleftarrow{\sim} \Omega_{\mathrm{nc}}, \qquad I \mapsto \omega'_I \mapsto \omega_I.$$

Since $\varepsilon = (-1)^n$, we have from (2.9)

$$\mu^\varepsilon = (z \mapsto (z, z, , \dots, z, z)) \in X_*(\mathrm{T}_{\mathrm{GSO}}) \simeq X_*(T_\infty),$$

a priori up to the Ω_{c}-action, but μ^ε is Ω_{c}-invariant. From this, we compute for $\lambda = (a_0, a_1, \dots, a_n) \in X^*(T_\infty) \simeq X^*(\mathrm{T}_{\mathrm{GSO}})$:

$$(\omega_I \star \lambda) \circ (\mu^\varepsilon)^{-1} = (\omega'_I \star \lambda) \circ (\mu^\varepsilon)^{-1} = -a_0 - \sum_{i \in I}(a_i + n - i) + \sum_{\text{all } i}(n-i) = -a_0 - \sum_{i\in I} a_i + \sum_{i \notin I}(n-i).$$

Thus the claim for $\varepsilon = (-1)^n$ follows.

Keep $\varepsilon = (-1)^n$ and let us prove the claim for $P^{-\varepsilon}(n)$. Since $n - |I|$ is odd, we no longer have $\omega'_I \in \Omega$ but instead have $\omega''_I := \theta^\circ \omega'_I = \omega'_I \theta^\circ \in \Omega$. Replacing ω'_I with ω''_I in the previous paragraph, we obtain ω_I and analogous bijections

$$\mathscr{P}^{-\varepsilon}(n) \xrightarrow{\sim} \Omega/\Omega_{\mathrm{c}} \xleftarrow{\sim} \Omega_{\mathrm{nc}}, \qquad I \mapsto \omega''_I \mapsto \omega_I.$$

It follows from $\mu^{-\varepsilon} = \theta^\circ \mu^\varepsilon$ and Ω_{c}-invariance of μ^ε that

$$(\omega_I \star \lambda) \circ (\mu^{-\varepsilon}) = (\theta^\circ(\omega_I \star \lambda)) \circ (\mu^\varepsilon) = (\omega'_I \star \lambda) \circ (\mu^\varepsilon).$$

The proof is now done since the computation of $(\omega'_I \star \lambda) \circ (\mu^\varepsilon)^{-1}$ in the preceding case goes through verbatim. □

10. Construction of GSpin_{2n}-valued Galois representations

We continue in the setting of §8 and §9. The goal of this section is to attach GSpin_{2n}-valued Galois representations of Γ_E to the automorphic representations of $G^* = \mathrm{GSO}_{2n}^{E/F}$ under consideration. The main input comes from the cohomology of Shimura varieties studied in the last section. Write $\mathrm{std}\colon \mathrm{GSpin}_{2n} \hookrightarrow \mathrm{GL}_{2n}$ for the composite of $\mathrm{pr}\colon \mathrm{GSpin}_{2n} \to \mathrm{GSO}_{2n}$ and the inclusion $\mathrm{GSO}_{2n} \subset \mathrm{GL}_{2n}$.

Let π be a cuspidal automorphic representation of $G^*(\mathbb{A}_F)$. Let ϕ_{π_y} denote the L-parameter of π_y for $y \in \mathcal{V}_\infty$. Throughout this section, we assume that

- **(St)** for some finite F-place $\mathfrak{q}_{\mathrm{St}}$ the local representation $\pi_{\mathfrak{q}_{\mathrm{St}}}$ is isomorphic to the Steinberg representation up to a character twist,
- **(coh)** the representation π_∞ is cohomological for some representation ξ of $(\operatorname{Res}_{F/\mathbb{Q}} G^*) \otimes_{\mathbb{Q}} \mathbb{C}$ (then ξ satisfies condition (cent) by [49, Lem. 7.1] as before).

Choose $\pi^\flat$ a cuspidal automorphic representation of $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ contained in $\pi|_{\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)}$ (see [51]). We observe that $\pi^\flat$ satisfies conditions (St°) and (coh°) of §6 thanks to Lemma 7.1 and 7.2. Consider the following analogue of (std-reg°) for π

(std-reg) $\mathrm{std} \circ \phi_{\pi_y}|_{W_{\overline{F}_y}}$ is regular at every $y \in \mathcal{V}_\infty$.

In addition to (St) and (coh), the following is also assumed throughout:

- Either (std-reg) holds for π, or Hypothesis 6.11 is true for $\pi^\flat$.

So Hypothesis 6.11 comes into play only when (std-reg) does not hold.

Condition (std-reg) is equivalent to the one given in the introduction via local Langlands for real groups, e.g., see [12, §2.3]. If (std-reg) is imposed on π, then (std-reg°) follows from (coh°). By [53, §3, (iv)], we have that $\phi_{\pi^\flat,y} = \mathrm{pr}^\circ \circ \phi_{\pi,y}$ at each $y \in \mathcal{V}_\infty$. We can also see (std-reg°) from this and (std-reg).

Lemma 10.1. *In addition to (St) and (coh) for π, assume either (std-reg) for π or Hypothesis 6.11 for $\pi^\flat$. Then $\pi^\flat$ is tempered at all places, and π is essentially tempered at all places.*

Proof. This follows from Proposition 6.1 if (std-reg) holds. Otherwise, the same proposition implies $\pi^\flat$ is tempered at infinite places, and Hypothesis 6.11 asserts that $\pi^\flat$ is tempered at finite places. The last assertion easily follows from the temperedness of $\pi^\flat$. (See the proof of [49, Lem. 2.7].) □

The right hand side of (8.4) is easily extended to a model of $\mathrm{GSO}_{2n}^{E/F}$ over $\mathcal{O}_F$ (by replacing E, F with $\mathcal{O}_E, \mathcal{O}_F$). Similarly we have a model of $\mathrm{SO}_{2n}^{E/F}$ closed in the model of $\mathrm{GSO}_{2n}^{E/F}$, defined by the condition $\lambda = 1$. At each F-prime $\mathfrak{q}$ not above 2 and unramified in E, we have the hyperspecial group $H_\mathfrak{q} := \mathrm{GSO}_{2n}^{E/F}(\mathcal{O}_{F_\mathfrak{q}})$, whose intersection with $\mathrm{SO}_{2n}^{E/F}(F_\mathfrak{q})$ is the hyperspecial subgroup $H_{0,\mathfrak{q}} := \mathrm{SO}_{2n}^{E/F}(\mathcal{O}_{F_\mathfrak{q}})$ in the latter. We will fix these choices of hyperspecial subgroups for $\mathrm{GSO}_{2n}^{E/F}$ and $\mathrm{SO}_{2n}^{E/F}$. At each $\mathfrak{q} \in \mathrm{Unr}(\pi)$ (so that $\pi_v^{H_\mathfrak{q}}$ is nontrivial), we can thus find an irreducible $\mathrm{SO}_{2n}^{E/F}(F_\mathfrak{q})$-subrepresentation in π_v with nonzero $H_{0,\mathfrak{q}}$-fixed vectors. Consequently, after translating $\pi^\flat$ inside of π by a suitable $g \in \mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$, we may assume that $\pi^\flat_\mathfrak{q}$ is unramified at every $\mathfrak{q}$ not above S_{bad} (with respect to the hyperspecial subgroups above).

Thanks to Theorem 6.5 if (std-reg) is assumed, or instead by Hypothesis 6.11, we have a Galois representation

$$\rho_{\pi^\flat} \colon \Gamma_{F,S_{\mathrm{bad}}} \to \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \mathrm{Gal}(E/F),$$

whose restriction to $\Gamma_{E,S_{\mathrm{bad}}}$ satisfies, writing $\mathfrak{q} := \mathfrak{p} \cap F$ for each $\mathfrak{p}$,

$$\rho_{\pi^\flat}(\mathrm{Frob}_\mathfrak{p})_{\mathrm{ss}} \overset{\circ}{\sim} \iota\phi_{\pi^\flat_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{p}) \in \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell), \tag{10.1}$$

for all E-places $\mathfrak{p} \notin S^E_{\mathrm{bad}}$. Here $\overset{\circ}{\sim}$ indicates O_{2n}-conjugacy (instead of SO_{2n}-conjugacy).

Let $H \subset \mathrm{SO}_{2n}$ denote the Zariski closure of the image of $\rho_{\pi^\flat} \colon \Gamma_{E,S_{\mathrm{bad}}} \to \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$. By Proposition 5.5, either H is connected or $H = H^0 \times Z(\mathrm{SO}_{2n})$. Therefore, via $\{\pm 1\} = Z(\mathrm{SO}_{2n})$, we can find a Galois character

$$\eta \colon \Gamma_{E,S_{\mathrm{bad}}} \to \{\pm 1\} \tag{10.2}$$

such that the product morphism $\eta\rho_{\pi^\flat}$ has Zariski dense image in H^0. In particular if $H^0 = H$ we take $\eta = 1$. We define the character

$$\widetilde{\eta} \colon \Gamma_{E,S_{\mathrm{bad}}} \to \langle z^+ \rangle \subset \mathrm{GSpin}_{2n} \tag{10.3}$$

to be the character so that the composition $\widetilde{\eta} \colon \Gamma_{E,S_{\mathrm{bad}}} \to \langle z^+ \rangle \xrightarrow{\sim}_{\mathrm{pr}^\circ} \{\pm 1\}$ is equal to η.

Recall that G is an inner form of $G^* = \mathrm{GSO}_{2n}^{E/F}$ giving rise to the Shimura data $(\mathrm{Res}_{F/\mathbb{Q}} G, X^\pm)$ studied earlier. By [49, Prop. 6.3], there exists a cuspidal automorphic representation $\pi^\natural$ of $G(\mathbb{A}_F)$ such that

- $\pi^\natural_{\mathfrak{q}'} \simeq \pi_{\mathfrak{q}'}$ at every finite prime $\mathfrak{q}'$ where $\pi_{\mathfrak{q}'}$ is unramified (we have $G_{\mathfrak{q}'} \simeq G^*_{\mathfrak{q}'}$ at such $\mathfrak{q}'$),
- $\pi^\natural_{\mathfrak{q}_{\mathrm{St}}}$ is a character twist of the Steinberg representation,

- $\pi^\natural_\infty$ is ξ-cohomological.

The first condition and Lemma 10.1 imply that $\pi^\natural$ satisfies condition (temp) of §9. Theorem 9.6 yields semisimple representations $\rho^{\mathrm{Sh},\varepsilon}_{\pi^\natural}$ of $\Gamma_{E,S}$ for $\varepsilon\in\{\pm1\}$ such that its dual $\rho^{\mathrm{Sh},\varepsilon,\vee}_{\pi^\natural}$ has the following property:

$$\rho^{\mathrm{Sh},\varepsilon,\vee}_{\pi^\natural}(\mathrm{Frob}_\mathfrak{p})_{\mathrm{ss}}\sim\iota q_\mathfrak{p}^{-n(n-1)/4}\big(i_{a_\pi}\circ\mathrm{spin}^\varepsilon(\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{p}))\big)\in\mathrm{GL}_{a_\pi 2^{n-1}}(\overline{\mathbb{Q}}_\ell),\qquad \mathfrak{p}\notin S^E, \tag{10.4}$$

where S is a finite set of rational primes containing S_{bad}, large enough, so that Theorem 9.6 holds for both $\varepsilon=+$ and $\varepsilon=-$. We define $\rho^{\mathrm{Sh},\varepsilon}_{\pi}:=\rho^{\mathrm{Sh},\varepsilon}_{\pi^\natural}$ for $\varepsilon\in\{\pm\}$ (which depends not on the choice of $\pi^\natural$ but only on π by (10.4)), and

$$\widetilde{\rho}^{\mathrm{Sh},\vee}_\pi:=\rho^{\mathrm{Sh},+,\vee}_\pi\oplus\big(\eta\otimes\rho^{\mathrm{Sh},-,\vee}_\pi\big).$$

Then $\widetilde{\rho}^{\mathrm{Sh},\vee}_\pi$ is a $\Gamma_{E,S}$-representation of dimension $a_\pi 2^n$, where $a_\pi:=a(\pi^\natural)$. We set

$$\widetilde{\mathrm{spin}}(\cdot):=\mathrm{spin}^+(\cdot)\oplus(\eta\otimes\mathrm{spin}^-(\cdot))$$

when the input is a GSpin_{2n}-valued Galois representation or a local L-parameter, and write $\widetilde{\mathrm{spin}}^a(\cdot)$ for the a-fold self-direct sum of $\widetilde{\mathrm{spin}}(\cdot)$. (So $\widetilde{\mathrm{spin}}=\mathrm{spin}$ if $\eta=1$.) We have

$$\widetilde{\rho}^{\mathrm{Sh},\vee}_\pi(\mathrm{Frob}_\mathfrak{p})_{\mathrm{ss}}\sim\iota q_\mathfrak{p}^{-n(n-1)/4}\widetilde{\mathrm{spin}}^{a_\pi}(\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{p}))\in\mathrm{GL}_{a_\pi 2^n}(\overline{\mathbb{Q}}_\ell),\qquad \mathfrak{p}\notin S^E. \tag{10.5}$$

Then $\rho^{\mathrm{Sh},\vee}_\pi$, $\widetilde{\rho}^{\mathrm{Sh},\vee}_\pi$, are a $\Gamma_{E,S}$-representation of dimension $a_\pi 2^n$, where $a_\pi:=a(\pi^\natural)$.

When $*$ is a map (resp. an element), we use $\overline{*}$ to denote the composition with the adjoint map (resp. the image under the adjoint map) that is clear from the context.

Proposition 10.2. *There exists a continuous semisimple representation*

$$\rho^C_\pi:\Gamma_{E,S}\to\mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$$

(with C standing for a cohomological normalization) such that we have

$$\forall\mathfrak{p}\notin S^E:\qquad \widetilde{\mathrm{spin}}\big(\rho^C_\pi(\mathrm{Frob}_\mathfrak{p})_{\mathrm{ss}}\big)\sim\iota q_\mathfrak{p}^{-n(n-1)/4}\widetilde{\mathrm{spin}}(\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{p}))\in\mathrm{GL}_{2^n}(\overline{\mathbb{Q}}_\ell), \tag{10.6}$$

$$\forall\mathfrak{p}\notin S^E_{\mathrm{bad}}:\qquad \mathrm{pr}^\circ\rho^C_\pi(\mathrm{Frob}_\mathfrak{p})_{\mathrm{ss}}\overset{\circ}{\sim}\iota\,\mathrm{pr}^\circ\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{p})\in\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell). \tag{10.7}$$

Proof. Consider the diagram

$$\begin{tikzcd}
\Gamma_{E,S} \arrow[rr, bend left, "\widetilde{\rho}^{\mathrm{Sh},\vee}_\pi"] \arrow[dr, bend right, "\eta\rho_{\pi^\flat}"'] & \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell) \arrow[r, hook, "\mathrm{spin}^{a_\pi}"'] \arrow[d, "\mathrm{pr}^\circ"'] & \mathrm{GL}_{a_\pi 2^n}(\overline{\mathbb{Q}}_\ell) \arrow[d] \\
 & \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \arrow[r, hook, "\overline{\mathrm{spin}^{a_\pi}}"'] & \mathrm{PGL}_{a_\pi 2^n}(\overline{\mathbb{Q}}_\ell).
\end{tikzcd} \tag{10.8}$$

At each prime $\mathfrak{p}$ of E not above S, we obtain from (10.1) that

$$\begin{aligned}&\overline{\mathrm{spin}^{a_\pi}}((\eta\rho_{\pi^\flat})(\mathrm{Frob}_\mathfrak{p})_{\mathrm{ss}})\sim\iota\overline{\mathrm{spin}^{a_\pi}}((\eta\phi_{\pi^\flat_\mathfrak{q}})(\mathrm{Frob}_\mathfrak{p}))\\ =\ &\iota\overline{\mathrm{spin}^{a_\pi}((\tilde\eta\phi_{\pi^\flat_\mathfrak{q}})(\mathrm{Frob}_\mathfrak{p}))}\sim\overline{\widetilde{\rho}^{\mathrm{Sh},\vee}_\pi(\mathrm{Frob}_\mathfrak{p})_{\mathrm{ss}}}\in\mathrm{PGL}_{a_\pi 2^n}(\overline{\mathbb{Q}}_\ell).\end{aligned} \tag{10.9}$$

Recall that $\eta\rho_{\pi^\flat}$ has connected image. So (10.9) implies, via [49, Prop. 4.6, Ex. 4.7], the existence of $g\in\mathrm{GL}_{a_\pi 2^n}(\overline{\mathbb{Q}}_\ell)$ such that

$$\overline{\widetilde{\rho}^{\mathrm{Sh},\vee}_\pi}=g\big(\overline{\mathrm{spin}^{a_\pi}}(\eta\rho_{\pi^\flat})\big)g^{-1}:\Gamma_{E,S}\to\mathrm{PGL}_{a_\pi 2^n}(\overline{\mathbb{Q}}_\ell).$$

Replace $\widetilde{\rho}^{\mathrm{Sh},\vee}_\pi$ by $g^{-1}\widetilde{\rho}^{\mathrm{Sh},\vee}_\pi g$ so that $\overline{\widetilde{\rho}^{\mathrm{Sh},\vee}_\pi}=\overline{\mathrm{spin}^{a_\pi}}(\eta\rho_{\pi^\flat})$. From Diagram (10.8) we deduce that

$$\widetilde{\rho}^{\mathrm{Sh},\vee}_\pi(\Gamma_{E,S})\subset\mathrm{pr}^{\circ,-1}\big((\eta\rho_{\pi^\flat})(\Gamma_{E,S})\big)\subset\mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell),$$

where GSpin_{2n} is viewed as a subgroup of $\mathrm{GL}_{a_\pi 2^n}$ via spin^{a_π}. That is, there exists a representation $\widetilde{\rho}^C_\pi:\Gamma_{E,S}\to\mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$ such that

$$\mathrm{spin}^{a_\pi}(\widetilde{\rho}^C_\pi)=\widetilde{\rho}^{\mathrm{Sh},\vee}_\pi\quad\text{and}\quad\mathrm{pr}^\circ\widetilde{\rho}^C_\pi=\eta\otimes\rho_{\pi^\flat}.$$

Define $\rho_\pi^C := \tilde\eta\widetilde{\rho}_\pi^C$. Then it follows that

$$\widetilde{\mathrm{spin}}^{a_\pi}(\rho_\pi^C) = \widetilde{\rho}_\pi^{\mathrm{Sh},\vee} \quad \text{and} \quad \mathrm{pr}^\circ\rho_\pi^C = \rho_{\pi^\flat}.$$

Thanks to (10.1) and (10.5), ρ_π^C satisfies (10.7) and (10.6). The proof is complete. □

Remark 10.3. The bottom row in (10.8) cannot be replaced with PSO_{2n}. (If it did, since $\overline{\rho}_{\pi^\flat}$ has connected image in PSO_{2n} by Proposition 5.5, the argument above would work without introducing the η-twist.) For instance, observe that $\mathrm{GSpin}_{2n} \xrightarrow{\mathrm{spin}} \mathrm{GL}_{2^n} \to \mathrm{PGL}_{2^n}$ does not factor through PSO_{2n} since spin^+ and spin^- have different central characters.

We can refine (10.6) by separating spin^+ and spin^-, which is a key intermediate step towards the main theorem. Our argument is quite delicate and sensitive to the underlying group-theoretic structures.

Proposition 10.4. *Up to replacing ρ_π^C by $\eta\theta(\rho_\pi^C)$ if necessary, we have the following. For every $\mathfrak{p}\notin S^E$ and $\varepsilon\in\{+,-\}$*

$$(10.10)\quad \forall\mathfrak{p}\notin S^E: \qquad \mathrm{spin}^\varepsilon\rho_\pi^C(\mathrm{Frob}_\mathfrak{p})_{\mathrm{ss}} \sim \iota q_\mathfrak{p}^{-n(n-1)/4}\mathrm{spin}^\varepsilon\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{p}) \in \mathrm{GL}_{2^{n-1}}(\overline{\mathbb{Q}}_\ell),$$

$$(10.11)\ \forall\mathfrak{p}\notin S^E_{\mathrm{bad}}: \qquad \mathrm{pr}^\circ\rho_\pi^C(\mathrm{Frob}_\mathfrak{p})_{\mathrm{ss}} \overset{\circ}{\sim} \iota\mathrm{pr}^\circ\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{p}) \in \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell).$$

where we write $\mathfrak{q}$ for the prime of F below $\mathfrak{p}$.

