ON THE COHOMOLOGICAL BASE CHANGE FOR UNITARY SIMILITUDE GROUPS (APPENDIX TO WUSHI GOLDRING’S PAPER)

SUG WOO SHIN

1. Appendix

This appendix is devoted to the proof of Theorem 1.1 on the automorphic base change for unitary similitude groups. The relationship with other results in the literature is explained between Theorem 1.1 and Remark 1.2. We wish to thank Wushi Goldring and Sophie Morel for their valuable comments on this appendix.

Let \( F \) be a CM field and \( F^+ \) its maximal totally real subfield so that \( [F : F^+] = 2 \). Let \( n \geq 1 \). Let \( G^1 \) be a unitary group over \( F^+ \) associated to a hermitian form on \( n \)-dimensional \( F \)-vector space. Let \( G \) be the associated unitary similitude group over \( \mathbb{Q} \) with multipliers in \( \mathbb{Q}^\times \) so that the multiplier map \( G \to \mathbb{G}_m \) has kernel \( \text{Res}_{F^+/\mathbb{Q}} G^1 \). We assume that

- \( F \) contains an imaginary quadratic subfield \( E \) (so that \( F = EF^+ \)).

However we do not assume that \( G \) is quasi-split at all finite places, nor do we impose any condition on \( G(\mathbb{R}) \). Let \((\xi, V)\) be an irreducible algebraic representation of \( G \over \mathbb{C} \). Let \( \pi \) be a discrete automorphic representation of \( G(\mathbb{A}) \) such that \( \pi_\infty \) is \( \xi \)-cohomological. The latter means that there exists \( j \geq 0 \) such that

\[
H^j(g, \xi, \pi_\infty \otimes \xi) \neq 0. \tag{1.1}
\]

Let \( S_{\text{ram}} \) be the set of finite places \( v \) of \( \mathbb{Q} \) such that either \( G \) or \( \pi \) is ramified at \( v \). Let \( G := \text{Res}_{E/\mathbb{Q}} G \). There is an \( L \)-embedding

\[
BC : L^1 \to G \rtimes \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to L^1 G \simeq (\hat{G} \rtimes \hat{G}) \rtimes \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \tag{1.2}
\]
given by \( g \times \sigma \mapsto (g, g) \rtimes \sigma \). The corresponding functoriality is usually referred to as (automorphic) base change. Although the global base change is expected to exist unconditionally, \cite[Cor 5.3]{Lab} and \cite[Prop 8.5.3]{Mor10} seem to be the best results available so far. Our modest goal is to make a small improvement on their work so that Goldring’s result applies without unnecessary restriction.

The global base change is believed to be compatible with local base change, which can be constructed explicitly and unconditionally at almost all places. There are two cases to consider.

- At finite places outside \( S_{\text{ram}} \): According to the unramified Langlands correspondence, (1.2) induces

\[
BC^{S_{\text{ram}}, \infty} : \text{Irr}^\text{ur}(G(\mathbb{A}_{S_{\text{ram}}, \infty})) \to \text{Irr}^\text{ur}(G(\mathbb{A}_{S_{\text{ram}}, \infty}))
\]

as well as a \( \mathbb{C} \)-algebra morphism \( BC^* : \mathcal{H}^\text{ur}(G(\mathbb{A}_{S_{\text{ram}}, \infty})) \to \mathcal{H}^\text{ur}(G(\mathbb{A}_{S_{\text{ram}}, \infty})) \) such that

\[
\text{tr} \pi(BC^* \phi) = \text{tr} BC^{S_{\text{ram}}, \infty}(\pi)(\phi), \quad \forall \pi \in \text{Irr}^\text{ur}(G(\mathbb{A}_{S_{\text{ram}}, \infty})), \phi \in \mathcal{H}^\text{ur}(G(\mathbb{A}_{S_{\text{ram}}, \infty})). \tag{1.3}
\]

- At a finite place \( v \) split in \( E \): Using the isomorphism \( G(\mathbb{Q}_v) \simeq G(\mathbb{Q}_v) \times G(\mathbb{Q}_v) \), define \( BC_v : \text{Irr}(G(\mathbb{Q}_v)) \to \text{Irr}(G(\mathbb{Q}_v)) \) by \( BC_v(\pi) := \pi \otimes \pi \). There is a corresponding algebra morphism \( BC' : \mathcal{H}(G(\mathbb{Q}_v)) \to \mathcal{H}(G(\mathbb{Q}_v)) \) such that

\[
\text{tr} \pi(BC'_* \phi) = \text{tr} BC^{S_{\text{ram}}, \infty}(\pi)(\phi), \quad \forall \pi \in \text{Irr}(G(\mathbb{Q}_v)), \phi \in \mathcal{H}(G(\mathbb{Q}_v)). \tag{1.4}
\]

The main theorem of appendix is the following. Let \( \chi(\cdot) \) signify the central character of a representation.