Proof. Recall from §1 that we often write G_0 to mean $G_0(\overline{\mathbb{Q}}_\ell)$ when G_0 is a reductive group over $\overline{\mathbb{Q}}_\ell$. Moreover we assume $\mathfrak{p}\notin S^E$ throughout, without repeating this condition.

The assertion (10.11) follows from (10.7) (and it is invariant under conjugation by an element of GPin_{2n}). The main thing to prove is (10.10). For simplicity, write $\rho := \rho_\pi^C$, $\check\rho^{\mathrm{Sh},\varepsilon} := \rho_\pi^{\mathrm{Sh},\varepsilon,\vee}$, $\rho^\circ := \mathrm{pr}^\circ\rho_\pi^C$, and $a := a_\pi$. From (10.5) and (10.6) we have

$$(10.12)\qquad \check\rho^{\mathrm{Sh},+}\oplus(\eta\otimes\check\rho^{\mathrm{Sh},-}) \simeq \left(\mathrm{spin}^+\rho\oplus(\eta\otimes\mathrm{spin}^-\rho)\right)^{\oplus a}.$$

Write $Z := Z(\mathrm{GSpin}_{2n})$ and H for the Zariski closure of $\mathrm{im}(\rho^\circ)$ in SO_{2n}. Then H contains a regular unipotent element by Corollary 6.8. We divide into two cases based on Proposition 5.5.

Case 1. Assume $\mathrm{spin}^\varepsilon\rho_\pi^C$ is irreducible for $\varepsilon=-$. This happens when H^0 is SO_{2n}, $i^\circ_{\mathrm{std}}(\mathrm{SO}_{2n-1})$, or $n=4$ and $H^0 = \mathrm{spin}^\circ(\mathrm{Spin}_7)$ (possibly after conjugation in GSpin_{2n}). In the first two subcases $\mathrm{spin}^+\rho$ is also irreducible; for irreducibility in the third case, see Lemma 5.2.

If $\mathrm{spin}^+\rho\simeq\eta\otimes\mathrm{spin}^-\rho$ then it is clear from (10.12) that $\check\rho^{\mathrm{Sh},+}\simeq\eta\otimes\check\rho^{\mathrm{Sh},-}\simeq(\mathrm{spin}^+\rho)^{\oplus a}\simeq(\eta\otimes\mathrm{spin}^-\rho)^{\oplus a}$. So the proposition follows from Theorem 9.6.

Henceforth assume that $\mathrm{spin}^+\rho\not\simeq\eta\otimes\mathrm{spin}^-\rho$.

We claim that $\mathrm{spin}^-\rho(\gamma)_{\mathrm{ss}}$ is regular in $\mathrm{GL}_{2^{n-1}}$ on a density 1 set of $\gamma\in\Gamma$. Define X^- to be the subset of $h\in H(\overline{\mathbb{Q}}_\ell)$ such that the semisimple part of $\overline{\mathrm{spin}}^-(h)$ is non-regular in $\mathrm{PGL}_{2^{n-1}}$. Then X^- is Zariski-closed and conjugation-invariant in H. To show $H\neq X^-$, let $\tilde H\subset\mathrm{GSpin}_{2n}$ be the preimage of H, so that $\tilde H^0$ equals Spin_{2n}, $i_{\mathrm{std}}(\mathrm{Spin}_{2n-1})$, or $\overline{\mathrm{spin}}(\mathrm{Spin}_7)$ in the three cases, respectively. Then the restriction of spin^- via $\tilde H^0\hookrightarrow\mathrm{GSpin}_{2n}$ is an irreducible representation with distinct weight vectors. (When $\tilde H^0 = i_{\mathrm{std}}(\mathrm{Spin}_{2n-1})$, the restriction is the spin representation of Spin_{2n-1} by Proposition 4.5.) So some element h_0 of $\tilde H^0$ maps to a regular element of $\mathrm{GL}_{2^{n-1}}$ under spin^-. It follows that some element of H^0 maps to a regular element of $\mathrm{PGL}_{2^{n-1}}$. We know that $\widetilde H$ is a subgroup of $Z(\mathrm{GSpin}_{2n})\widetilde H\subset\mathrm{GSpin}_{2n}$, thus by multiplying h_0 by elements in the center, we obtain in each connected component of H an element with regular image in $\mathrm{GL}_{2^{n-1}}$. In particular for each connected component C of H we have $X^-\cap C\neq C$ and thus $\dim X^- < \dim H$. Therefore the set of γ such that $\rho^\circ(\gamma)\notin X^-$ has density 1 according to Lemma 1.1, and in this case $\overline{\mathrm{spin}^-\rho(\gamma)_{\mathrm{ss}}} = \overline{\mathrm{spin}}^-(\rho^\circ(\gamma)_{\mathrm{ss}})$ is regular. The claim is verified.

Given a square matrix g, let $\mathscr{EV}(g)$ for the multi-set of its eigenvalues. Since $\mathrm{spin}^+\rho\not\simeq\eta\otimes\mathrm{spin}^-\rho$, there exists $\gamma\in\Gamma$ such that

- $\mathrm{spin}^-\rho(\gamma)$ has distinct eigenvalues,
- $\mathscr{EV}(\eta(\gamma)\mathrm{spin}^-\rho(\gamma))\neq\mathscr{EV}(\mathrm{spin}^+\rho(\gamma))$.

In particular there exists an eigenvalue α of $\eta(\gamma)\mathrm{spin}^-\rho(\gamma)$ which is not an eigenvalue of $\mathrm{spin}^+\rho(\gamma)$. Then α appears as an eigenvalue with multiplicity a on the right hand side of (10.12). We know from Theorem 9.6 that each eigenvalue of $\check{\rho}^{\mathrm{Sh},+}(\gamma)$ and $\eta(\gamma)\check{\rho}^{\mathrm{Sh},-}(\gamma)$ appears with multiplicity divisible by a. Thus α is an eigenvalue of either $\check{\rho}^{\mathrm{Sh},+}(\gamma)$ or $\eta(\gamma)\check{\rho}^{\mathrm{Sh},-}(\gamma)$ but not both. This implies, together with Theorem 9.6 and the irreducibility of $\mathrm{spin}^-\rho$, that (i) $(\eta\otimes\mathrm{spin}^-\rho)^{\oplus a}\simeq\eta\otimes\check{\rho}^{\mathrm{Sh},-}$ and $(\mathrm{spin}^+\rho)^{\oplus a}\simeq\check{\rho}^{\mathrm{Sh},+}$, or (ii) $(\eta\otimes\mathrm{spin}^-\rho)^{\oplus a}\simeq\check{\rho}^{\mathrm{Sh},+}$ and $(\mathrm{spin}^+\rho)^{\oplus a}\simeq\eta\otimes\check{\rho}^{\mathrm{Sh},-}$. In case (i), Equation (10.10) follows from Theorem 9.6. If (ii) occurs, replace ρ with $\eta\otimes(\vartheta\rho\vartheta^{-1})$, where $\vartheta\in\mathrm{GPin}_{2n}$ is as in (3.7). (Here $\mathrm{im}(\eta)=\{\pm1\}$ is viewed as the subgroup of $\ker(\mathrm{pr}^\circ)=\mathbb{G}_m$.) Then equations (10.6) and (10.7) are still true (as $\mathrm{pr}^\circ(\eta)=1$). Hence if we run the current proof again, we will be in Case 1(i). We are done in Case 1.

Case 2. We now assume $H^0\subset i^\circ_{\mathrm{std}}(\mathrm{SO}_{2n-1})$, which covers the cases $H^0=i^\circ_{\mathrm{reg}}(\mathrm{PGL}_2)$, $H^0=i^\circ_{\mathrm{std}}(G_2)$ and $n=4$. By Proposition 5.5 we have $H\subset H^0Z(\mathrm{SO}_{2n})$, and ρ has image in the group H_{2n-1} from (5.4). By (10.2), $\eta\rho^\circ$ has dense image in H^0, and by (10.3), $\widetilde{\rho}$ has image in $\mathrm{GSpin}_{2n-1}\subset H_{2n-1}$. In particular η is equal to $\kappa_0\circ\rho$, with κ_0 from (5.6). From (5.7) we obtain

$$\theta\rho_{\mathfrak{p}}\sim\eta_{\mathfrak{p}}\rho_{\mathfrak{p}}\in\mathrm{GSpin}_{2n},\tag{10.13}$$

where we write $\rho_{\mathfrak{p}}:=\rho(\mathrm{Frob}_{\mathfrak{p}})_{\mathrm{ss}}$ and $\eta_{\mathfrak{p}}:=\eta(\mathrm{Frob}_{\mathfrak{p}})$. Similarly we write $\widetilde{\eta}_{\mathfrak{p}}:=\widetilde{\eta}(\mathrm{Frob}_{\mathfrak{p}})$ and $\phi_{\mathfrak{p}}:=\iota q_{\mathfrak{p}}^{-n(n-1)/4}\phi_{\pi_{\mathfrak{q}}}(\mathrm{Frob}_{\mathfrak{p}})$. We claim

$$\theta\phi_{\mathfrak{p}}\sim\eta_{\mathfrak{p}}\phi_{\mathfrak{p}}\in\mathrm{GSpin}_{2n}.\tag{10.14}$$

By Equation (10.1) we have $\mathrm{pr}^\circ\rho_{\mathfrak{p}}\overset{\circ}{\sim}\mathrm{pr}^\circ\phi_{\mathfrak{p}}\in\mathrm{SO}_{2n}$. Multiplying η, we obtain $\mathrm{pr}^\circ(\widetilde{\eta}_{\mathfrak{p}}\phi_p)\overset{\circ}{\sim}\mathrm{pr}^\circ(\widetilde{\eta}_p\rho_{\mathfrak{p}})$. By assumption, $\mathrm{pr}^\circ(\widetilde{\eta}_p\rho_{\mathfrak{p}})\in i^\circ_{\mathrm{std}}(\mathrm{SO}_{2n-1})$. Hence $\widetilde{\eta}_{\mathfrak{p}}\phi_{\mathfrak{p}}=gxg^{-1}$ for some $g\in\mathrm{GPin}_{2n}$ and $x\in i_{\mathrm{std}}(\mathrm{GSpin}_{2n-1})$. We have $\theta(x)=x$ and thus

$$\theta(\widetilde{\eta}_{\mathfrak{p}}\phi_{\mathfrak{p}})=\theta(g)x\theta(g)^{-1}=(\theta(g)g^{-1})\widetilde{\eta}_{\mathfrak{p}}\phi_{\mathfrak{p}}(g\theta(g)^{-1})$$

As $\theta(g)g^{-1}\in\mathrm{GSpin}_{2n}$, this implies that $\theta(\widetilde{\eta}_{\mathfrak{p}}\phi_{\mathfrak{p}})\sim\widetilde{\eta}_{\mathfrak{p}}\phi_{\mathfrak{p}}$. Since $\theta(\widetilde{\eta}_{\mathfrak{p}})=\eta_{\mathfrak{p}}\widetilde{\eta}_{\mathfrak{p}}$, (10.14) follows.

In (10.6) we established

$$\mathrm{spin}^+(\rho_{\mathfrak{p}})\oplus\eta_{\mathfrak{p}}\mathrm{spin}^-(\rho_{\mathfrak{p}})\sim\mathrm{spin}^+(\phi_{\mathfrak{p}})\oplus\eta_{\mathfrak{p}}\mathrm{spin}^-(\phi_{\mathfrak{p}}),$$

which implies by (10.13) and (10.14) $\mathrm{spin}^{+,\oplus2}(\rho_{\mathfrak{p}})\sim\mathrm{spin}^{+,\oplus2}(\phi_{\mathfrak{p}})$. It follows that $\mathrm{spin}^+(\rho_{\mathfrak{p}})\sim\mathrm{spin}^+(\phi_{\mathfrak{p}})$, Similarly we deduce $\mathrm{spin}^-(\rho_{\mathfrak{p}})\sim\mathrm{spin}^-(\phi_{\mathfrak{p}})$. ☐

From now on, we replace, if necessary, ρ^C_π by $\eta\theta(\rho^C_\pi)$ so that the conclusion of Proposition 10.4 holds for ρ^C_π.

Proposition 10.5. *We have that (writing $\mathfrak{q}:=\mathfrak{p}\cap F$)*

$$\forall\mathfrak{p}\notin S^E:\quad\rho^C_\pi(\mathrm{Frob}_{\mathfrak{p}})_{\mathrm{ss}}\sim\iota q_{\mathfrak{p}}^{-n(n-1)/4}\phi_{\pi_{\mathfrak{q}}}(\mathrm{Frob}_{\mathfrak{p}})\in\mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell).\tag{10.15}$$

Proof. We first establish the claim that $\chi_\ell^{n(n-1)/2}\iota\omega_\pi=\mathcal{N}\rho^C_\pi$, where χ_ℓ is the cyclotomic character and we view ω_π as a Galois character via class field theory. In view of Lemma 5.6(i), it suffices to check that

$$\chi_\ell^{n(n-1)/2}\iota\omega_\pi\cdot\mathrm{spin}^\varepsilon(\rho^C_\pi)\simeq\mathcal{N}\rho^C_\pi\cdot\mathrm{spin}^\varepsilon(\rho^C_\pi),\quad\varepsilon\in\{\pm1\}.\tag{10.16}$$

By Lemma 4.2 we have

$$\mathrm{spin}^\varepsilon(\rho^C_\pi)\simeq(\mathrm{spin}^{(-1)^n\varepsilon})^\vee(\rho^C_\pi)\otimes\mathcal{N}\rho^C_\pi.\tag{10.17}$$

Let $\mathfrak{p}\notin S^E$, and write shorthand $\rho_{\mathfrak{p}}:=\rho^C_\pi(\mathrm{Frob}_{\mathfrak{p}})_{\mathrm{ss}}$ and $\phi_{\mathfrak{p}}:=\iota q_{\mathfrak{p}}^{-n(n-1)/4}\phi_{\pi_{\mathfrak{q}}}(\mathrm{Frob}_{\mathfrak{p}})$. We apply (10.10) and compute using Lemma 4.2 again (but now locally)

$$\mathrm{spin}^\varepsilon(\rho_{\mathfrak{p}})\simeq\mathrm{spin}^\varepsilon(\phi_{\mathfrak{p}})\simeq(\mathrm{spin}^{(-1)^n\varepsilon})^\vee(\phi_{\mathfrak{p}})\otimes\mathcal{N}(\phi_{\mathfrak{p}})\simeq(\mathrm{spin}^{(-1)^n\varepsilon})^\vee(\rho_{\mathfrak{p}})\otimes\mathcal{N}(\phi_{\mathfrak{p}})\tag{10.18}$$

We now appeal to functoriality of the Satake isomorphism (unramified local Langlands correspondence) with respect to $\mathbb{G}_{\mathrm{m}}\hookrightarrow\mathrm{GSO}_{2n}$ (dual to $\mathcal{N}:\mathrm{GSpin}_{2n}\to\mathbb{G}_{\mathrm{m}}$), to get $\mathcal{N}(\phi_{\mathfrak{p}})=\chi_\ell^{n(n-1)/2}(\mathrm{Frob}_{\mathfrak{p}})\iota\omega_\pi(\mathrm{Frob}_p)$. Therefore

$$\mathrm{spin}^\varepsilon(\rho^C_\pi)\simeq(\mathrm{spin}^{(-1)^n\varepsilon})^\vee(\rho^C_\pi)\otimes\chi_\ell^{n(n-1)/2}\iota\omega_\pi.$$

Comparing with (10.17), we obtain (10.16). At this point we have established that

$$\begin{aligned} \mathrm{spin}^\varepsilon \rho_{\mathfrak p} &\sim \mathrm{spin}^\varepsilon \phi_{\mathfrak p} \in \mathrm{GL}_{2^{n-1}}(\overline{\mathbb Q}_\ell) && \text{(Prop. 10.4)},\\ \mathrm{pr}^\circ \rho_{\mathfrak p} &\overset{\circ}{\sim} \mathrm{pr}^\circ \phi_{\mathfrak p} \in \mathrm{SO}_{2n}(\overline{\mathbb Q}_\ell) && \text{(Prop. 10.4)},\\ \mathcal N \rho_{\mathfrak p} &= \mathcal N \phi_{\mathfrak p} \in \mathbb G_{\mathrm m}(\overline{\mathbb Q}_\ell) && \text{(claim above)} \end{aligned} \tag{10.19}$$

By [49, Lem. 1.1, table] a semi-simple element γ of $\mathrm{GSpin}_{2n}(\overline{\mathbb Q}_\ell)$ is determined up to conjugacy by the conjugacy classes of $\mathrm{spin}^+\gamma, \mathrm{spin}^-\gamma \in \mathrm{GL}_{2^{n-1}}$, $\mathrm{std}\gamma \in \mathrm{GL}_{2n}$ and $\mathcal N\gamma \in \mathbb G_{\mathrm m}$. We complete the proof by noting that the two sides of (10.15) become conjugate under spin^+, spin^-, std, and $\mathcal N$ by (10.19). $\square$

11. Compatibility at unramified places

We continue in the setup of §10 with the same running assumptions. We determined the image of Frobenius under ρ_π^C at each prime away from some finite set S. Now we compute the image at the finite places $\mathfrak p \nmid \ell$ above $S\backslash S_{\mathrm{bad}}$. The argument follows that of [49, §10] but there are new technical difficulties due to half-spin representations and the automorphism θ.

Proposition 11.1. *Let $\mathfrak p$ be a prime of E not lying above $S_{\mathrm{bad}} \cup \{\ell\}$. Then ρ_π^C is unramified at $\mathfrak p$. Moreover writing $\mathfrak q := \mathfrak p \cap F$,*

$$\rho_\pi^C(\mathrm{Frob}_{\mathfrak p})_{\mathrm{ss}} \sim \iota q_{\mathfrak p}^{-n(n-1)/4} \phi_{\pi_{\mathfrak q}}(\mathrm{Frob}_{\mathfrak p}) \in \mathrm{GSpin}_{2n}(\overline{\mathbb Q}_\ell).$$

Proof. Fix $\mathfrak p$ as in the statement. Let p denote the prime of $\mathbb Q$ below $\mathfrak p$. Let $\pi^\natural$ be a transfer of π from $G^*(\mathbb A_F)$ to $G(\mathbb A_F)$ as in the paragraph above (10.4). Let $B(\pi^\natural)$ be the set of cuspidal automorphic representations τ of $G(\mathbb A_F)$ such that

- $\tau_{\mathfrak q_{\mathrm{St}}}$ and $\pi^\natural_{\mathfrak q_{\mathrm{St}}}$ are isomorphic up to a twist by an unramified character,
- $\tau^{\infty,\mathfrak q_{\mathrm{St}},p}$ and $\pi^{\natural,\infty,\mathfrak q_{\mathrm{St}},p}$ are isomorphic,
- τ_p is unramified,
- τ_∞ is ξ-cohomological

We define an equivalence relation $\approx$ on the set $B(\pi^\natural)$ by declaring that $\tau_1 \approx \tau_2$ if and only if $\tau_2 \in A(\tau_1)$. (Recall the definition of $A(\tau_1)$ from above (9.3); notice that $\tau_1 \approx \tau_2$ if and only if $\tau_{1,\mathfrak q} \simeq \tau_{2,\mathfrak q}$.) To simplify notation, we will write B for a set of representatives for $B(\pi^\natural)/\approx$.

For $\varepsilon \in \{+,-\}$, define (true) representations of Γ_E by $\rho_B^{\mathrm{Sh},\varepsilon} := \sum_{\tau\in B} \rho_\tau^{\mathrm{Sh},\varepsilon}$ (see Theorem 9.6). Put $b(\pi^\natural) := \sum_{\tau \in B} a(\tau) \in \mathbb Z_{>0}$. Since $\rho_\tau^{\mathrm{Sh},\varepsilon,\vee}$ satisfies (10.4) for each $\tau \in B$, we deduce the following on the dual of $\rho_B^{\mathrm{Sh},\varepsilon}$ by comparing the images of Frobenius conjugacy classes at all but finitely many places via (10.4) and (10.10):

$$\rho_B^{\mathrm{Sh},\varepsilon,\vee} \simeq i_{b(\pi^\natural)} \circ \mathrm{spin}^\varepsilon \circ \rho_\pi^C. \tag{11.1}$$

We adapt the argument of Theorem 9.6. Consider the function f on $G(\mathbb A_F)$ of the form $f = f_\infty f_{\mathfrak q_{\mathrm{St}}} \mathbf 1_{K_p} f^{\infty,\mathfrak q_{\mathrm{St}},p}$, where f_∞ and $f_{\mathfrak q_{\mathrm{St}}}$ are as in that argument, and $f^{\infty,\mathfrak q_{\mathrm{St}},p}$ is such that, for all automorphic representations τ of $G(\mathbb A_F)$ with $(\tau^\infty)^K \neq 0$ and $\mathrm{Tr}\,\tau_\infty(f_\infty) \neq 0$, we have:

$$\mathrm{Tr}\,\tau^{\infty,\mathfrak q_{\mathrm{St}},p}(f^{\infty,\mathfrak q_{\mathrm{St}},p}) = \begin{cases} 1 & \text{if } \tau^{\infty,\mathfrak q_{\mathrm{St}},p} \simeq \pi^{\natural,\infty,\mathfrak q_{\mathrm{St}},p},\\ 0 & \text{otherwise.}\end{cases} \tag{11.2}$$

Arguing as in Theorem 9.6 we obtain

$$\iota^{-1}\mathrm{Tr}\,(\mathrm{Frob}_{\mathfrak p}^j, \rho_B^{\mathrm{Sh},\varepsilon}) = \sum_{\tau\in B} a(\tau)\mathrm{Tr}\,\tau_p(f_p^{(j)}) = \sum_{\tau \in B} a(\tau) q_{\mathfrak p}^{jn(n-1)/4} \mathrm{Tr}\,(\mathrm{spin}^{\varepsilon,\vee}(\phi_{\tau_{\mathfrak q}}))(\mathrm{Frob}_{\mathfrak p}^j). \tag{11.3}$$

Define $\rho_B^{\mathrm{Sh}} := \rho_B^{\mathrm{Sh},+} \oplus \rho_B^{\mathrm{Sh},-}$. Applying (11.3) for both $\varepsilon = \pm$ and taking the sum, we obtain

$$\iota^{-1}\mathrm{Tr}\,(\mathrm{Frob}_{\mathfrak p}^j, \rho_B^{\mathrm{Sh}}) = \sum_{\tau\in B} a(\tau) q_{\mathfrak p}^{jn(n-1)/4} \mathrm{Tr}\,(\mathrm{spin}^{\vee}(\phi_{\tau_{\mathfrak q}}))(\mathrm{Frob}_{\mathfrak p}^j). \tag{11.4}$$

Since Xu [81, Thm. 1.8] describes global L-packets for (not only GSp_{2n} but) quasi-split forms of GSO_{2n}, the argument for [49, Lem. 10.2] goes through unchanged, except Corollary 9.7

replaces [49, Cor. 8.4]. This argument shows that $\pi^\natural$ and $\tau\otimes\omega$ belong to the same global packet in Xu's paper for an automorphic quadratic character $\omega\colon \mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)\to\mathbb{C}^\times$. Since each member of the packet in [81] is a θ-orbit of representations, this tells us that either $\pi^\natural_x\simeq\tau_x\otimes\omega_x$ or $\theta(\pi^\natural_x)\simeq\tau_x\otimes\omega_x$ at almost all places x (where both $\pi^\natural_x$, τ_x, and ω_x are unramified). Since $\pi^\natural_x\simeq\tau_x$ at almost all x,

$$\pi^\natural_x\simeq\pi^\natural_x\otimes\omega_x \text{ or } \theta(\pi^\natural_x)\simeq\pi^\natural_x\otimes\omega_x. \tag{11.5}$$

Let us define characters $\chi^\varepsilon:\Gamma_E\to\{\pm1\}$ from ω via $\mathrm{spin}^\varepsilon$ as follows. Via class field theory and Galois cohomology (applying [54, Lem. A.1] to $\mathrm{GSO}_{2n}^{E/F}$; see also [79]) we assign to ω the continuous character

$$W_F\to Z(\mathrm{GSpin}_{2n})\rtimes\Gamma_{E/F},$$

whose restriction to W_E factors through a character $c\colon\Gamma_E\to Z(\mathrm{GSpin}_{2n})$. We then define $\chi^\varepsilon:=\mathrm{spin}^\varepsilon(c)$. We deduce

$$\chi^+\mathrm{spin}^+(\rho^C_\pi)\oplus\chi^-\mathrm{spin}^-(\rho^C_\pi)\simeq\mathrm{spin}(\rho^C_\pi), \tag{11.6}$$

by using (11.5) to verify that the semisimplfication of the image of Frobenius matches at almost all places.

By Lemma 5.6(*iii*) we have $\chi^+=\chi^-$. Set $\chi:=\chi^+$. The same lemma tells us that $\chi=1$ or that ρ^C_π has image in the group H_{2n-1} from (5.4) and $\chi=\kappa_0\circ\rho^C_\pi$.

First case. Suppose that $\chi=1$ for every $\tau\in B$. Then, for each $\tau\in B$ there is some $l\in\mathbb{Z}/2\mathbb{Z}$ we have

$$\mathrm{spin}^{\varepsilon,\vee}(\phi_{\tau_{\mathfrak q}})\simeq\mathrm{spin}^{\varepsilon,\vee}(\theta^l(\omega_{\mathfrak q}\phi_{\pi_{\mathfrak q}}))\simeq\chi^{\varepsilon(-1)^l}\mathrm{spin}^{\varepsilon(-1)^l,\vee}(\phi_{\pi_{\mathfrak q}})=\mathrm{spin}^{\varepsilon(-1)^l,\vee}(\phi_{\pi_{\mathfrak q}}).$$

As l does not depend on ε, we obtain from (11.4) that

$$\iota^{-1}\mathrm{Tr}\,\rho^{\mathrm{Sh}}_B(\mathrm{Frob}^j_{\mathfrak p})=b(\pi^\natural)q_{\mathfrak p}^{jn(n-1)/4}\mathrm{Tr}\,(\mathrm{spin}^\vee(\phi_{\pi_{\mathfrak q}}))(\mathrm{Frob}^j_{\mathfrak p}),\qquad j\gg1.$$

Thus $\rho^{\mathrm{Sh}}_B(\mathrm{Frob}_{\mathfrak p})_{\mathrm{ss}}\sim\iota q_{\mathfrak p}^{n(n-1)/4}i_{b(\pi^\natural)}\circ\mathrm{spin}^\vee(\phi_{\pi_{\mathfrak q}})(\mathrm{Frob}_{\mathfrak p})$. Comparing the dual of this with (11.1), we deduce that

$$\mathrm{spin}\rho^C_\pi(\mathrm{Frob}_{\mathfrak p})_{\mathrm{ss}}\sim\iota q_{\mathfrak p}^{-n(n-1)/4}\mathrm{spin}(\phi_{\pi_{\mathfrak q}})(\mathrm{Frob}_{\mathfrak p}).$$

Since we also know the conjugacy relation with std and $\mathcal{N}$ in place of spin from (10.11) and Proposition 10.5 (and the argument at (10.16) in its proof), we use Lemma 5.7 to conclude that

$$\rho^C_\pi(\mathrm{Frob}_{\mathfrak p})_{\mathrm{ss}}\sim\iota q_{\mathfrak p}^{-n(n-1)/4}\theta^k\phi_{\pi_{\mathfrak q}}(\mathrm{Frob}_{\mathfrak p})\in\mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell),\quad\text{ for some }k\in\mathbb{Z}/2\mathbb{Z}. \tag{11.7}$$

If $k=0$, then (11.7) implies the theorem. So we assume $k=1$ in the rest of the argument.

We now distinguish between those $\tau\in B$ according to whether or not their Satake parameter at $\mathfrak q$ becomes conjugate to that of π under spin^+ and spin^-: Write

$$B_{\mathrm{good}}:=\{\tau\in B\,|\,\mathrm{spin}^\varepsilon(\phi_{\tau_{\mathfrak q}})\simeq\mathrm{spin}^\varepsilon(\phi_{\pi_{\mathfrak q}}),\ \varepsilon\in\{+,-\}\} \tag{11.8}$$

and $B_{\mathrm{bad}}:=B-B_{\mathrm{good}}$. Thus (11.3) implies

$$\begin{aligned} q_{\mathfrak p}^{-jn(n-1)/4}\iota^{-1}b(\pi^\natural)\mathrm{Tr}\,\mathrm{spin}^\varepsilon(\rho^C_\pi)(\mathrm{Frob}^j_{\mathfrak p})=&\sum_{\tau\in B_{\mathrm{good}}}a(\tau)\mathrm{Tr}\,(\mathrm{spin}^\varepsilon(\phi_{\pi_{\mathfrak q}}))(\mathrm{Frob}^j_{\mathfrak p})\\ &+\sum_{\tau\in B_{\mathrm{bad}}}a(\tau)\mathrm{Tr}\,(\mathrm{spin}^{-\varepsilon}(\phi_{\pi_{\mathfrak q}}))(\mathrm{Frob}^j_{\mathfrak p}).\end{aligned} \tag{11.9}$$

Equation (11.7) and (11.9) imply that $\mathrm{spin}^{-\varepsilon}(\phi_{\pi_{\mathfrak q}})^{b(\pi^\natural)}\simeq\mathrm{spin}^\varepsilon(\phi_{\pi_{\mathfrak q}})^{b_0}\oplus\mathrm{spin}^{-\varepsilon}(\phi_{\pi_{\mathfrak q}})^{b_1}$, as $W_{E_{\mathfrak p}}$-representations, where $b_0=\sum_{\tau\in B_{\mathrm{good}}}a(\tau)$, $b_1=\sum_{\tau\in B_{\mathrm{bad}}}a(\tau)$ and $b(\pi^\natural)=b_0+b_1$. Thus

$$\mathrm{spin}^{-\varepsilon}(\phi_{\pi_{\mathfrak q}})^{b(\pi^\natural)-b_1}\simeq\mathrm{spin}^\varepsilon(\phi_{\pi_{\mathfrak q}})^{b_0}. \tag{11.10}$$

As $\pi^\natural$ contributes to B_{good}, we have $b_0=b(\pi^\natural)-b_1>0$. Thus $\mathrm{spin}^{-\varepsilon}(\phi_{\pi_{\mathfrak q}})\simeq\mathrm{spin}^\varepsilon(\phi_{\pi_{\mathfrak q}})$, and $\phi_{\pi_{\mathfrak q}}\sim\theta\phi_{\pi_{\mathfrak q}}$, in which case the proposition follows from (11.7). Here we applied Lemma 1.1 of [49] and the fact that $\mathrm{spin}^\pm$, std, $\mathcal{N}$ are fundamental representations (see table above Lemma 1.1 in [*loc. cit*]).

Second case. Suppose that $\chi \neq 1$ for some $\tau \in B$. As explained, then ρ_π^C has image in the group H_{2n-1} from (5.4). We obtain from (11.4) and (11.1) that

$$\iota^{-1}b(\pi^\natural)\mathrm{Tr}\left(\mathrm{spin}(\rho_\pi^C)\oplus\chi\mathrm{spin}(\rho_\pi^C)\right)(\mathrm{Frob}_\mathfrak{p}^j) = \tag{11.11}$$
$$= \sum_{\tau\in B} a(\tau)(1+\chi(\mathrm{Frob}_\mathfrak{p})^j)q_\mathfrak{p}^{jn(n-1)/4}\mathrm{Tr}\left(\mathrm{spin}(\phi_{\tau_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{p})^j)\right).$$

For each $\tau \in B$ there exist $a, b \in \mathbb{Z}/2\mathbb{Z}$ such that for both $\varepsilon \in \{\pm1\}$ we have $\mathrm{spin}^\varepsilon(\phi_{\tau_\mathfrak{p}}) \simeq \chi^b\mathrm{spin}^{(-1)^a\varepsilon}(\phi_{\pi_\mathfrak{p}})$. Thus we have

$$(1\oplus\chi)\otimes\mathrm{spin}(\phi_{\tau_\mathfrak{p}}) \simeq (\chi^b\oplus\chi^{b+1})\otimes\left[\mathrm{spin}^{(-1)^a}(\phi_{\pi_\mathfrak{p}})\oplus\mathrm{spin}^{-(-1)^a}(\phi_{\pi_\mathfrak{p}})\right] \simeq (1\oplus\chi)\otimes\mathrm{spin}(\phi_{\pi_\mathfrak{p}}).$$

as $W_{E_\mathfrak{p}}$-representations. In particular, on the right hand side of (11.11) we may replace $\phi_{\tau_\mathfrak{p}}$ by $\phi_{\pi_\mathfrak{p}}$. We have $\chi\mathrm{spin}^+(\rho_\pi^C) \simeq \mathrm{spin}^-(\rho_\pi^C)$ by Lemma 5.6(ii) and so $\chi\mathrm{spin}(\rho_\pi^C) \simeq \mathrm{spin}(\rho_\pi^C)$. By removing the multiplicity $b(\pi^\natural)$ on both sides of (11.11) we thus find that

$$\mathrm{spin}(\rho_\pi^C)^{\oplus 2}|_{W_{E_\mathfrak{p}}} \simeq [1\oplus\chi]\otimes\mathrm{spin}(\phi_{\pi_\mathfrak{q}})\otimes|\cdot|_\mathfrak{q}^{-jn(n-1)/4}. \tag{11.12}$$

We claim that in fact also $\chi\mathrm{spin}^+(\phi_{\pi_\mathfrak{q}}) \simeq \mathrm{spin}^-(\phi_{\pi_\mathfrak{q}})$. If true, (11.12) would imply that

$$\mathrm{spin}(\rho_\pi^C)|_{W_{E_\mathfrak{p}}} \simeq \mathrm{spin}(\phi_{\pi_\mathfrak{q}})\otimes|\cdot|_\mathfrak{q}^{-jn(n-1)/4}. \tag{11.13}$$

We check the claim. As pr° surjects $Z(\mathrm{GSpin}_{2n})$ onto $Z(\mathrm{SO}_{2n})$, we see that pr° induces an isomorphism from the component group of H_{2n-1} to the component group of $H_{2n-1}^\circ = \mathrm{SO}_{2n-1}Z(\mathrm{SO}_{2n})$. Consequently, χ (which equals $\kappa_0\circ\rho_\pi^C$ by Lemma 5.6(iii)) is equal to the composition

$$\Gamma_E \overset{\rho_{\pi^\flat}^C}{\to} H_{2n-1}^\circ \twoheadrightarrow \{\pm1\}.$$

We know that $\mathrm{pr}^\circ(\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{p})) \overset{\circ}{\sim} \mathrm{pr}^\circ(\rho_\pi^C(\mathrm{Frob}_\mathfrak{p})_{\mathrm{ss}}) \in \mathrm{SO}_{2n}$ since they become conjugate after applying std. Therefore $\chi|_{W_{E_\mathfrak{p}}}$ equals

$$W_{E_\mathfrak{p}} \overset{\phi_{\pi^\flat_\mathfrak{q}}}{\to} H_{2n-1}^\circ \twoheadrightarrow \{\pm1\}$$

and hence equals $W_{E_\mathfrak{p}} \overset{\phi_{\pi_\mathfrak{q}}}{\to} H_{2n-1} \twoheadrightarrow \{\pm1\}$ (the argument is similar to the one below (10.14)), which in turn implies that $\chi\mathrm{spin}^+(\phi_{\pi_\mathfrak{q}}) \simeq \mathrm{spin}^-(\phi_{\pi_\mathfrak{q}})$. Hence the claim is proved, and (11.13) holds true as observed above.

We thus find (11.7) again. If $k = 0$ in that equation, we are done. Now assume $k = 1$. Define $B_{\mathrm{good}}, B_{\mathrm{bad}}$ as in (11.8). As $\chi\mathrm{spin}^+(\phi_{\pi_\mathfrak{q}}) \simeq \mathrm{spin}^-(\phi_{\pi_\mathfrak{q}})$, it follows from (11.6) that for each $\tau \in B_{\mathrm{bad}}$, we have $\mathrm{spin}^\varepsilon(\phi_{\tau_\mathfrak{q}}) \simeq \mathrm{spin}^{-\varepsilon}(\phi_{\pi_\mathfrak{q}})$ for both signs $\varepsilon \in \{\pm1\}$. Thus we obtain (11.9) with $\#B_{\mathrm{good}} > 0$ again. By the same argument as in (11.10) we deduce that $\phi_{\pi_\mathfrak{q}} \sim \theta\phi_{\pi_\mathfrak{q}}$, in which case the proposition follows from (11.7). □

12. The main theorem

In this section we prove Theorem A (Theorem 12.5), the main result of this paper. Before doing this, we switch the normalization for π from (coh) to (L-coh), and extend the Galois action from Γ_E to Γ_F.

As in Theorem A, let π be a cuspidal automorphic representation of $G^*(\mathbb{A}_F)$ satisfying (St) and (L-coh). Fix a cuspidal automorphic representation $\pi^\flat$ of $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ which embeds in $\pi|_{\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)}$ as it is possible by [51]. Assume either (std-reg) for π or Hypothesis 6.11 for an $\mathrm{SO}_{2n}(\mathbb{A}_F)$-subrepresentation $\pi^\flat$ of π. Define $\tilde\pi := \pi|\mathrm{sim}|^{-n(n-1)/4}$. Then $\tilde\pi$ is ξ-cohomological and will play the role of π in Sections 10 and 11. Naturally $\pi^\flat$ is a subrepresentation of $\tilde\pi|_{\mathrm{SO}_{2n}(\mathbb{A}_F)}$ since $|\mathrm{sim}|$ is trivial when restricted to $\mathrm{SO}_{2n}(\mathbb{A}_F)$.

Let S^F (resp. S^E) be the finite set of places of F (resp. E) above $S := S_{\mathrm{bad}} \cup \{\ell\}$. Fix an infinite place $y_\infty \in \mathcal{V}_\infty$ and also fix a finite place $\mathfrak{q}$ as in (St). (Recall that the group G, Shimura varieties, and the resulting GSpin_{2n}-valued Galois representations in earlier sections depend on the choice of y_∞ and $\mathfrak{q}$.) From Propositions 10.2 and 11.1, we obtain

$$\rho^C_{\tilde\pi}\colon \Gamma_{E,S} \to \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$$

such that for every $\mathfrak{p} \notin S^E$, writing $\mathfrak{q} := \mathfrak{p}|_F$, we have

$$\rho^C_{\tilde\pi}(\mathrm{Frob}_\mathfrak{p})_{\mathrm{ss}} \sim \iota q_\mathfrak{p}^{-n(n-1)/4}\phi_{\tilde\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{p}) = \iota\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{p}). \tag{12.1}$$

Let us explain the definition of ρ_π on $\Gamma_{F,S}$. If n is even (thus $E = F$) then we simply take $\rho_\pi := \rho^C_{\tilde\pi}$. In case n is odd (so $[E:F] = 2$), write $c_{y_\infty} \in \Gamma$ for the complex conjugation corresponding to y_∞ (canonical up to conjugacy). In order to apply Lemma A.1, we check

Lemma 12.1. *When n is odd, we have ${}^{c_{y_\infty}}\rho^C_{\tilde\pi} \simeq \theta \circ \rho^C_{\tilde\pi}$.*

Proof. In light of Proposition 5.8, it is enough to check this locally, namely that

$$\rho^C_{\tilde\pi}(c_{y_\infty}\mathrm{Frob}_\mathfrak{p} c_{y_\infty}^{-1})_{\mathrm{ss}} \sim \theta\circ\rho^C_{\tilde\pi}(\mathrm{Frob}_\mathfrak{p})_{\mathrm{ss}} \qquad \text{in } \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$$

for almost all primes $\mathfrak{p}$ of E. For each $\mathfrak{p}$, write $\mathfrak{q} := \mathfrak{p}\cap F$. Firstly if $\mathfrak{q}$ splits in E as $\mathfrak{p}c(\mathfrak{p})$ then we use (12.1) to deduce that

$$\rho^C_{\tilde\pi}(c_{y_\infty}\mathrm{Frob}_\mathfrak{p} c_{y_\infty}^{-1})_{\mathrm{ss}} \sim \rho^C_{\tilde\pi}(\mathrm{Frob}_{c(\mathfrak{p})})_{\mathrm{ss}} \sim \iota\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_{c(\mathfrak{p})}) \sim \iota\theta(\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{p})) \sim \theta(\rho^C_{\tilde\pi}(\mathrm{Frob}_\mathfrak{p})).$$

(To see the third conjugacy relation, we argue as follows. From (8.4) we see that an element of $\mathrm{GSO}^{E/F}_{2n,F_\mathfrak{q}}$ has the form $(g,\theta(g))$ with $g \in \mathrm{GSO}_{2n,E_\mathfrak{p}}$ and that $\mathrm{GSO}^{E/F}_{2n,F_\mathfrak{q}}$ is isomorphic to $\mathrm{GSO}_{2n,E_\mathfrak{p}}$ and $\mathrm{GSO}_{2n,E_{c(\mathfrak{p})}}$ by the projection map onto the first and second components, respectively. Likewise the dual group of $\mathrm{GSO}^{E/F}_{2n,F_\mathfrak{q}}$ is naturally the subgroup of $\mathrm{GSpin}_{2n}\times\mathrm{GSpin}_{2n}$ consisting of elements of the form $(g,\theta(g))$, the two components corresponding to $\mathfrak{p}$ and $c(\mathfrak{p})$. It follows that $\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_{c(\mathfrak{p})}) \sim \theta(\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{p}))$.)