**Theorem 1.1.** For \( \pi \) and \( S_{\text{ram}} \) as above, there exists an automorphic representation \( \Pi = \psi \otimes \Pi^1 \) of \( G(\mathbb{A}) \simeq GL_1(\mathbb{A}_E) \times GL_n(\mathbb{A}_F) \) such that

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(i) $\Pi^\text{Sram,\infty} \simeq BC^S\text{ram,\infty}(\pi^\text{Sram,\infty})$, 
(ii) $\Pi_v \simeq BC_\pi(\pi_v)$ for any place $v \in S_{\text{ram}}$ which splits in $E$, 
(iii) The infinitesimal character of $\Pi_\infty$ is the same as that of $(\xi \otimes \xi)'$ of $G(\mathbb{C}) \simeq G(\mathbb{C}) \times G(\mathbb{C})$. 
(iv) $\chi|\mathbb{A}_E^* = \psi'/\psi$ and $(\Pi^1)' \simeq \Pi^1 \circ c$. 
(v) $\Pi^1$ is isomorphic to an isobaric sum $\Pi_1 \oplus \cdots \oplus \Pi_r$ for some $r \geq 1$ and discrete automorphic representations $\Pi_i$ such that $\Pi_i' \simeq \Pi_i \circ c$.

The theorem is due to Labesse ([Lab, Cor 5.3]) if the following two conditions hold.

- $\xi$ has regular highest weight or $G(\mathbb{R})$ is compact modulo center,
- $[F^+: \mathbb{Q}] \geq 2$.

(Labesse makes it clear in his footnote 1 that the second condition can be removed with additional work. He works with unitary groups rather than their similitude groups but it should not be difficult to carry over his results.) In the other cases his method does not apply as his condition (*) in the corollary 5.3 is hardly satisfied. The failure of (*) causes the trouble that the coefficients in a certain sum are no longer nonnegative and have alternating signs. His argument relies on the nonvanishing of that sum, which is not obvious when there are alternating signs.

The purity of weight for intersection cohomology is what enables us to get around the above difficulty coming from alternating signs. This strategy (already used by [CL99, Thm A.4.2, Prop A.4.3], based on a result of Kottwitz for a simpler Shimura variety) was adopted in [Mor10, Cor 8.5.3], which led to the proof of Theorem 1.1 modulo the fact that it proves (i) and (ii) outside an unspecified finite set of finite primes unless $G$ is quasi-split over $\mathbb{Q}$. (Strictly speaking, Morel works in the setting $F^+ = \mathbb{Q}$. However her method yields a similar result without that assumption. On the other hand, see [Mor10, Rem 8.5.4] for a case when the unspecified set can be specified.) Thus our contribution may be seen as getting rid of the unspecified set from the picture. For this it could have sufficed to supply necessary changes and complements to Morel’s proof. However, for the reader’s convenience and completeness of the argument, we decided to rewrite the proof, sketchy as it may sometimes be. No originality is claimed on our part.

Remark 1.2. Eventually the above theorem should be a consequence of the most general base change result for $G$, which would follow from a full stabilization of the (twisted) invariant trace formula for $G$ and $G$ and their endoscopic groups, with all the complicated terms. As such a general result would have to await some years to come, we find it reasonable to prove here a simple case, namely Theorem 1.1, especially when it has an immediate arithmetic application.

1.1. Proof of Theorem 1.1. We will freely adopt the notation and terminology of [Mor10]. (The reader may refer to the index at the end of that book.) Occasionally we adopt a few things from [Shi11] as well. The symbols $\xi$ and $V$ will be used interchangeably. (The former is used in [Shi11] while the latter in [Mor10].) One notable difference from [Mor10] is our selection of notation for groups, which is as follows:

- $G$ is a unitary similitude group as above, and $H$ denotes its elliptic endoscopic group.
- $G := \text{Res}_{E/\mathbb{Q}} G \times_{\mathbb{Q}} E$, $H := \text{Res}_{E/\mathbb{Q}} H \times_{\mathbb{Q}} E$.

(Compare this with the two different uses of $G$ and $H$ in [Mor10]. For instance see §2.3 and §8.4 in that book.)