Secondly if $\mathfrak{q}$ is inert in E then $c_{y_\infty}\mathrm{Frob}_\mathfrak{p}c_{y_\infty}^{-1} \sim \mathrm{Frob}_\mathfrak{p}$. Thus we need to check that the conjugacy class of $\rho^C_{\tilde\pi}(\mathrm{Frob}_\mathfrak{p})_{\mathrm{ss}}$ is θ-invariant. Writing $\theta(\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{q})) = s \rtimes c \in \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)\rtimes\Gamma_{E/F}$,

$$\theta(\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{p})) \sim \theta(\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{q}^2)) = s\theta(s) \sim \theta(s)s \qquad \text{in } \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell).$$

This implies the desired θ-invariance via (12.1). The proof is complete. □

We are assuming that n is odd. By Lemmas 12.1 and A.1, we extend $\rho^C_{\tilde\pi}$ to a Galois representation to be denoted ρ_π:

$$\rho_\pi\colon \Gamma_{F,S} \to \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)\rtimes\Gamma_{E/F}. \tag{12.2}$$

There are two choices up to conjugacy (Example A.6). We choose one arbitrarily and possibly modify the choice below.

We return to treating both parities of n. We fixed $\pi^\flat$ above. Theorem 6.5, or Hypothesis 6.11 if (std-reg) is not assumed, supplies us with

$$\rho_{\pi^\flat}\colon \Gamma_{F,S} \to \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)\rtimes\Gamma_{E/F}$$

such that $\rho_{\pi^\flat}(\mathrm{Frob}_\mathfrak{q})_{\mathrm{ss}} \overset{\circ}{\sim} \iota\phi_{\pi^\flat_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{q})$ for $\mathfrak{q},\mathfrak{p}$ as above. Thanks to (12.1) and the unramified Langlands functoriality with respect to $\mathrm{SO}_{2n} \to \mathrm{GSO}^{E/F}_{2n}$ (whose dual morphism is pr°),

$$\rho_{\pi^\flat}(\mathrm{Frob}_\mathfrak{p})_{\mathrm{ss}} \overset{\circ}{\sim} \iota\phi_{\pi^\flat_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{p}) \sim \iota\,\mathrm{pr}^\circ(\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{p})) \sim \mathrm{pr}^\circ(\rho_\pi(\mathrm{Frob}_\mathfrak{p})_{\mathrm{ss}}).$$

Thus the conjugacy classes at the left and right ends are $\mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$-conjugate, under the identification $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)\rtimes\Gamma_{E/F} = \mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$. Since O_{2n} is acceptable, $\rho_{\pi^\flat}|_{\Gamma_{E,S}}$ and $\mathrm{pr}^\circ\circ\rho_\pi|_{\Gamma_{E,S}}$ are $\mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$-conjugate. Replacing $\rho_{\pi^\flat}$ by an $\mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$-conjugate, we may and will assume that

$$\rho_{\pi^\flat}|_{\Gamma_{E,S}} = \mathrm{pr}^\circ\circ\rho_\pi|_{\Gamma_{E,S}}$$

without disturbing the validity of (SO-i) through (SO-v) in Theorem 6.5. When n is odd, we take an extra step as follows. Observe that $\rho_{\pi^\flat}$ and $\mathrm{pr}^\circ\circ\rho_\pi$ are two $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)\rtimes\Gamma_{E/F}$-valued representations of $\Gamma_{F,S}$ extending (12.3). If they are not equal then $\mathrm{pr}^\circ\circ\rho_\pi = \rho_{\pi^\flat}\otimes\chi_{E/F}$ by

Example A.5 with $\chi_{E/F}:\Gamma_F \twoheadrightarrow \Gamma_{E/F} \xrightarrow{\sim} \{\pm1\}$. Then we go back to (12.2) and replace ρ_π with $\rho_\pi\otimes\chi$, where χ is as in Example A.6; this does not affect the discussion between (12.2) and here. Since $\mathrm{pr}^\circ\circ\chi=\chi_{E/F}$, this ensures that

$$\rho_{\pi^\flat}=\mathrm{pr}^\circ\circ\rho_\pi. \tag{12.3}$$

As in §2, let $(s_0,s_1,\dots,s_n)\in(\overline{\mathbb{Q}}_\ell^\times)^{n+1}$ denote an element of $T_{\mathrm{GSpin}}(\overline{\mathbb{Q}}_\ell)\subset\mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$. This element maps to $\mathrm{diag}(s_1,\dots,s_n,s_1^{-1},\dots,s_n^{-1})\in\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ under pr°, and maps to $s_0^2s_1s_2\cdots s_n$ under the spinor norm $\mathcal{N}$.

Lemma 12.2. *At every infinite place y of F, the following are $\mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$-conjugate:*

$$\rho_\pi(c_y)\sim\begin{cases}(a,\underbrace{1,\dots,1}_{n/2},\underbrace{-1,\dots,-1}_{n/2}),\ a\in\{\pm1\}, & n:\text{even},\\ (1,\underbrace{1,\dots,1}_{(n-1)/2},\underbrace{-1,\dots,-1}_{(n-1)/2},1)\rtimes c, & n:\text{odd}.\end{cases} \tag{12.4}$$

where the right hand side lies in $\mathrm{T}_{\mathrm{GSpin}}(\overline{\mathbb{Q}}_\ell)\rtimes\mathrm{Gal}(E/F)$.

Proof. In light of (12.3) (which is valid for both odd and even n as discussed above) and Theorem 6.5 (SO-v) (or Hypothesis 6.11) which describes $\rho_{\pi^\flat}(c_y)$, the following are $\mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$-conjugate:

$$\mathrm{pr}^\circ(\rho_\pi(c_y))\sim\begin{cases}\mathrm{diag}(\underbrace{1,\dots,1}_{n/2},\underbrace{-1,\dots,-1}_{n/2},\underbrace{1,\dots,1}_{n/2},\underbrace{-1,\dots,-1}_{n/2}), & n:\text{even},\\ \mathrm{diag}(\underbrace{1,\dots,1}_{(n-1)/2},\underbrace{-1,\dots,-1}_{(n-1)/2},1,\underbrace{1,\dots,1}_{(n-1)/2},\underbrace{-1,\dots,-1}_{(n-1)/2},1)\rtimes\theta, & n:\text{odd}.\end{cases}$$

Therefore $\rho_\pi(c_y)$ is a lift of the right hand side (up to $\mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$-conjugacy) via pr°. Moreover $\rho_\pi(c_y)^2=\rho_\pi(c_y^2)=1$. We claim that these two conditions imply (12.4).

This is straightforward when n is even. Now suppose that n is odd. Evidently the right hand side of (12.4) satisfies the two conditions. Any other lift of order 2 can only differ (possibly after conjugation) from the right hand side of (12.4) by scalars $\{\pm1\}$. (Use Lemma 3.1 (ii) and the order two condition.) This implies (12.4) since every $g\in\mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)\rtimes c$ is conjugate to $-g$; indeed, $-g=\zeta g\zeta^{-1}$ if $\zeta\in Z_{\mathrm{Spin}}(\overline{\mathbb{Q}}_\ell)$ is an element of order 4, noting that $\theta(\zeta)=\zeta^{-1}$. □

Let $\omega_\pi: F^\times\backslash\mathbb{A}_F^\times\to\mathbb{C}^\times$ denote the central character of π. By abuse of notation, we still write ω_π (depending on the choice of ι) for the ℓ-adic character of Γ_F corresponding to ω_π via class field theory (as in [33, pp.20–21]). To make ω_π explicit, recall that $\tilde\pi=\pi|\mathrm{sim}|^{-n(n-1)/4}$ is ξ-cohomological. By condition (cent), the central character of ξ is $z\mapsto z^w$ on $F_y^\times$ at every real place y of F, for an integer w independent of y. Therefore (recalling sim is the squaring map on the center)

$$\omega_{\pi,y}(z)=z^{-w}|z|^{n(n-1)/2}=\mathrm{sgn}(z)^w|z|^{-w+n(n-1)/2},\qquad z\in F_y^\times.$$

Then $\omega_\pi|\cdot|^{w-n(n-1)/4}$ is a finite-order Hecke character which is sgn^w at every real place. Hence $\omega_\pi=\chi_{\mathrm{cyc}}^{-w+n(n-1)/2}\chi_0$, where χ_{cyc} is the ℓ-adic cyclotomic character, and χ_0 a finite-order character with $\chi_0(c_y)=(-1)^w$ at each real place y. The upshot is that

$$\omega_\pi(c_y)=(-1)^{-w+n(n-1)/2}(-1)^w=(-1)^{n(n-1)/2},\qquad y:\text{real place of }F. \tag{12.5}$$

We are ready to upgrade (12.1) to a compatibility at places of F for odd n (thus $[E:F]=2$).

Corollary 12.3. *We have $\mathcal{N}\circ\rho_\pi=\omega_\pi$. Moreover, at every finite place $\mathfrak{q}$ of F not above $S_{\mathrm{bad}}\cup\{\ell\}$,*

$$\rho_\pi(\mathrm{Frob}_{\mathfrak{q}})_{\mathrm{ss}}\sim\iota\phi_{\pi_{\mathfrak{q}}}(\mathrm{Frob}_{\mathfrak{q}}).$$

Remark 12.4. The corollary is certainly not automatic from (12.1) since the unramified base change from $G^*(F_{\mathfrak{q}})$ to $G^*(E_{\mathfrak{p}})$ is not injective when $\mathfrak{q}$ does not split in E. Curiously our proof crucially relies on the image of complex conjugation. We have not found a local or global proof only using properties at finite places.

Proof. Via the unramified Langlands functoriality with respect to the central embedding $\mathbb{G}_m \hookrightarrow \mathrm{GSO}_{2n}^{E/F}$, (12.1) implies that $\mathcal{N} \circ \rho_\pi|_{\Gamma_E} = \omega_\pi|_{\Gamma_E}$. If n is even then $E = F$ so there is no more to prove as the latter assertion is already true by (12.1).

Henceforth assume that n is odd (so $[E : F] = 2$). Then either $\mathcal{N} \circ \rho_\pi = \omega_\pi$ or $\mathcal{N} \circ \rho_\pi = \omega_\pi \otimes \chi_{E/F}$, where $\chi_{E/F} \colon \Gamma_F \twoheadrightarrow \Gamma_{E/F} \xrightarrow{\sim} \{\pm 1\}$. To exclude the latter case, let y be a real place of F. We have $\mathcal{N}(\rho_\pi(c_y)) = (-1)^{(n-1)/2}$ from Lemma 12.2, and $\omega_\pi(c_y) = (-1)^{n(n-1)/2}$ from (12.5), but clearly $\chi_{E/F}(c_y) = -1$. Then the only possibility is that $\mathcal{N} \circ \rho_\pi = \omega_\pi$.

We prove the second assertion. If $\mathfrak{q}$ splits in E, this follows immediately from (12.1) for $\rho_\pi|_{\Gamma_{E,S}}$. Henceforth assume that $\mathfrak{q}$ is inert in E. We have seen that $\mathrm{pr}^\circ \circ \rho_\pi|_{\Gamma_{E,S}} = \rho_{\pi^\flat}|_{\Gamma_{E,S}}$. Theorem 6.5 (SO-i) (or Hypothesis 6.11) tells us that

$$\rho_{\pi^\flat_{\mathfrak{q}}}(\mathrm{Frob}_{\mathfrak{q}})_{\mathrm{ss}} \sim \iota\phi_{\pi^\flat_{\mathfrak{q}}}(\mathrm{Frob}_{\mathfrak{q}}) = \iota\mathrm{pr}^\circ(\phi_{\pi_{\mathfrak{q}}}(\mathrm{Frob}_{\mathfrak{q}})).$$

(Note that the outer automorphism ambiguity disappears as it is absorbed by the SO_{2n}-conjugacy on the nontrivial coset of $\mathrm{SO}_{2n} \rtimes \Gamma_{E/F}$; since $\mathfrak{q}$ is inert in E, the image of $\mathrm{Frob}_{\mathfrak{q}}$ in $\Gamma_{E/F}$ is nontrivial.) Therefore $\rho_\pi(\mathrm{Frob}_{\mathfrak{q}})_{\mathrm{ss}} \sim z\iota\phi_{\pi_{\mathfrak{q}}}(\mathrm{Frob}_{\mathfrak{q}})$ for some $z \in \overline{\mathbb{Q}}_\ell^\times$. Taking the spinor norm,

$$\mathcal{N}(z) = (\mathcal{N} \circ \rho_\pi(\mathrm{Frob}_{\mathfrak{q}})_{\mathrm{ss}})\mathcal{N}(\iota\phi_{\pi_{\mathfrak{q}}}(\mathrm{Frob}_{\mathfrak{q}}))^{-1} = \omega_\pi(\mathrm{Frob}_{\mathfrak{q}})\omega_\pi(\mathrm{Frob}_{\mathfrak{q}})^{-1} = 1.$$

It follows that $z \in \{\pm 1\}$. Since every $g \in \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes c$ is conjugate to $-g$ (proof of Lemma 12.2), we conclude that $\rho_\pi(\mathrm{Frob}_{\mathfrak{q}})_{\mathrm{ss}}$ is conjugate to $\iota\phi_{\pi_{\mathfrak{q}}}(\mathrm{Frob}_{\mathfrak{q}})$. $\square$

Theorem 12.5. *Theorem A is true.*

Proof. Let π be as in the theorem. We fix an automorphic representation $\pi^\flat$ of $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ in $\pi|_{\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)}$, take $\rho_{\pi^\flat} : \Gamma_F \to \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$ to be as in Theorem 6.5, or Hypothesis 6.11 if (std-reg) is false, and define

$$\rho_\pi : \Gamma_F \to \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F} \tag{12.6}$$

such that $\rho_{\pi^\flat} = \mathrm{pr}^\circ \circ \rho_\pi$ as explained at the start of this section. We can inflate ρ_π to a representation $\Gamma_F \to \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_F$ of Theorem A, but we work with ρ_π in the form of (12.6) as this is harmless for verifying the theorem.

The equality $\rho_{\pi^\flat} = \mathrm{pr}^\circ \circ \rho_\pi$ and Corollary 12.3 imply (A2). Corollary 12.3 exactly gives (A1). Item (A4) is straightforward from Lemma 12.2. To see (A5), note that the image of ρ_π in $\mathrm{PSO}_{2n}(\overline{\mathbb{Q}}_\ell)$ is the same as the image of $\rho_{\pi^\flat}$ in the same group. The Zariski closure of the image is (possibly disconnected and) reductive since $\rho_{\pi^\flat}$ is semisimple and contains a regular unipotent element by Corollary 6.8. Hence (A5) is implied by Proposition 5.5. Now ρ_π also contains a regular unipotent in the image, so (A6) and the uniqueness of ρ_π up to conjugacy are consequences of Proposition 5.8.

It remains to verify (A3). We begin with part (b). If $\pi_{\mathfrak{q}}$ has nonzero invariants under a hyperspecial (resp. Iwahori) subgroup, then $\pi^\flat_{\mathfrak{q}}$ and $\omega_{\pi,\mathfrak{q}}$ enjoy the same property. Therefore (b) follows from (A2) and Theorem 6.5 (SO-iv). To prove part (c), write $\mathfrak{p}$ for a place of E above $\mathfrak{q}$. Since $\mathfrak{p}$ is unramified over E, it suffices to check that $\rho_\pi|_{\Gamma_E}$ is crystalline at $\mathfrak{p}$. Moreover we may assume that $F \neq \mathbb{Q}$ by the automorphic base change of [49, Prop. 6.6] and (A6). (If $F = \mathbb{Q}$ then replace F with a real quadratic field F' in which ℓ is split, and E with EF'. By (A6), $\rho_\pi|_{\Gamma_{F'}} \simeq \rho_{\pi_{F'}}$, where $\pi_{F'}$ is the base change of π to F' constructed in *loc. cit.* Thus the question is now about $\rho_{\pi_{F'}}$.) Now that $F \neq \mathbb{Q}$, the Shimura varieties in §9 are proper, and $\rho_{\pi^\natural}^{\mathrm{Sh},\varepsilon}$ is crystalline at all places above ℓ by [57]. Since $\mathrm{spin} \circ \rho_\pi|_{\Gamma_E}$ embeds in $\rho_{\pi^\natural}^{\mathrm{Sh},+} \oplus \rho_{\pi^\natural}^{\mathrm{Sh},-}$ (which is isomorphic to the $a(\pi^\natural)$-fold direct sum of $\mathrm{spin} \circ \rho_\pi$), and since spin is faithful, we deduce that $\rho_\pi|_{\Gamma_E}$ is crystalline at $\mathfrak{p}$ as desired.

Finally we prove (A3), part (a). We first claim that if two cocharacters $\mu_1, \mu_2 \in X_*(T_{\mathrm{GSpin}})$ become conjugate after composition with each of $\mathrm{spin}^{+,\vee}$, $\mathrm{spin}^{-,\vee}$, std°, and $\mathcal{N}$, then μ_1 and μ_2 are GSpin_{2n}-conjugate. To see this, note that a semi-simple conjugacy class γ in $\mathrm{GSpin}_{2n}(\mathbb{C})$ is determined by the conjugacy classes $\mathrm{spin}^\pm(\gamma)$, $\mathcal{N}(\gamma)$ and $\mathrm{std}(\gamma)$ by [49, Lem. 1.3] (thus also determined by $\mathrm{spin}^{\pm,\vee}(\gamma)$, $\mathcal{N}(\gamma)$ and $\mathrm{std}(\gamma)$) and the table above Lemma 1.1 therein. The same statement holds for the cocharacters via the Weyl group-equivariant isomorphism $X_*(T_{\mathrm{GSpin}})\otimes_{\mathbb{Z}}\mathbb{C}^\times \to T_{\mathrm{GSpin}}(\mathbb{C})$, proving the claim.

Our second claim is that for every $y : F \hookrightarrow \mathbb{C}$,

$$\mathrm{spin}^\varepsilon(\mu_{\mathrm{HT}}(\rho_{\pi,\mathfrak{q}}, \iota y)) \sim \mathrm{spin}^\varepsilon(\iota\mu_{\mathrm{Hodge}}(\xi_y) - \tfrac{n(n-1)}{4}\mathrm{sim}), \qquad \varepsilon\in\{\pm\}. \tag{12.7}$$

Accept this for now. The representations std° and $\mathcal{N}$ factor over the isogeny $(\mathcal{N}, \mathrm{pr}^\circ)$: $\mathrm{GSpin}_{2n} \to \mathbb{G}_{\mathrm{m}} \times \mathrm{SO}_{2n}$, so it follows easily from (A2) and Theorem 6.5 (SO-iii) that (12.7) holds with std° and $\mathcal{N}$ in place of $\mathrm{spin}^\varepsilon$. Thus we can conclude by the first claim.

To complete the proof of (A3)(a), we check the second claim (12.7). A base-change argument as in the preceding paragraph allows us to assume that $F \neq \mathbb{Q}$. Recall that we chose an embedding $y_\infty : F \hookrightarrow \mathbb{C}$ in the definition of G and the Shimura data $(\mathrm{Res}_{F/\mathbb{Q}}G, X^\varepsilon)$. It follows from Proposition 9.8 (applicable as $F \neq \mathbb{Q}$) that

$$\mu_{\mathrm{HT}}(\mathrm{spin}^{\varepsilon,\vee}\circ\rho_\pi, \iota y_\infty) \;\sim\; \mathrm{spin}^{\varepsilon,\vee}\circ\Big(\iota\mu_{\mathrm{Hodge}}(\xi_{y_\infty}) - \tfrac{n(n-1)}{4}\mathrm{sim}\Big), \tag{12.8}$$

We can repeat the construction from the beginning of §12 up to now, with $y : F \hookrightarrow \mathbb{C}$ in place of y_∞. Write $\rho_\pi(y) : \Gamma_F \to \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$ for the resulting Galois representation. From (A1) and (A6) of Theorem A (which have already been verified) to ρ_π and $\rho_\pi(y)$, we deduce that $\rho_\pi \simeq \rho_\pi(y)$. Applying Proposition 9.8 to $\rho_\pi(y)$, we see that (12.8) holds with y in place of y_∞.

Now the left hand side of (12.8) equals $\mathrm{spin}^{\varepsilon,\vee}\circ\mu_{\mathrm{HT}}(\rho_\pi, \iota y)$ by construction of Hodge–Tate cocharacters, so we are done with proving (12.7) as desired. □

Remark 12.6. Lemma 12.2 tells us that ρ_π is totally odd. Our result also shows that $\rho_\pi(c_y)$ is as predicted by [12, Conj. 3.2.1, 3.2.2] for every infinite place y of F. Indeed, as explained in §6 of their paper, their conjectures are compatible with the functoriality. Considering the L-morphism ${}^L\mathrm{GSO}_{2n}^{E/F} \to {}^L\mathrm{SO}_{2n}^{E/F}$ dual to the inclusion $\mathrm{SO}_{2n}^{E/F} \hookrightarrow \mathrm{GSO}_{2n}^{E/F}$, we reduce the question to the case of $\mathrm{SO}_{2n}^{E/F}$ in view of the characterization of $\rho_\pi(c_y)$ in terms of $\mathrm{pr}^\circ(\rho_{\pi_\pi}(c_y))$. The latter is conjugate to $\rho_{\pi^\flat}$, which is as conjectured by *loc. cit.* by Remark 6.6.

Remark 12.7. It was easier to determine the Hodge–Tate cocharacter in the GSp-case [49] , thanks to the absence of nontrivial outer automorphisms. In particular we did not need to prove the analogue of Proposition 9.8. Compare with the proof of Theorem 9.1 (iii.a') of *loc. cit.*

13. Refinement for SO_{2n}-valued Galois representations

As an application of our results we improve upon Theorem 6.5 in this section by removing the outer ambiguity in the images of Frobenius conjugacy classes.

Let E/F be a quadratic CM extension of F in case n is odd, and $E := F$ for n even. Let $\mathrm{SO}_{2n}^{E/F}$ be the corresponding group defined above (6.2). If $\pi^\flat$ (resp. π) is an automorphic representation of $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ (resp. $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$), we write $S_{\mathrm{bad}}(\pi^\flat)$ (resp $S_{\mathrm{bad}}(\pi)$) for the set of rational prime numbers p, such that $p = 2$, p ramifies in E, or $\pi_p^\flat$ (resp. π_p) is a ramified representation. For other notation, we refer to Section 1.

In order to extend a given cohomological representation $\pi^\flat$ of $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ to a *cohomological* representation π of $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$, the following condition on the central character $\omega_{\pi^\flat} : \mu_2(F)\backslash\mu_2(\mathbb{A}_F) \to \mathbb{C}^\times$ is necessary in view of condition (cent) of §9. (If π is ξ-cohomological with $w \in \mathbb{Z}$ as in (cent) then all $\omega_{\pi^\flat,y}$ are trivial, resp. nontrivial, according as w is even, resp. odd.)

(**cent**$^\circ$) The sign character $\omega_{\pi^\flat,y} : \mu_2(F_y) = \{\pm1\} \to \mathbb{C}^\times$ does not depend on $y|\infty$.

Theorem 13.1. *Let $\pi^\flat$ be a cuspidal automorphic representation of* $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ *satisfying* (cent°), (coh°), (St°), and (std-reg°) *of §6. Then there exists a semisimple Galois representation (depending on ι)*

$$\rho_{\pi^\flat} = \rho_{\pi^\flat,\iota} : \Gamma_F \to \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$$

satisfying (SO-i)–(SO-v) *as in Theorem 6.5 as well as the following.*

(SO-i+) *For every finite prime* $\mathfrak{q}$ *of* F *not above* $S_{\mathrm{bad}}(\pi^\flat) \cup \{\ell\}$,

$$\iota\phi_{\pi^\flat_{\mathfrak{q}}} \sim \mathrm{WD}(\rho_{\pi^\flat}|_{\Gamma_{F_{\mathfrak{q}}}})^{\text{F-ss}},$$

as $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$*-parameters.*

(SO-iii+) *For every* $\mathfrak{q}|\ell$, *the representation* $\rho_{\pi^\flat,\mathfrak{q}}$ *is potentially semistable. For each* $y\colon F \hookrightarrow \mathbb{C}$ *such that* ιy *induces* $\mathfrak{q}$, *we have* $\mu_{\mathrm{HT}}(\rho_{\pi^\flat,\mathfrak{q}}, \iota y) \sim \iota\mu_{\mathrm{Hodge}}(\xi^\flat, y)$.

Condition (SO-i+) characterizes $\rho_{\pi^\flat}$ *uniquely up to* $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$*-conjugation.*

Remark 13.2. Statement (SO-*i+*) is stronger than (SO-*i*) in that the statement is up to $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$-conjugacy, but also weaker as it excludes the places above $S_{\mathrm{bad}}(\pi^\flat) \cup \{\ell\}$. Clearly (SO-*iii+*) strengthens (SO-*iii*). If we drop (std-reg°) from the assumption, then the theorem can be proved by the same argument but conditionally on Hypothesis 6.11.

Proof of Theorem 13.1. We have $\mu_2 = Z(\mathrm{SO}^{E/F}_{2n})$. We claim that the central character $\omega_{\pi^\flat}$ extends (via $\mu_2(\mathbb{A}_F) \subset \mathbb{A}_F^\times$) to a Hecke character

$$\chi\colon F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$$

such that $\chi_y(z) = z^w$ at every infinite place y, where $w = 0$ (resp. $w = 1$) if $\omega_{\pi^\flat,y}$ is trivial (resp. nontrivial) at every $y|\infty$.

To prove the claim, let E' be a quadratic CM extension of F. We start by extending $\omega_{\pi^\flat}$ to a (unitary) Hecke character $\chi' : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$ whose infinite components are trivial if $w = 0$ and the sign character if $w = 1$. If $w = 0$, then such a χ' exists since $\mu_2(F)\mu_2(F_\infty)\backslash\mu_2(\mathbb{A}_F)$ is a closed subgroup of $\overline{F^\times F_\infty^\times}\backslash\mathbb{A}_F^\times$, where the bar means the closure in $\mathbb{A}_F^\times$. If $w = 1$, consider the quadratic Hecke character $\chi_{E'/F}$ associated with E'/F via class field theory. Then $\omega_{\pi^\flat}(\chi_{E'/F}|_{\mu_2(\mathbb{A}_F)})$ extends to a Hecke character with trivial components at ∞ by the $w = 0$ case. Multiplying $\chi_{E'/F}$, we obtain a desired choice of χ'. Whether $w = 0$ or $w = 1$, we now see that $\chi := \chi'|\cdot|^w$ has desired components at ∞, where $|\cdot|$ is the absolute value character on $\mathbb{A}_F^\times$. The claim is proved.

Consider the multiplication map $f\colon \mathrm{GL}_1 \times \mathrm{SO}^{E/F}_{2n} \to \mathrm{GSO}^{E/F}_{2n}$. Let $\xi^\flat$ be such that $\pi^\flat$ is $\xi^\flat$-cohomological. Write ς for the algebraic character $z \mapsto z^w$ of GL_1 over F. Then $(\varsigma, \xi^\flat)$ descends to an algebraic representation ξ of $\mathrm{GSO}^{E/F}_{2n}$ via f.

Let us extend $\pi^\flat$ to an irreducible admissible $\mathrm{GSO}^{E/F}_{2n}(\mathbb{A}_F)$-representation, by decomposing $\pi^\flat = \otimes'_v \pi^\flat_v$ and taking an irreducible subrepresentation π_v of

$$\mathrm{Ind}^{\mathrm{GSO}^{E/F}_{2n}(F_v)}_{\mathrm{GL}_1(F_v)\mathrm{SO}^{E/F}_{2n}(F_v)} \chi_v\pi^\flat_v,$$

which is semisimple [80, pp.1832–1833]. Take π_v to be unramified for almost all v, and define $\pi := \otimes'_v \pi_v$. Lemma 5.4 of [81] states that

$$\sum_{\omega\in X/YX(\pi)} m(\pi\otimes\omega) = \sum_{g\in \mathrm{GSO}^{E/F}_{2n}(\mathbb{A}_F)/\widetilde{G}(\pi)\mathrm{GSO}^{E/F}_{2n}(F)} m\big((\pi^\flat)^g\big),$$

where X is the set of characters of $\mathrm{GSO}^{E/F}_{2n}(\mathbb{A}_F)/\mathrm{SO}^{E/F}_{2n}(\mathbb{A}_F)Z(\mathrm{GSO}^{E/F}_{2n})(\mathbb{A}_F)$, and ω in the formula is represented by such characters. We refer to *loc. cit.* for some undefined notation that is not important for us but content ourselves with pointing out that both sides are finite sums. Since $m(\pi^\flat) > 0$, the right hand side is positive. Thus the left hand side is positive, and thus we may (and do) twist π so that it is discrete automorphic.

We now check that π satisfies the conditions of Theorem A. Since $\pi^\flat_\infty$ is $\xi^\flat$-cohomological, by construction π_∞ is cohomological according to Lemma 7.2. By Lemma 7.1, π satisfies (St) thus also cuspidal. Condition (std-reg) is implied by (std-reg°) on $\pi^\flat$. Hence we have a Galois representation

$$\rho_\pi\colon \Gamma \to \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}.$$

such that for every finite F-place $\mathfrak{q}$ not above $S_{\mathrm{bad}}(\pi)\cup\{\ell\}$,

$$\rho_\pi(\mathrm{Frob}_{\mathfrak{q}})_{\mathrm{ss}} \sim \iota\phi_{\pi_{\mathfrak{q}}}(\mathrm{Frob}_{\mathfrak{q}}) \in \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)\rtimes\Gamma_{E/F}. \tag{13.1}$$

As in the preceding section, we can arrange that $\rho_{\pi^\flat} = \mathrm{pr}^\circ\circ\rho_\pi$ (not just up to outer automorphism). The Satake parameter of $\pi^\flat_{\mathfrak{q}}$ is equal to the composition of the Satake parameter of $\pi_{\mathfrak{q}}$ with the natural surjection (cf. [81, Lem. 5.2])

$$(\mathrm{pr}^\circ,\mathrm{id}) : \mathrm{GSpin}_{2n}(\mathbb{C})\rtimes\Gamma \to \mathrm{SO}_{2n}(\mathbb{C})\rtimes\Gamma.$$

In particular (SO-i+) follows from (13.1) for the places not above $S_{\mathrm{bad}}(\pi)\cup\{\ell\}$. Similarly (SO-$iii$+) follows from Theorem A (A3)(a).

At this point we have not yet completely proved (SO-i+), as the inclusion $S_{\mathrm{bad}}(\pi^\flat)\subset S_{\mathrm{bad}}(\pi)$ is strict in general. Thus it remains to treat $\mathfrak{q}$ above a prime $p\in S_{\mathrm{bad}}(\pi)\backslash S_{\mathrm{bad}}(\pi^\flat)$. Consider for n odd (resp. even) the obvious hyperspecial subgroup (recall $\mathfrak{q}\nmid 2$)

$$K_{\mathfrak{q}} := \begin{cases} \{(g,\lambda)\in \mathrm{GL}_{2n}(\mathcal{O}_E\otimes_{\mathcal{O}_F}\mathcal{O}_{F_{\mathfrak{q}}})\times\mathcal{O}_{F_{\mathfrak{q}}}^\times \mid \overline{g} = \vartheta^\circ g\vartheta^\circ, g^t\left(\begin{smallmatrix}0 & 1_n\\ 1_n & 0\end{smallmatrix}\right)g = \lambda\left(\begin{smallmatrix}0 & 1_n\\ 1_n & 0\end{smallmatrix}\right), \det(g)=\lambda^n\} \\ \text{resp.} \\ \left\{(g,\lambda)\in \mathrm{GL}_{2n}(\mathcal{O}_{F_{\mathfrak{q}}})\times\mathcal{O}_{F_{\mathfrak{q}}}^\times \mid g^{\mathrm{t}}\cdot\left(\begin{smallmatrix} & 1_n\\ 1_n & \end{smallmatrix}\right)\cdot g = \lambda\cdot\left(\begin{smallmatrix} & 1_n\\ 1_n & \end{smallmatrix}\right), \det(g)=\lambda^n\right\}\end{cases}$$

of $\mathrm{GSO}^{E/F}_{2n}(F_{\mathfrak{q}})$. Define $K_{0\mathfrak{q}}$ to be the kernel of the similitudes mapping $K_{\mathfrak{q}}\to\mathcal{O}^\times_{F_{\mathfrak{q}}}$, $(g,\lambda)\mapsto\lambda$. Then $\pi_{\mathfrak{q}}$ is a ramified representation of $\mathrm{GSO}^{E/F}_{2n}(F_{\mathfrak{q}})$, but has nonzero $K_{0\mathfrak{q}}$-fixed vectors, on which $K_{\mathfrak{q}}$ acts through nontrivial characters of $K_{\mathfrak{q}}/K_{0\mathfrak{q}}\simeq\mathcal{O}^\times_{F_{\mathfrak{q}}}$. We fix one such character $\chi^0_{\mathfrak{q}}$ of $K_{\mathfrak{q}}$, and do this at every $\mathfrak{q}$ above p. Now we globalize $\{\chi^0_{\mathfrak{q}}\}_{\mathfrak{q}|p}$ to a Hecke character $\chi : F^\times\backslash\mathbb{A}_F^\times\to\mathbb{C}^\times$ whose restriction to each $\mathcal{O}^\times_{F_{\mathfrak{q}}}$ is $\chi^0_{\mathfrak{q}}$ and whose archimedean components are trivial. (This is possible by [17, Lem. 4.1.1].) Define $\pi' := \pi\otimes\chi^{-1}$. Then π' also satisfies the conditions of Theorem A. Moreover, $p\notin S_{\mathrm{bad}}(\pi')$ by construction. Therefore (13.1) is true at each $\mathfrak{q}|p$, with π' in place of π. Then (SO-i+) for $\mathfrak{q}$ follows as before. □

14. Automorphic multiplicity one

Let E/F be a quadratic CM extension of F in case n is odd, and $E := F$ for n even. Let $\mathrm{SO}^{E/F}_{2n}$ and $\mathrm{GSO}^{E/F}_{2n}$ be as before. If π (resp. $\pi^\flat$) is an automorphic representation of $\mathrm{GSO}^{E/F}_{2n}(\mathbb{A}_F)$ (resp. $\mathrm{SO}^{E/F}_{2n}(\mathbb{A}_F)$), we write $m(\pi)$ (resp. $m(\pi^\flat)$) for its automorphic multiplicity. In this section we will show that $m(\pi^\flat)$ and $m(\pi)$ are 1 for certain classes of automorphic representations of $\mathrm{SO}^{E/F}_{2n}(\mathbb{A}_F)$ and $\mathrm{GSO}^{E/F}_{2n}(\mathbb{A}_F)$ (and some inner forms of those groups). To do this we combine our results with Arthur's result on multiplicities for $\mathrm{SO}^{E/F}_{2n}$, and Xu's result on multiplicities for $\mathrm{GSO}^{E/F}_{2n}$.

Let $\pi^\flat$ be a discrete automorphic representation of $\mathrm{SO}^{E/F}_{2n}(\mathbb{A}_F)$. Arthur gives in the discussion below [1, Thm. 1.5.2] the following result towards the computation of $m(\pi^\flat)$. Let $\psi=\psi_1\boxplus\cdots\boxplus\psi_r$ be the global (formal) parameter of $\pi^\flat$ [1, §1.4] (cf. Section 6). Technically, ψ is an automorphic representation $\pi^\sharp$ of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ given as an isobaric sum of discrete automorphic representations $\pi^\sharp_i$ of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$, with $\pi^\sharp_i$ representing the formal parameter ψ_i. In terms of these parameters Arthur proves a decomposition of the form

$$\mathrm{L}^2_{\mathrm{disc}}(\mathrm{SO}^{E/F}_{2n}(F)\backslash\mathrm{SO}^{E/F}_{2n}(\mathbb{A}_F)) \xrightarrow{\sim} \bigoplus_{\psi\in\widetilde{\Psi}_2(\mathrm{SO}^{E/F}_{2n})}\ \bigoplus_{\tau\in\widetilde{\Pi}_\psi(\varepsilon_\psi)} m_\psi\tau$$

as an $\widetilde{\mathcal{H}}(\mathrm{SO}^{E/F}_{2n})$-Hecke module. It takes us too far afield to recall all the notation here, but we emphasize that $\widetilde{\mathcal{H}}(\mathrm{SO}^{E/F}_{2n})$ is the restricted tensor product of the local algebras $\widetilde{\mathcal{H}}(\mathrm{SO}^{E/F}_{2n}(F_v))$ consisting of θ°-invariant functions [1, before (1.5.3)]. Similarly, the local packet $\widetilde{\Pi}_{\psi_v}(\varepsilon_\psi)$ consists of θ°-orbits of representations.

Assume $\pi^\flat \not\simeq \pi^\flat \circ \theta^\circ$ for the moment. Both $\pi^\flat$ and $\pi^\flat \circ \theta^\circ$ map to the same global parameter ψ, and are isomorphic as $\widetilde{\mathcal{H}}(\mathrm{SO}_{2n}^{E/F})$-modules. Arthur proves $m_\psi \le 2$ for all ψ. Thus

$$m(\pi^\flat) + m(\pi^\flat \circ \theta^\circ) \le m_\psi \le 2. \tag{14.1}$$

On the other hand, θ° acts on $\mathrm{L}^2_{\mathrm{disc}}(\mathrm{SO}_{2n}^{E/F}(F)\backslash \mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F))$, so if $\pi^\flat$ appears, then $\pi^\flat \circ \theta^\circ$ also appears. Hence $m(\pi^\flat), m(\pi^\flat \circ \theta^\circ) \ge 1$, forcing $m(\pi^\flat) = 1$ and $m(\pi^\flat \circ \theta^\circ) = 1$.

From now on we impose the assumption (std-reg$^\circ$) on $\pi^\flat$. At the infinite F-places v the infinitesimal character of $\pi_v^\flat$ is then not fixed by θ°. In particular $\pi^\flat \not\simeq \pi^\flat \circ \theta^\circ$. By the preceding paragraph, we have $m(\pi^\flat) = 1$, $m(\pi^\flat \circ \theta^\circ) = 1$, and $m_\psi = 2$.

Proposition 14.1. *Let π be a cuspidal automorphic representation of $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$ satisfying (L-coh), (St) and (std-reg). Then $m(\pi) = 1$.*

Proof. (cf. [49, Thm. 12.1]). Let $\pi^\flat$ be a cuspidal automorphic representation of $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ contained in π. Then $\pi^\flat$ satisfies (coh$^\circ$) and (St$^\circ$) as explained at the start of §10. Let $Y(\pi)$ be the set of continuous characters $\omega\colon \mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F) \to \mathbb{C}^\times$ which are trivial on the subgroup $\mathrm{GSO}_{2n}^{E/F}(F)\mathbb{A}_F^\times \mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ of $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$ and such that $\pi \simeq \pi \otimes \omega$. Xu [81, Prop. 1.7] proves that

$$m(\pi) = m_{\widetilde{\psi}} |Y(\pi)/\alpha(\mathcal{S}_\phi)|, \tag{14.2}$$

where $\widetilde{\psi}$ is the global parameter of π as defined in [81, Sect. 3] ($\widetilde{\psi}$ is denoted $\widetilde{\phi}$ there), and $\alpha(\mathcal{S}_\phi)$ will not be important for us.

We claim that $Y(\pi) = \{\mathbf{1}\}$ in (14.2). Let $\omega \in Y(\pi)$ and let $\chi\colon \Gamma \to \overline{\mathbb{Q}}_\ell^\times$ be the corresponding character via class field theory. As $\chi\rho_\pi$ and ρ_π have conjugate Frobenius images at almost all places, we obtain $\chi\rho_\pi \simeq \rho_\pi$ by Proposition 5.8, and thus $\chi = 1$ by Lemma 5.6. Hence $Y(\pi) = \{\mathbf{1}\}$.

Let ψ denote the Arthur parameter of $\pi^\flat$. In [81, Cor. 5.10], Xu states that $m_{\widetilde{\psi}} = m_\psi/\#\Sigma_Y(\pi)$, where $\Sigma_Y(\pi) := \Sigma_0/\Sigma_0(\pi, Y)$, where Σ_0 is the 2-group $\{1, \theta\}$, and $\Sigma_0(\pi, Y)$ is the group of $\theta' \in \Sigma_0$ such that $\pi \otimes \omega \simeq \pi^{\theta'}$ for some $\omega \in Y(\pi)$. We saw below (14.1) that $m_\psi = 2$. It is enough to check that $\#\Sigma_Y(\pi) = 2$, which would imply $m_{\widetilde{\psi}} = 1$. As $Y(\pi) = \{\mathbf{1}\}$ this reduces to $\pi \not\simeq \pi^\theta$. Again by (std-reg) the infinitesimal character of π_v for $v|\infty$ is not fixed by θ, so this is indeed true. □

Let G be the inner form of $\mathrm{GSO}_{2n}^{E/F}$ which was constructed in (8.7) and used in our Shimura data. We close this section with computing automorphic multiplicities for this G. In particular we prove that the multiplicities $a(\cdot)$ appearing in Section 9 are in fact equal to 1.

Proposition 14.2. *Let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$, satisfying (coh), (St) and (str-reg). Then $m(\pi) = 1$.*

Proof. The proof is the same as the argument for [49, Thm. 12.2]. The main point is that automorphic representations τ^* of $G^*(\mathbb{A}_F) = \mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$ contributing to the analogue of [*loc. cit.*, Eq. (12.2)] have automorphic multiplicity 1. Notice that [81, Thm. 1.8] may be used again, together with the existence of Galois representations (our Theorem A) to prove that for all π^* and τ^* contributing to [49, Eq. (12.2)] we have $\tau^*_{\mathfrak{q}\mathrm{St}} \simeq \pi^*_{\mathfrak{q}\mathrm{St}}$. □

15. Meromorphic continuation of spin L-functions

Let $n \in \mathbb{Z}_{\ge 3}$, and $\mathfrak{e}$ be as in (0.2). Let π be a cuspidal automorphic representation of $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$ unramified away from a finite set of places S satisfying (St), (L-coh), and (spin-reg). This implies (std-reg) for π. Indeed, if the image of $(s_0, s_1, \ldots, s_n) \in T_{\mathrm{GSpin}}$ under $\mathrm{spin}^\varepsilon$ is regular for some $\varepsilon \in \mathfrak{e}$ then $s_1, \ldots, s_n$ must be mutually distinct, as the weights in $\mathrm{spin}^\varepsilon$ are described as the Weyl orbit(s) of (2.9).

Proposition 15.1. *Assume that π satisfies (St), (L-coh), and (spin-reg). Let $n \in \mathbb{Z}_{\geq 3}$. There exist a number field M_π and a semisimple representation*

$$R^{\varepsilon}_{\pi,\lambda} : \Gamma \to \mathrm{GL}_{2^n/|\mathfrak{e}|}(\overline{M}_{\pi,\lambda})$$

for each finite place λ of M_π such that the following hold for every $\varepsilon \in \mathfrak{e}$. (Write ℓ for the rational prime below λ.)

(1) At each place $\mathfrak{q}$ of F not above $S_{\mathrm{bad}} \cup \{\ell\}$, we have

$$\mathrm{char}(R^{\varepsilon}_{\pi,\lambda}(\mathrm{Frob}_\mathfrak{q})) = \mathrm{char}(\mathrm{spin}^{\varepsilon}(\iota\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{q}))) \in M_\pi[X].$$

(2) $R^{\varepsilon}_{\pi,\lambda}|_{\Gamma_\mathfrak{q}}$ is de Rham for every $\mathfrak{q}|\ell$. Moreover it is crystalline if π_v is unramified and $\mathfrak{q} \notin S^F_{\mathrm{bad}}$.

(3) For each $\mathfrak{q}|\ell$ and each $y : F \hookrightarrow \mathbb{C}$ such that ιy induces $\mathfrak{q}$, we have $\mu_{\mathrm{HT}}(R^{\varepsilon}_{\pi,\lambda}|_{\Gamma_\mathfrak{q}}, \iota y) = \iota(\mathrm{spin}^{\varepsilon} \circ \mu_{\mathrm{Hodge}}(\phi_{\pi_y}))$. In particular $\mu_{\mathrm{HT}}(R^{\varepsilon}_{\pi,\lambda}|_{\Gamma_\mathfrak{q}}, \iota y)$ is a regular cocharacter for each y.

(4) $R^{\varepsilon}_{\pi,\lambda}$ is pure.

(5) $R^{\varepsilon}_{\pi,\lambda}$ maps into $\mathrm{GSp}_{2^n/|\mathfrak{e}|}(\overline{M}_{\pi,\lambda})$ if $n \equiv 2, 3 \pmod 4$ (resp. $\mathrm{GO}_{2^n/|\mathfrak{e}|}(\overline{M}_{\pi,\lambda})$ if $n \equiv 0, 1 \pmod 4$) for a nondegenerate alternating (resp. symmetric) pairing on the underlying $2^n/|\mathfrak{e}|$-dimensional space over $\overline{M}_{\pi,\lambda}$. The multiplier character $\mu^{\varepsilon}_\lambda : \Gamma \to \mathrm{GL}_1(\overline{M}_{\pi,\lambda})$ (so that $R^{\varepsilon}_{\pi,\lambda} \simeq (R^{\varepsilon}_{\pi,\lambda})^\vee \otimes \mu^{\varepsilon}_\lambda$) is totally of sign $(-1)^{n(n-1)/2}$ and associated with ω_π via class field theory and ι_λ.

Proof. Let M be the field of definition of ξ, which is a finite extension of $\mathbb{Q}$ in $\mathbb{C}$. We can choose M_π to be the field of definition for the π^∞-isotypic part in the (compact support) Betti cohomology of $H^\bullet(\mathrm{Sh}^+(\mathbb{C}), \mathcal{L}_\xi) \oplus H^\bullet(\mathrm{Sh}^-(\mathbb{C}), \mathcal{L}_\xi)$ with M-coefficient. Then M_π is a finite extension of M in $\mathbb{C}$. For each prime ℓ and a finite place λ of M_π above ℓ, extend $M \hookrightarrow \mathbb{C}$ to an isomorphism $\overline{M}_{\pi,\lambda} \simeq \mathbb{C}$. Identifying $\overline{M}_{\pi,\lambda} \simeq \overline{\mathbb{Q}}_\ell$, we have $\iota_\lambda : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$. Take

$$R^{\varepsilon}_{\pi,\lambda} := \mathrm{spin}^{\varepsilon} \circ \rho_{\pi,\iota_\lambda}.$$

Then (1), (2), and (3) follow from (A2) and (A3) of Theorem A, respectively. Part (4) follows from (SO-ii) of Theorem 6.5 via (A2). The first part of (5) holds true since $\mathrm{spin}^{\varepsilon} : \mathrm{GSpin}_{2n} \to \mathrm{GL}_{2^{n-1}}$ is an irreducible representation preserving a nondegenerate symplectic (resp. symmetric) pairing up to scalar if n is 2 (resp. 0) mod 4, and since $\mathrm{spin} : \mathrm{GPin}_{2n} \to \mathrm{GL}_{2^n}$ is irreducible and preserves a nondegenerate symplectic (resp. symmetric) pairing up to scalar if n is 3 (resp. 1) mod 4. Indeed, the irreducibility is standard and the rest follows from Lemma 4.2 (with the pairing given as in the lemma). Lemma 4.2 also tells us that $\mu^{\varepsilon}_\lambda = \mathscr{N} \circ \rho_{\pi,\iota_\lambda}$. By (A2), $\omega_\pi = \mathscr{N} \circ \rho_{\pi,\iota_\lambda}$ so $\mu^{\varepsilon}_\lambda$ is associated with ω_π. As in the proof of part 5 of [49, Prop. 13.1], $\omega_\pi \otimes |\cdot|^{n(n-1)/2}$ corresponds to an even Galois character of Γ. (We change $n(n+1)/2$ in [49] to $n(n-1)/2$ here due to the difference in the definition of (L-coh).) It follows that $\mu_{\lambda,y}(c_y) = (-1)^{n(n-1)/2}$ for every $y|\infty$. □

Now we apply potential automorphy results to the weakly compatible system of $R^{\varepsilon}_{\pi,\lambda}$.

Theorem 15.2. *Theorem D is true.*

Proof. This follows from [65, Thm. A], which can be applied to the weakly compatible system $\{R^{\varepsilon}_{\pi,\lambda}\}$ thanks to the preceding proposition. □

Remark 15.3. We cannot appeal to the potential automorphy as in [3, Thm. A] as $R^{\varepsilon}_{\pi,\lambda}$ may be reducible. The point of [65] is to replace the irreducibility hypothesis with a purity hypothesis (guaranteed by (iv) of Proposition 15.1). We take advantage of this.

Appendix A. Extending a Galois representation

Here we investigate the problem of extending a $\widehat{G}$-valued Galois representation to an ${}^LG$-valued representation over a quadratic extension.

We freely use the notation and terminology of §1. Let E be a CM quadratic extension over a totally real field F in an algebraic closure $\overline{F}$. Set $\Gamma = \Gamma_F := \mathrm{Gal}(\overline{F}/F)$, $\Gamma_E := \mathrm{Gal}(\overline{F}/E)$, and $\Gamma_{E/F} := \mathrm{Gal}(E/F) = \{1, c\}$. Let G be a quasi-split group over F which splits over E. Let $\theta \in \mathrm{Aut}(\widehat{G})$ denote the action of c on $\widehat{G}$ (with respect to a pinning over F). By $\widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$, we mean the L-group relative to E/F, namely the semi-direct product such that $cgc = \theta(g)$ for $g \in \widehat{G}(\overline{\mathbb{Q}}_\ell)$.

Fix an infinite place y of F. Write $c_y \in \Gamma_F$ for the corresponding complex conjugation (well defined up to conjugacy). Let $\rho' : \Gamma_E \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ be a Galois representation. Define

$$ {}^{c_y}\rho'(\gamma) := \rho'(c_y \gamma c_y^{-1}). $$

(Of course $c_y^{-1} = c_y$.) We will sometimes impose the following hypotheses.

(H1) $\mathrm{Cent}_{\widehat{G}}(\mathrm{im}(\rho')) = Z(\widehat{G})$.

(H2) The map $Z(\widehat{G}) \to Z(\widehat{G})^\theta$ given by $z \mapsto z\theta(z)$ is a surjection on $\overline{\mathbb{Q}}_\ell$-points.

Lemma A.1. *Consider the following statements.*

(1) ρ' extends to some $\rho : \Gamma_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$.

(2) ${}^{c_y}\rho' \simeq \theta \circ \rho'$.

(3) there exists $g \in \widehat{G}(\overline{\mathbb{Q}}_\ell)$ such that $g\theta(g) = 1$ and $\rho'(c_y\gamma c_y^{-1}) = g\theta(\rho'(\gamma))g^{-1}$ for every $\gamma \in \Gamma_E$.

Then (3)⇔(1)⇒(2). In particular if ρ is as in (1) then the element g such that $\rho(c_y) = g \rtimes c$ enjoys the property of (3). If (H1) and (H2) are satisfied, then we also have (2)⇒(3), so all three statements are equivalent.

Remark A.2. We recommend [5, Section A.11] as a useful guide to similar ideas.

Remark A.3. Often (2) is the condition to verify to extend a Galois representation, as we did in Lemma 12.1 of this paper.

Proof. (3)⇔(1): First we show (3)⇒(1). Define ρ by $\rho|_{\Gamma_E} := \rho'$ and $\rho(\gamma c_y) := \rho'(\gamma)gc$ ($\gamma \in \Gamma_E$). Then

$$ \rho(c_y^2) = gcgc = g\theta(g) = 1, $$
$$ \rho(c_y\gamma c_y^{-1}) = {}^{c_y}\rho'(\gamma) = g\theta(\rho'(\gamma))g^{-1}, $$

and using this, one checks that ρ is a homomorphism on the entire Γ. A similar computation shows (1)⇒(3) for g such that $\rho(c_y) = g \rtimes c$.

(1)⇒(2): Write $\rho(c_y) = gc$ with $g \in \widehat{G}(\overline{\mathbb{Q}}_\ell)$. For every $\gamma \in \Gamma_E$,

$$ {}^{c_y}\rho'(\gamma) = \rho(c_y\gamma c_y^{-1}) = gc\rho'(\gamma)c^{-1}g^{-1} = g\theta(\rho'(\gamma))g^{-1}. $$

(2)⇒(3), assuming (H1) and (H2): There exists $g \in \widehat{G}(\overline{\mathbb{Q}}_\ell)$ such that

$$ \rho'(c_y\gamma c_y^{-1}) = g\theta(\rho'(\gamma))g^{-1}, \qquad \gamma \in \Gamma_E. \tag{A.1} $$

Putting $c_y\gamma c_y^{-1}$ in place of γ, we obtain

$$ \rho'(\gamma) = \rho'(c_y^2\gamma c_y^{-2}) = g\theta(g\theta(\rho'(\gamma))g^{-1})g^{-1} = g\theta(g)\rho'(\gamma)(g\theta(g))^{-1}. $$

Hence $g\theta(g) \in Z(\widehat{G})$ as $\mathrm{Cent}_{\widehat{G}}(\rho') = Z(\widehat{G})$ by (H1). As a central element,

$$ g\theta(g) = g^{-1}(g\theta(g))g = \theta(g)g = \theta(g\theta(g)), $$

namely $g\theta(g) \in Z(\widehat{G})^\theta$. By (H2), $g\theta(g) = z\theta(z)$ for some $z \in Z(\widehat{G})$. Replacing g with gz^{-1}, we can arrange that

$$ g\theta(g) = 1. $$

This does not affect (A.1) so we are done. □

Lemma A.4. *Assume (H1). Then the set of $\widehat{G}$-conjugacy classes of extensions of ρ' to Γ is an $H^1(\Gamma_{E/F}, Z(\widehat{G}))$-torsor if nonempty.*

Proof. Fix an extension ρ_0 of ρ', which exists by Lemma A.1. If ρ is another extension of ρ', then set $z := \rho_0(c_y)\rho(c_y)^{-1}$. Writing $\rho_0(c_y) = g_0 \rtimes c$ and $\rho(c_y) = g \rtimes c$, we have $zg = g_0$, and both g_0, g satisfy the condition of Lemma A.1 (3). It follows that z centralizes $\theta(\mathrm{im}(\rho'))$, hence $z \in Z(\widehat{G})$, and also that $z\theta(z) = 1$. Thus z defines a $Z(\widehat{G})$-valued 1-cocycle on $\Gamma_{E/F}$, and by reversing the process, such a cocycle determines an extension of ρ'.

Let ρ_z be the extension given by $z \in Z(\widehat{G})$ such that $z\theta(z) = 1$. It remains to show that $\rho_z \sim \rho_0$ if and only if $z = \theta(x)/x$ for some $x \in Z(\widehat{G})$. If $\rho_z \sim \rho_0$ then $\rho_z = \mathrm{Int}(x)\rho_0$ for some $x \in \widehat{G}$. By (H1), $x \in Z(\widehat{G})$. Evaluating at c_y, we obtain $z^{-1}\rho_0(c_y) = x\rho_0(c_y)x^{-1}$. Therefore $z = \theta(x)/x$. The converse direction is shown similarly by arguing backward. □

We illustrate assumptions (H1), (H2), and the lemmas in the following examples.

Example A.5. Consider $\widehat{G} = \mathrm{SO}_{2n}$ $(n \geq 3)$ with θ being the conjugation by $\vartheta^\circ \in \mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell) - \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ as in (2.4). Assume that $\mathrm{im}(\rho')$ contains a regular unipotent of $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$. In this case $Z(\widehat{G}) = Z(\widehat{G})^\theta = \{\pm 1\}$. Then (H2) is trivially false but (H1) is true. To see this, by assumption, $\mathrm{std} \circ \rho'$ is either irreducible or the direct sum of an irreducible $(2n-1)$-dimensional representation and a character. In the former case (H1) is clear by Schur's lemma. In the latter case, again by Schur's lemma, a centralizer of $\mathrm{im}(\rho')$ in $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ is contained in $\left(\begin{smallmatrix} a \cdot 1_{2n-1} & 0 \\ 0 & b \end{smallmatrix}\right)$ with $a, b \in \{\pm 1\}$ up to $\mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$-conjugacy. Since the determinant equals 1, we deduce that $a = b$, i.e., the centralizer belongs to $Z(\widehat{G})$.

We easily compute $Z^1(\Gamma_{E/F}, Z(\widehat{G})) = H^1(\Gamma_{E/F}, Z(\widehat{G})) \simeq \mathbb{Z}/2\mathbb{Z}$, the nontrivial element sending c to -1. In fact if ρ extends ρ' in the setup of the preceding lemmas, the other extension is easily described as $\rho \otimes \chi_{E/F}$, where $\chi_{E/F} : \Gamma \twoheadrightarrow \Gamma_{E/F} \xrightarrow{\sim} \{\pm 1\}$.

Example A.6. The main case of interest for us is when

- $\widehat{G} = \mathrm{GSpin}_{2n}$ $(n \geq 3)$,
- θ is the conjugation by an element of $\mathrm{GPin}_{2n}(\overline{\mathbb{Q}}_\ell) - \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$,
- $\mathrm{im}(\rho')$ contains a regular unipotent.

Since $Z(\widehat{G})^\theta = \mathbb{G}_m$ (identified with invertible scalars in the Clifford algebra underlying $\widehat{G}$ as a GSpin group; see §3), assumption (H2) is satisfied. (The squaring map $\mathbb{G}_m \to \mathbb{G}_m$ is clearly surjective on $\overline{\mathbb{Q}}_\ell$-points.) To check (H1), $\mathrm{Cent}_{\widehat{G}}(\mathrm{im}(\rho'))$ is contained in the preimage of $\mathrm{Cent}_{\mathrm{SO}_{2n}}(\mathrm{im}(\rho'^{,\circ}))$ via $\mathrm{pr}^\circ : \mathrm{GSpin}_{2n} \to \mathrm{SO}_{2n}$. Since the latter centralizer is $\{\pm 1\} \subset \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$, we see that $\mathrm{Cent}_{\widehat{G}}(\mathrm{im}(\rho')) \subset \mathrm{pr}'^{,-1}(\{\pm 1\}) = Z(\widehat{G})$.

In the coordinates for $Z(\widehat{G})$ of Lemma 2.5, $Z^1(\Gamma_{E/F}, Z(\widehat{G})) = \{(s_0, s_1) : s_1 \in \{\pm 1\},\ s_1 = s_0^2\} \simeq \mu_4$, of which coboundaries are $\{(\pm 1, 1)\} \simeq \mu_2$. (The first identification is given by taking the image of c.) Hence $H^1(\Gamma_{E/F}, Z(\widehat{G})) \simeq \mathbb{Z}/2\mathbb{Z}$. Let $\zeta = (\zeta_4, -1) \in Z(\widehat{G})$, where ζ_4 is a primitive fourth root of unity, cf. Lemma 3.7. If ρ is an extension of ρ', then the other extension (up to $\widehat{G}$-conjugacy) is described as $\rho \otimes \chi$, where $\chi : \Gamma \to Z(\widehat{G}) \rtimes \{1, c\}$ is inflated from $\Gamma_{E/F} \xrightarrow{\sim} \{1, \zeta \rtimes c\}$. Notice that $\mathrm{pr}^\circ \circ \chi = \chi_{E/F}$, for $\chi_{E/F}$ as in the preceding example.

Example A.7. When studying Galois representations arising from automorphic representations on a unitary group U_n in n variables, two target groups appear in the literature: the group $\mathcal{G}_n$ in [17, §2.1] and the C-group of U_n in [12]; the two are isogenous as explained in [12, §8.3]. The latter is the L-group of a $\mathbb{G}_m$-extension of U_n; it does not satisfy (H2). The former is not an L-group, but still a semi-direct product $(\mathrm{GL}_n \times \mathrm{GL}_1) \rtimes \Gamma_{E/F}$, with $c(g, \mu) = (\mu g^{-t}, \mu)$ for an anti-diagonal matrix $\Phi_n \in \mathrm{GL}_n$. As such, the discussion in this appendix goes through for $\mathcal{G}_n$. An easy computation shows that $\mathcal{G}_n$ satisfies (H2) and that $H^1(\Gamma_{E/F}, Z(\mathrm{GL}_n \times \mathrm{GL}_1)) = \{1\}$ for the given Galois action. Thus provided that ρ' satisfies (H1) (e.g., if ρ' is irreducible), an extension of ρ' exists if and only if ${}^{c_y}\rho' \simeq \theta \circ \rho'$, and the extension is unique up to conjugacy. Compare this with [17, Lem. 2.1.4] (which allows a general coefficient field of characteristic 0).

Appendix B. On local A-packets of even special orthogonal groups

In this appendix we study the A-packets of the trivial and Steinberg representations of quasi-split forms of SO_{2n}, with $n \geq 3$, often following the notation and formulation of [1].

Let F be a finite extension of $\mathbb{Q}_p$. Suppose that $E = F$ or that E is a quadratic extension of F. Let $\chi_{E/F} : F^\times \to \{\pm 1\}$ denote the quadratic character associated with E/F via class field theory. Let $G := \mathrm{SO}_{2n}^{E/F}$ denote the quasi-split form of the split group SO_{2n} over F twisted by $\chi_{E/F}$. Write $\tilde{\mathrm{Out}}_{2n}(G) := \mathrm{O}_{2n}(\mathbb{C})/\mathrm{SO}_{2n}(\mathbb{C})$ for the outer automorphism group of $\mathrm{SO}_{2n}(\mathbb{C})$. Denote by $\mathbf{1}$ and St the trivial and Steinberg representations of $G(F)$. We aim to identify local A-packets containing $\mathbf{1}$ or St.

Let $\mathcal{L}_F := W_F \times \mathrm{SU}(2)$ denote the local Langlands group. Let $|\cdot| : W_F \to \mathbb{R}^\times_{>0}$ denote the absolute value character sending a geometric Frobenius element to the inverse of the residue cardinality of F. By abuse, keep writing $|\cdot|$ for its pullback to $\mathcal{L}_F$ via projection.

Denote by $\Psi^+(G)$ the set of isomorphism classes of extended A-parameters, that is, continuous morphisms $\psi : \mathcal{L}_F \times \mathrm{SU}(2) \to {}^LG$ such that $\psi|_{\mathcal{L}_F}$ is an L-parameter. (Two A-parameters are considered isomorphic if they are in the same $\widehat{G}$-orbit.) An extended A-parameter $\psi \in \Psi^+(G)$ gives rise to an L-parameter:

$$\phi_\psi : \mathcal{L}_F \to {}^LG, \qquad \gamma \mapsto \psi(\gamma, \mathrm{diag}(|\gamma|^{1/2}, |\gamma|^{-1/2})).$$

Write $\Psi(G)$ for the subset of $\Psi^+(G)$ consisting of $\psi \in \Psi^+(G)$ such that the image of $\psi(\mathcal{L}_F)$ in $\mathrm{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_{E/F}$ is bounded. (Such a property is $\widehat{G}$-invariant.) The set of $\tilde{\mathrm{Out}}_{2n}(G)$-orbits in $\Psi^+(G)$ (resp. $\Psi(G)$) is denoted by $\tilde{\Psi}^+(G)$ (resp. $\tilde{\Psi}(G)$). The group $\mathcal{L}_F \times \mathrm{SU}(2)$ admits the involution permuting the two $\mathrm{SU}(2)$-components (acting as the identity on W_F). This involution induces an involution

$$\psi \mapsto \widehat{\psi} \quad \text{on each of} \quad \tilde{\Psi}^+(G) \text{ and } \tilde{\Psi}(G).$$

We say $\psi \in \tilde{\Psi}^+(G)$ is square-integrable if $g\psi g^{-1} = \psi$ for at most finitely many elements $g \in \widehat{G}$. Then ψ lies in $\tilde{\Psi}(G)$. To see this, let $w \in W_F$ be a lift of (geometric) Frobenius. Then $\psi(w)^m$ centralizes the image of ψ for some $m \in \mathbb{Z}_{\geq 1}$ as in [19, proof of Lem. 8.4.3]. It follows that, replacing m with a suitable multiple, $\psi(w)^m$ has trivial image in $\mathrm{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_{E/F}$. Write $I_F \subset W_F$ for the inertia subgroup. Since $I_F \times \mathrm{SU}(2) \times \mathrm{SU}(2) \subset \mathcal{L}_F \times \mathrm{SU}(2)$ has already bounded image in $\mathrm{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_{E/F}$ under ψ, we see that $\psi \in \Psi(G)$. Denote by $\Psi_2(G)$ the subset of $\Psi(G)$ consisting of square-integrable members.

Define $\psi_{\mathrm{triv}} \colon \mathcal{L}_F \times \mathrm{SU}(2) \to {}^LG$ as follows. On $\mathcal{L}_F$ it is the composite map $\mathcal{L}_F \twoheadrightarrow W_F \twoheadrightarrow \Gamma_{E/F} \subset {}^LG$ through the natural projections. On $\mathrm{SU}(2)$ (outside $\mathcal{L}_F$), ψ_{triv} is a principal embedding $i_{\mathrm{pri}} \colon \mathrm{SU}(2) \to \widehat{G}$ that is θ°-invariant, i.e., i_{pri} commutes with the L-action of $\Gamma_{E/F}$ on $\widehat{G}$. (Such an i_{pri} into $\widehat{G}$ can be realized as the $\mathrm{SU}(2)$-representation $\mathrm{Sym}^{2n-2} \oplus \mathbf{1}$ into $\mathrm{GL}_{2n-1} \times \mathrm{GL}_1$, where the latter is identified with the centralizer of the element $\vartheta^\circ \in \mathrm{GL}_{2n}$ from §3. Write $\psi_{\mathrm{St}} := \widehat{\psi}_{\mathrm{triv}}$. Then $\psi_{\mathrm{triv}}, \psi_{\mathrm{St}} \in \Psi(G)$ and they are $\tilde{\mathrm{Out}}_{2n}(G)$-stable.

To every $\psi \in \tilde{\Psi}(G)$, Arthur [1, Thm. 1.5.1] assigned an A-packet $\tilde{\Pi}(\psi)$, which is a certain finite multi-set consisting of $\tilde{\mathrm{Out}}_{2n}(G)$-orbits of irreducible unitary representations of $G(F)$. Below *loc. cit.* he also defines $\tilde{\Pi}(\psi)$ for $\psi \in \tilde{\Psi}^+(G)$, consisting of $\tilde{\mathrm{Out}}_{2n}(G)$-orbits of parabolically induced representations of $G(F)$ (which need not be irreducible or unitary).

By a globalization $(\dot{E}/\dot{F}, \mathfrak{q}, \dot{G})$ of $(E/F, G)$ as above, we mean an extension of number fields $\dot{E}/\dot{F}$, a finite place $\mathfrak{q}$, and a quasi-split form $\dot{G}$ of the split SO_{2n} over $\dot{F}$ such that $\dot{E}_{\mathfrak{q}} \simeq E$, $\dot{F}_{\mathfrak{q}} \simeq F$, and $\dot{G}_{\mathfrak{q}} \simeq G$. It is an elementary fact that such a globalization always exists. Recall that a (formal) global parameter $\dot{\psi} \in \tilde{\Psi}(\dot{G})$ gives rise to a parameter $\dot{\psi}_v \in \tilde{\Psi}^+(\dot{G}_{\dot{F}_v})$ and a packet $\tilde{\Pi}(\dot{\psi}_v)$ at each place v of $\dot{F}$.

Proposition B.1. *Let $\psi \in \tilde{\Psi}^+(G)$. The following are true.*

(1) $\tilde{\Pi}(\psi_{\mathrm{triv}}) = \{\mathbf{1}\}$ and $\tilde{\Pi}(\psi_{\mathrm{St}}) = \{\mathrm{St}\}$.

(2) Assume $\psi \in \tilde{\Psi}(G)$. If $\mathbf{1}$ (resp. St*) is a member of $\tilde{\Pi}(\psi)$ then $\psi = \psi_{\mathrm{triv}}$ (resp. $\psi = \psi_{\mathrm{St}}$).*

(3) *Assume that* $\psi = \dot{\psi}_{\mathfrak{q}} \in \tilde{\Psi}^+(G)$ *for global data* $(\dot{E}/\dot{F}, \mathfrak{q}, \dot{G})$ *and* $\dot{\psi}$ *as above. If* $\mathbf{1}$ *(resp.* St*) is a subquotient of a member of* $\tilde{\Pi}(\psi)$ *then* $\psi = \psi_{\mathrm{triv}}$ *(resp.* $\psi = \psi_{\mathrm{St}}$*).*

Remark B.2. We use (3) in the main text. Part (3) would be subsumed by (2) if the generalized Ramanujan conjecture for general linear groups was known, cf. proof of (3) below.

Proof. (1) According to [1, Lem. 7.1.1] (and the discussion following it), the involution $\psi \mapsto \widehat{\psi}$ changes members of A-packets by the Aubert involution, which carries $\mathbf{1}$ to St and vice versa. So it suffices to treat the case of $\psi = \psi_{\mathrm{triv}}$.

We know that $\mathcal{S}_\psi = \{1\}$ and that the L-packet $\tilde{\Pi}(\phi_\psi)$ embeds in $\tilde{\Pi}(\psi)$. Hence $\tilde{\Pi}(\psi) = \tilde{\Pi}(\phi_\psi)$ and it is a singleton. Clearly ϕ_ψ is a spherical parameter in the sense of [1, §6.1]. Hence $\Pi(\phi_\psi)$ contains the irreducible spherical representation corresponding to ϕ_ψ via the spherical local Langlands correspondence recalled in *loc. cit.*[13]

Hence it boils down to observing that $\phi_{\psi_{\mathrm{triv}}}$ corresponds to $\mathbf{1}$. (This is easy to see from the fact that $\mathbf{1}$ is a subquotient of $\mathrm{Ind}_B^G(\delta_B^{-1/2})$.)

(2)⇒(3). Before proving (2), let us assume (2) and verify (3) for each $(E/F, G)$. We fix a Borel subgroup B and a maximal torus T of G over F such that $T \subset B$. Since ψ comes from a global parameter, $\psi|_{\mathcal{L}_F}$ is constrained by what local components of cuspidal automorphic representations of general linear groups can be. Following the same observations as in [81, App. A] we can express ψ concretely as follows:

$$\psi = \psi_{G_-} \oplus \bigoplus_{i=1}^{r} (|\cdot|^{a_i}\psi_i \oplus |\cdot|^{-a_i}\psi_i^\vee), \qquad 0 < a_r < \cdots < a_1 < 1/2.$$

where $\psi_{G_-} \in \tilde{\Psi}(\mathrm{SO}_{2n_-}^{E/F})$ and $\psi_i \in \Psi(\mathrm{GL}_{n_i})$ such that $n_- + \sum_{i=1}^r n_i = n$, and if we take

$$M = \mathrm{SO}_{2n_-}^{E/F} \times \prod_{i=1}^{r} \mathrm{GL}_{n_i} \tag{B.1}$$

to be a Levi subgroup of a standard parabolic subgroup P, we have

$$\psi_M := \psi_{G_-} \times (\textstyle\prod_{i=1}^r \psi_i) \in \tilde{\Psi}(M).$$

(Namely ψ_M is not just an "extended" parameter of M.) Actually in *loc. cit.*, a weaker inequality $a_r \le \cdots \le a_1$ holds for the exponents but this is because he wants ψ_i to be simple parameters. We only require ψ_i to be bounded parameters, so the simple parameters with the same exponent will go into a single ψ_i in our case.

Our goal is to show that $M = G$. Indeed, it would immediately imply that ψ belongs to $\tilde{\Psi}(G)$ (not just $\tilde{\Psi}^+(G)$) so we can conclude by (2). If $M \ne G$, i.e., if $r \ge 1$, we will show a contradiction.

By construction (see [1, §1.5]) the packet $\tilde{\Pi}(\psi)$ consists of ($\mathrm{Out}_{2n}(G)$-orbits of) the normalized induced representations

$$\mathrm{Ind}_P^G(\sigma \otimes \chi) \tag{B.2}$$

for σ in the A-packet $\Pi(\psi_M)$ for M, where the character $\chi : M(F) \to \mathbb{C}^\times$ is defined to be trivial on $\mathrm{SO}_{2n_-}^{E/F}(F)$ and $|\det|^{a_i}$ on $\mathrm{GL}_{n_i}(F)$.

Now suppose that $\mathbf{1}$ or St is a subquotient of a member of $\tilde{\Pi}(\psi)$. Then for some $\sigma \in \Pi(\psi_M)$, the irreducible subquotients of (B.2) have the supercuspidal support $(T, \delta_B^{-1/2})$, which is the supercuspidal support of both $\mathbf{1}$ and St (up to the Weyl group action). Write $\sigma = \sigma_- \otimes (\otimes_{i=1}^r \sigma_i)$

[13] This assertion is stated on p.304 of [1] in the spherical case (not just the unramified case) without proof. Although this may be obvious to the experts, let us verify the assertion for the L-packet constructed in *loc. cit.* If ϕ is generic and spherical, then the L-packet $\Pi(\phi)$ consists of all irreducible constituents of the principal series representation determined by ϕ; see [1, Prop. 2.4.3]. The principal series representation contains exactly one-dimensonal space of spherical vectors (the dimension is at most one by the Iwasawa decomposition, and a nonzero vector is easily exhibited, cf. [32, §3]). Now the assertion results from exactness of the functor taking spherical subspaces. For spherical non-generic ϕ, the construction of $\Pi(\phi)$ in *loc. cit.* is reduced to the generic case via Langlands quotients as usual. Therefore the assertion follows from [74, Lem. 7.10.(a)].

according to (B.1). Then σ is a subquotient of a principal series representation. The unitary representation σ_i of $\mathrm{GL}_{n_i}(F)$ corresponds to an A-parameter in $\Psi(\mathrm{GL}_{n_i})$, so it has the form

$$\sigma_i = \boxplus_{j=1}^{n_i} \chi_{i,j}|\cdot|^{e_{i,j}}, \qquad e_{i,j} \in \tfrac{1}{2}\mathbb{Z}, \tag{B.3}$$

where $\chi_{i,j}$ is a unitary character, and $\boxplus$ denotes the operation corresponding to the direct sum under the local Langlands correspondence. If we write $T = \mathrm{SO}_{2n_0}^{E/F} \times \prod_{k=1}^{n-n_0} \mathrm{GL}_1$ (if $E = F$ then $n_0 = 0$; otherwise $n_0 = 1$), then accordingly (up to the Weyl action)

$$\delta_B^{-1/2} = \mathbf{1} \otimes (\otimes_{k=1}^{n-1} |\cdot|^{(k-n)/2}).$$

On the other hand, (B.3) tells us that the supercuspidal support of (B.2) is represented by a character

$$\varsigma_0 \otimes (\otimes_k \varsigma_k)$$

of $T(F)$ such that each $\chi_{i,j}|\cdot|^{e_{i,j}+a_i}$ appears in the multi-set $\{\varsigma_k\}$. Comparing the two supports, we must have that $e_{i,j} + a_i \in \frac{1}{2}\mathbb{Z}$ since the half-integrality of exponents is invariant under the Weyl action. However this is impossible since $e_{i,j} \in \frac{1}{2}\mathbb{Z}$ and $0 < a_i < \frac{1}{2}$ (and the index set for i, j is nonempty). This proves that $M = G$ as desired.

(2) As in the proof of (1), it is enough to consider the case $\mathbf{1} \in \tilde{\Pi}(\psi)$ by Aubert involution. We deduce $\psi = \psi_{\mathrm{triv}}$ by dividing into two cases.

Case 1: when G is unramified. Let $s_\psi \in \widehat{G}$ denote the image of $-1 \in \mathrm{SU}(2)$ (outside $\mathcal{L}_F$) under ψ. Let us show that $\psi|_{W_F}$ is unramified, using the fact that $\tilde{\Pi}(\psi)$ contains an unramified representation, namely $\mathbf{1}$. In fact we induct on rank for unramified even special orthogonal groups to prove the unique existence of an unramified representation in $\tilde{\Pi}(\psi)$, where we allow more general $\psi \in \tilde{\Psi}(G)$, for the moment, which are unramified on W_F. The base case of induction is obvious. Next we recall the character identities from [1, Thm. 2.2.1], which have the form (cf. the second and last displayed formulas in the proof of [59, Lem. 5.4] for the analogues for unitary groups)

$$\sum_{\sigma \in \tilde{\Pi}(\psi)} 1_K(\sigma) = \tilde{1}_{K^{2n}}(\tilde{\pi}_{\psi^{2n}}), \qquad \text{if } s_\psi \in Z(\widehat{G}), \tag{B.4}$$

$$\sum_{\sigma \in \tilde{\Pi}(\psi)} 1_K(\sigma) = 1_{K'}(\psi'), \qquad \text{if } s_\psi \notin Z(\widehat{G}), \tag{B.5}$$

where the setup and notation are as in [1]. We partly explain the notation.

- ψ^{2n} is the A-parameter for GL_{2n} obtained from ψ via the standard embedding.
- $\tilde{\pi}_{\psi^{2n}}$ is the unique representation of $\mathrm{GL}_{2n}(F)$ in the A-packet of ψ^{2n}.
- When $s_\psi \notin Z(\widehat{G})$, we can find an unramified endoscopic group G' such that (G', s_ψ) is part of an elliptic endoscopic datum for G. (In particular, $\widehat{G}'$ is a connected centralizer of s_ψ in $\widehat{G}$.) Then G' is a product of two even special orthogonal groups [1, p.14]; ψ' is a parameter on G' giving rise to ψ.
- K^{2n}, K, K' are hyperspecial subgroups of $\mathrm{GL}_{2n}(F)$, $G(F)$, $G'(F)$, respectively,
- $\tilde{1}_{K^{2n}}(\tilde{\pi}_{\psi^{2n}})$ is a twisted trace.

The existence of an unramified member in $\tilde{\Pi}(\psi)$ tells us that the left hand side is nonzero. The resulting nonvanishing of the right hand side implies $\psi|_{W_F}$ is unramified: if $s_\psi \in Z(\widehat{G})$ then $\tilde{\pi}_{\psi^{2n}}$ is unramified from (B.4) so this follows; if $s_\psi \notin Z(\widehat{G})$, we conclude by induction via (B.5).

We have shown $\psi|_{W_F}$ to be unramified. Next we observe that $\tilde{\Pi}(\psi)$ contains a unique unramified representation by a similar induction. If $s_\psi \in Z(\widehat{G})$ then this is true because the right hand side of (B.4) is 1 in the normalization of *loc. cit.* If $s_\psi \notin Z(\widehat{G})$, this follows since the right hand side of (B.5) is equal to 1 by induction hypothesis.

As in the proof of (1), we have the embedding $\tilde{\Pi}(\phi_\psi) \hookrightarrow \tilde{\Pi}(\psi)$. Since ϕ_ψ is unramified, $\tilde{\Pi}(\phi_\psi)$ contains an unramified representation. Now we appeal to the special property that $\mathbf{1} \in \tilde{\Pi}(\psi)$.

By the uniqueness shown above, $\mathbf{1} \in \tilde{\Pi}(\psi)$ belongs to $\tilde{\Pi}(\phi_\psi)$. Therefore ϕ_ψ is the (unramified) L-parameter of $\mathbf{1}$. It follows that $\psi = \psi_{\mathrm{triv}}$.

Case 2: when G is ramified. Although it might be possible to give a local proof, our strategy is to make use of Case 1 via globalization. Our first claim is that $\psi \in \tilde{\Psi}_2(G)$ if $\mathbf{1} \in \tilde{\Pi}(\psi)$. If the claim was false, then there exists a proper Levi subgroup M of G such that ψ is the image of some $\psi_M \in \tilde{\Psi}_2(M)$. According to Arthur's construction of $\tilde{\Pi}(\psi)$ (see [1, Prop. 2.4.3]), $\mathbf{1}$ (as well as the other members of $\tilde{\Pi}(G)$) appears as a subquotient of a parabolic induction $\mathrm{Ind}_P^G(\sigma)$, where P is a parabolic subgroup of G with Levi factor M, and $\sigma \in \tilde{\Pi}(\psi_M)$. Since σ is unitary, $\mathrm{Ind}_P^G(\sigma)$

is also unitary, thus semisimple. From this and Frobenius reciprocity,

$$0 \neq \mathrm{Hom}_{G(F)}(\mathbf{1}, \mathrm{Ind}_P^G(\sigma)) = \mathrm{Hom}_{M(F)}(\mathrm{Jac}_P^G \mathbf{1}, \sigma),$$

where Jac_P^G denotes the normalized Jacquet module with respect to P. Since $\mathrm{Jac}_P^G \mathbf{1} = \delta_P^{-1/2}$ is a non-unitary character, this is a contradiction. The claim is proved.

As $\psi \in \tilde{\Psi}_2(G)$, we may mimic [1, Prop. 7.2.1] to globalize $(E/F, G, \psi)$ to $(\dot{E}/\dot{F}, \dot{G}, \dot{\psi})$ with $\dot{\psi} \in \tilde{\Psi}_2(\dot{G})$ such that the latter triple specializes to $(E/F, G, \psi)$ at $\mathfrak{q}$, and $\mathcal{S}_{\dot{\psi}} \xrightarrow{\sim} \mathcal{S}_\psi$. We may further arrange (possibly by choosing a new globalization) that there exists another prime $\mathfrak{q}' \neq \mathfrak{q}$ such that $(\dot{E}/\dot{F}, \dot{G}, \dot{\psi})$ specializes to $(E/F, G, \psi)$ at $\mathfrak{q}'$ as well. Since $\mathbf{1} \in \tilde{\Pi}(\psi) = \tilde{\Pi}(\dot{\psi}_{\mathfrak{q}})$, we can find a discrete automorphic representation $\dot{\pi}$ such that $\dot{\pi}_{\mathfrak{q}} \simeq \mathbf{1}$ by Arthur's multiplicity formula [1, Thm. 1.5.2]. (Let $\chi : \mathcal{S}_\psi \to \mathbb{C}^\times$ denote the character corresponding to $\mathbf{1}$ in [1, Thm. 1.5.1]. We can find $\dot{\pi} \in \tilde{\Pi}_{\dot{\psi}}$ such that $\dot{\pi}_{\mathfrak{q}} = \mathbf{1}$, $\dot{\pi}_v$ corresponds to the trivial character of $\mathcal{S}_{\dot{\psi}}$ at $v \neq \mathfrak{q}, \mathfrak{q}'$, and $\dot{\pi}_{\mathfrak{q}'}$ corresponds to $\epsilon_{\dot{\psi}}\chi^{-1}$. Then $\dot{\pi} \in \tilde{\Pi}_{\dot{\psi}}(\epsilon_{\dot{\psi}})$ by the choices.) As $\dot{\pi}_{\mathfrak{q}} = \mathbf{1}$, it follows from [41, Lem 6.2] that $\dot{\pi}$ is one-dimensional (up to outer automorphism).

Then $\dot{\pi}_v$ is trivial on the image of $\mathrm{Spin}_{2n}^{E/F}(F_v)$ by [50, §2.3] for each finite place v. By bounding the Galois cohomology of $\mu_2 = \ker(\mathrm{Spin}_{2n}^{E/F} \to \mathrm{SO}_{2n}^{E/F})$ uniformly across all finite places, we see that $\dot{\pi}$ is a finite order character. This implies that there exists a finite prime $\mathfrak{p} \neq \mathfrak{q}, \mathfrak{q}'$ such that $\dot{G}$ is unramified at $\mathfrak{p}$ and that $\dot{\pi}_{\mathfrak{p}} = \mathbf{1}$. Hence $\dot{\psi}_{\mathfrak{p}}$ is determined by part (3); in particular, $\dot{\psi}_{\mathfrak{p}}|_{\mathrm{SU}(2)}$ (outside $\mathcal{L}_F$) is principal. (Part (3) is applicable to $\dot{\psi}_{\mathfrak{p}}$ thanks to Case 1 above combined with our proof of (2)⇒(3) for unramified groups.) Thus the SU(2)-part of $\dot{\psi}$ is $\mathrm{Sym}^{2n-2} \oplus \mathbf{1}$, and the same is true for $\psi = \dot{\psi}_{\mathfrak{p}}$. Therefore

$$\psi = \chi_1 \otimes \mathrm{Sym}^{2n-2} \oplus \chi_2 \otimes \mathbf{1}$$

for quadratic characters $\chi_1, \chi_2 : W_F \to \{\pm 1\}$. This tells us that $\mathcal{S}_\psi = \{1\}$. As the dual parameter is tempered, we see, using compatibility with the Aubert involution, that $\tilde{\Pi}(\psi)$ is a singleton and coincides with $\tilde{\Pi}(\phi_\psi)$. The latter is the Langlands quotient of the principal series arising from ϕ_ψ (which factors through the L-group of a minimal Levi). On the other hand, $\mathbf{1} \in \tilde{\Pi}(\psi)$ by hypothesis. We conclude that $\psi = \psi_{\mathrm{triv}}$. □

References

[1] J. Arthur. *The endoscopic classification of representations*, volume 61 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups.

[2] W. L. Baily, Jr. and A. Borel. On the compactification of arithmetically defined quotients of bounded symmetric domains. *Bull. Amer. Math. Soc.*, 70:588–593, 1964.

[3] T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor. Potential automorphy and change of weight. *Ann. of Math. (2)*, 179(2):501–609, 2014.

[4] Hyman Bass. Clifford algebras and spinor norms over a commutative ring. *Amer. J. Math.*, 96:156–206, 1974.

[5] Joël Bellaïche and Gaëtan Chenevier. Families of Galois representations and Selmer groups. *Astérisque*, (324):xii+314, 2009.

[6] Joël Bellaïche and Gaëtan Chenevier. The sign of Galois representations attached to automorphic forms for unitary groups. *Compos. Math.*, 147(5):1337–1352, 2011.

[7] I. N. Bernstein and A. V. Zelevinsky. Induced representations of reductive $\mathfrak{p}$-adic groups. I. *Ann. Sci. École Norm. Sup. (4)*, 10(4):441–472, 1977.

[8] A. Borel and N. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, volume 67 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2000.

[9] N. Bourbaki. *Éléments de mathématique. Algèbre. Chapitre 9*. Springer-Verlag, Berlin, 2007. Reprint of the 1959 original.

[10] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 4–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.

[11] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 7–9*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005. Translated from the 1975 and 1982 French originals by Andrew Pressley.

[12] Kevin Buzzard and Toby Gee. The conjectural connections between automorphic representations and Galois representations. In *Automorphic forms and Galois representations. Vol. 1*, volume 414 of *London Math. Soc. Lecture Note Ser.*, pages 135–187. Cambridge Univ. Press, Cambridge, 2014.

[13] Ana Caraiani. Local-global compatibility and the action of monodromy on nearby cycles. *Duke Math. J.*, 161(12):2311–2413, 2012.

[14] Ana Caraiani and Bao V. Le Hung. On the image of complex conjugation in certain Galois representations. *Compos. Math.*, 152(7):1476–1488, 2016.

[15] Henri Carayol. Sur les représentations l-adiques associées aux formes modulaires de Hilbert. *Ann. Sci. École Norm. Sup. (4)*, 19(3):409–468, 1986.

[16] Gaëtan Chenevier. Subgroups of Spin(7) or SO(7) with each element conjugate to some element of G_2 and applications to automorphic forms. *Doc. Math.*, 24:95–161, 2019.

[17] L. Clozel, M. Harris, and R. Taylor. Automorphy for some l-adic lifts of automorphic mod l Galois representations. *Publ. Math. Inst. Hautes Études Sci.*, (108):1–181, 2008. With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras.

[18] Laurent Clozel. Motifs et formes automorphes: applications du principe de fonctorialité. In *Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988)*, volume 10 of *Perspect. Math.*, pages 77–159. Academic Press, Boston, MA, 1990.

[19] P. Deligne. Les constantes des équations fonctionnelles des fonctions L. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 501–597. Lecture Notes in Math., Vol. 349. Springer, Berlin, 1973.

[20] P. Deligne. Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 247–289. Amer. Math. Soc., Providence, R.I., 1979.

[21] P. Deligne and J. S. Milne. Tannakian categories. In *Hodge Cycles, Motives, and Shimura Varieties*, pages 101–228. Springer Berlin Heidelberg, Berlin, Heidelberg, 1982.

[22] Pierre Deligne. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.*, (52):137–252, 1980.

[23] Gerd Faltings. On the cohomology of locally symmetric Hermitian spaces. In *Paul Dubreil and Marie-Paule Malliavin algebra seminar, 35th year (Paris, 1982)*, volume 1029 of *Lecture Notes in Math.*, pages 55–98. Springer, Berlin, 1983.

[24] L. Fargues and P. Scholze. Geometrization of the local Langlands correspondence. *in preparation.*

[25] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[26] Wee Teck Gan, Benedict H. Gross, and Dipendra Prasad. Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups. Number 346, pages 1–109. 2012. Sur les conjectures de Gross et Prasad. I.

[27] B. Gross. Odd Galois representations. `http://www.math.harvard.edu/~gross`.

[28] B. H. Gross and M. Reeder. Arithmetic invariants of discrete Langlands parameters. *Duke Math. J.*, 154(3):431–508, 2010.

[29] Benedict H. Gross. On the motive of G and the principal homomorphism $\mathrm{SL}_2 \to \widehat{G}$. *Asian J. Math.*, 1(1):208–213, 1997.

[30] Benedict H. Gross. On minuscule representations and the principal SL_2. *Represent. Theory*, 4:225–244, 2000.

[31] Benedict H. Gross and Gordan Savin. Motives with Galois group of type G_2: an exceptional theta-correspondence. *Compositio Math.*, 114(2):153–217, 1998.

[32] Thomas J. Haines. On Satake parameters for representations with parahoric fixed vectors. *Int. Math. Res. Not. IMRN*, (20):10367–10398, 2015.

[33] M. Harris and R. Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[34] Michael Harris. Automorphic forms and the cohomology of vector bundles on Shimura varieties. In *Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988)*, volume 11 of *Perspect. Math.*, pages 41–91. Academic Press, Boston, MA, 1990.

[35] Michael Harris, Kai-Wen Lan, Richard Taylor, and Jack Thorne. On the rigid cohomology of certain Shimura varieties. *Res. Math. Sci.*, 3:Paper No. 37, 308, 2016.

[36] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.
[37] Tasho Kaletha. Supercuspidal L-packets. `http://www-personal.umich.edu/~kaletha/ssp.pdf`.
[38] Tasho Kaletha. The local Langlands conjectures for non-quasi-split groups. In *Families of automorphic forms and the trace formula*, Simons Symp., pages 217–257. Springer, [Cham], 2016.
[39] Tasho Kaletha. Regular supercuspidal representations. *J. Amer. Math. Soc.*, 32(4):1071–1170, 2019.
[40] Tasho Kaletha, Alberto Minguez, Sug Woo Shin, and Paul-James White. Endoscopic classification of representations: Inner forms of unitary groups. page 219 pages. `https://arxiv.org/abs/1409.3731`.
[41] Ju-Lee Kim, Sug Woo Shin, and Nicolas Templier. Asymptotic behavior of supercuspidal representations and Sato-Tate equidistribution for families. *Adv. Math.*, 362:106955, 57, 2020.
[42] Mark Kisin. mod p points on Shimura varieties of abelian type. *J. Amer. Math. Soc.*, 30(3):819–914, 2017.
[43] Mark Kisin, Sug Woo Shin, and Yihang Zhu. The stable trace formula for Shimura varieties of abelian type. *to appear in Annals of Math. Studies.*
[44] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.
[45] R. E. Kottwitz. Shimura varieties and twisted orbital integrals. *Math. Ann.*, 269(3):287–300, 1984.
[46] R. E. Kottwitz. Stable trace formula: elliptic singular terms. *Math. Ann.*, 275(3):365–399, 1986.
[47] R. E. Kottwitz. Shimura varieties and λ-adic representations. In *Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988)*, volume 10 of *Perspect. Math.*, pages 161–209. Academic Press, Boston, MA, 1990.
[48] R. E. Kottwitz. Points on some Shimura varieties over finite fields. *J. Amer. Math. Soc.*, 5(2):373–444, 1992.
[49] Arno Kret and Sug Woo Shin. Galois representations for general symplectic groups. *J. Eur. Math. Soc. (JEMS)*, 25(1):75–152, 2023.
[50] Arno Kret and Sug Woo Shin. H^0 of Igusa varieties via automorphic forms. *J. Ecole Polytech. Math.*, 10:1299–1390, 2023. `https://arxiv.org/abs/2102.10690`.
[51] Jean-Pierre Labesse and Joachim Schwermer. Central morphisms and cuspidal automorphic representations. *J. Number Theory*, 205:170–193, 2019.
[52] Kai-Wen Lan, Ruochuan Liu, and Xinwen Zhu. de Rham comparison and Poincaré duality for rigid varieties. *Peking Math. J.*, 6(1):143–216, 2023.
[53] R. P. Langlands. On the classification of irreducible representations of real algebraic groups. In *Representation theory and harmonic analysis on semisimple Lie groups*, volume 31 of *Math. Surveys Monogr.*, pages 101–170. Amer. Math. Soc., Providence, RI, 1989.
[54] Erez Lapid and Zhengyu Mao. A conjecture on Whittaker-Fourier coefficients of cusp forms. *J. Number Theory*, 146:448–505, 2015.
[55] M. Larsen. On the conjugacy of element-conjugate homomorphisms. II. *Quart. J. Math. Oxford Ser. (2)*, 47(185):73–85, 1996.
[56] Michael Larsen. On the conjugacy of element-conjugate homomorphisms. *Israel J. Math.*, 88(1-3):253–277, 1994.
[57] T. Lovering. Filtered F-crystals on Shimura varieties of abelian type. 2017. `https://arxiv.org/abs/1702.06611`.
[58] Keerthi Madapusi Pera. Toroidal compactifications of integral models of Shimura varieties of Hodge type. *Ann. Sci. Éc. Norm. Supér. (4)*, 52(2):393–514, 2019.
[59] Simon Marshall and Sug Woo Shin. Endoscopy and cohomology in a tower of congruence manifolds for $U(n,1)$. *Forum Math. Sigma*, 7:Paper No. e19, 46, 2019.
[60] J. S. Milne. Introduction to Shimura varieties. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 265–378. Amer. Math. Soc., Providence, RI, 2005.
[61] J. S. Milne. *Algebraic groups*, volume 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. The theory of group schemes of finite type over a field.
[62] Chung Pang Mok. Endoscopic classification of representations of quasi-split unitary groups. *Mem. Amer. Math. Soc.*, 235(1108):vi+248, 2015.
[63] S. Morel. *On the cohomology of certain noncompact Shimura varieties*, volume 173 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2010. With an appendix by Robert Kottwitz.
[64] Stefan Patrikis. Deformations of galois representations and exceptional monodromy. *Inventiones mathematicae*, 205(2):269–336, 2016.
[65] Stefan Patrikis and Richard Taylor. Automorphy and irreducibility of some l-adic representations. *Compos. Math.*, 151(2):207–229, 2015.
[66] Richard Pink. *Arithmetical compactification of mixed Shimura varieties*. Bonner Mathematische Schriften [Bonn Mathematical Publications], 209. Universität Bonn, Mathematisches Institut, Bonn, 1990. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1989.
[67] Richard Pink. On l-adic sheaves on Shimura varieties and their higher direct images in the Baily-Borel compactification. *Math. Ann.*, 292(2):197–240, 1992.

[68] V. Platonov and A. Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[69] Susana A. Salamanca-Riba. On the unitary dual of real reductive Lie groups and the $A_g(\lambda)$ modules: the strongly regular case. *Duke Math. J.*, 96(3):521–546, 1999.

[70] Jan Saxl and Gary M. Seitz. Subgroups of algebraic groups containing regular unipotent elements. *J. London Math. Soc. (2)*, 55(2):370–386, 1997.

[71] Peter Scholze. On torsion in the cohomology of locally symmetric varieties. *Ann. of Math. (2)*, 182(3):945–1066, 2015.

[72] J.-P. Serre. *Abelian l-adic representations and elliptic curves.* CRC Press, 1997.

[73] Jean-Pierre Serre. Exemples de plongements des groupes $\mathrm{PSL}_2(\mathbf{F}_p)$ dans des groupes de Lie simples. *Invent. Math.*, 124(1-3):525–562, 1996.

[74] M. Solleveld. On principal series representations of quasi-split reductive p-adic groups. `https://arxiv.org/abs/2304.06418`.

[75] T. A. Springer. Some arithmetical results on semi-simple Lie algebras. *Inst. Hautes Études Sci. Publ. Math.*, (30):115–141, 1966.

[76] Olivier Taïbi. Eigenvarieties for classical groups and complex conjugations in Galois representations. *Math. Res. Lett.*, 23(4):1167–1220, 2016.

[77] R. Taylor. The image of complex conjugation in l-adic representations associated to automorphic forms. *Algebra Number Theory*, 6(3):405–435, 2012.

[78] David A. Vogan, Jr. The local Langlands conjecture. In *Representation theory of groups and algebras*, volume 145 of *Contemp. Math.*, pages 305–379. Amer. Math. Soc., Providence, RI, 1993.

[79] J.-L. Waldspurger. Caractères automorphes d'un groupe réductif. `https://arxiv.org/abs/1608.07150`.

[80] Bin Xu. On a lifting problem of L-packets. *Compos. Math.*, 152(9):1800–1850, 2016.

[81] Bin Xu. L-packets of quasisplit $GSp(2n)$ and $GO(2n)$. *Math. Ann.*, 370(1-2):71–189, 2018.

[82] Alexander Frank Youcis. *The Langlands-Kottwitz Method and Deformation Spaces of p-Divisible Groups of Abelian.* ProQuest LLC, Ann Arbor, MI, 2019. Thesis (Ph.D.)–University of California, Berkeley.

[83] F. Zhang. Quaternions and matrices of quaternions. *Linear algebra and its applications*, 251:21–57, 1997.