Choose a Hecke character $\omega : \mathbb{A}_E^* \rightarrow \mathbb{C}^\times$ whose restriction to $\mathbb{A}_\mathbb{Q}^\times$ is the quadratic character associated to the extension $E/\mathbb{Q}$ via class field theory. Let $\text{Ram}(\omega)$ be the set of finite primes $v$ such that $\omega$ is ramified at a place dividing $v$. We may and will arrange that every prime $v$ in $\text{Ram}(\omega)$ splits in $E$. For each elliptic endoscopic group $H$ of $G$, one uses $\omega$ to fix an $L$-embedding $\eta : LH \rightarrow LG$ as in [Shi11, §3.2]. Then $\eta$ is unramified outside $S \cup \text{Ram}(\omega)$.

Let $p$ be a prime outside $S_{\text{ram}} \cup \text{Ram}(\omega)$ and $\varphi$ a prime of $F_0$ dividing $p$. Put $S := S_{\text{ram}} \cup \text{Ram}(\omega) \cup \{p\}$. As $G$ is unramified at $p$, it has a smooth reductive integral model over $\mathbb{Z}_p$. Choose a place $\lambda$ not divided by $p$. For $i \geq 0$, define a $\lambda$-adic vector space $H^i(\text{Sh}, V) := \lim_{\rightarrow} H^i(M^K(G, \mathcal{A})_{\overline{\mathbb{Q}}}, IC^K V_{\overline{\mathbb{Q}}})$,
where $K = K^p G(Z_p)$, and $K^p$ runs over all sufficiently small open compact subgroups $K^p$ of $G(\mathbb{A}^{p,\infty})$. Define

$$W_\lambda^+ := \sum_{2^j} H^j(S,h), \quad W_\lambda^- := \sum_{2^j} H^j(S,h), \quad W_\lambda = W_\lambda^+ - W_\lambda^-,$$

which are considered in the Grothendieck group of $\mathcal{H}(G(\mathbb{A}^{p,\infty})) \times \mathcal{H}(G(\mathbb{Q}_p),G(Z_p)) \times \text{Frob}_p^{\mathbb{Z}}$-modules (cf. [Mor10, Rem 6.3.3]). In view of Zucker's conjecture (proved by Looijenga, Saper-Stern and Looijenga-Rapoport) and the Matsushima-Borel-Casselman's formula, (1.1) implies that $H^j(S,h) \neq 0$. In particular,

$$W_\lambda^+ \neq 0 \text{ or } W_\lambda^- \neq 0. \quad (1.5)$$

It is primarily due to Beilinson, Deligne and Pink that $IC^K \mathcal{V}_\mathcal{T}$ is pure of some weight and the following is satisfied (cf. [Mor10, pp.112-113]): $IC^K \mathcal{V}_\mathcal{T}$ is pure of weight 0 there due to the assumption of the assumption that $V$ is pure of weight 0, but we do not impose it here).

**Lemma 1.3.** There exists some integer $a \in \mathbb{Z}$ such that for every $i \geq 0$, every eigenvalue $\alpha$ of Frobenius on $H^i(S,h)$ is a Weil $i + a$-number.

Corollary 6.3.2, Remark 6.3.3 and Proposition 8.3.1 of [Mor10] state that

**Proposition 1.4.** Let $f^\infty = f^{p,\infty} 1_{G(Z_p)}$ with $f^{p,\infty} \in \mathcal{H}(G(\mathbb{A}^{p,\infty}))$.

(i) One can construct a function $f^H = (f^H)_p f^H(\cdot) \in C^\infty_c(H(\mathbb{A}),\xi^{-1}_H)$ for each elliptic endoscopic triple $(H, s, \eta_0)$ such that for every sufficiently large integer $j > 0$,

$$\text{tr} \left( \Phi_p^j f^\infty | W_\lambda \right) = \sum_{(H, s, \eta_0) \in E(G)} \eta(G,H) ST_H(f^H(j)).$$

(ii) Suppose that $f^H \in C^\infty_c(H(\mathbb{A}),\xi^{-1}_H)$ and $\phi^H \in C^\infty_c(H^0(\mathbb{A}),\xi^{-1}_H)$ are associated in the sense of [Lab99, 3.2] and that $f^H_\xi$ and $\phi^H_\xi$ are as in [Mor10, Prop 8.3.1]. Then there is a constant $c \in \mathbb{R}^\times$ (independent of $\phi^H$ and $f^H$) such that

$$T^H(\phi^H) = c \cdot ST^H(f^H).$$

Now we are ready to start the proof. In the notation of diagram of [Shi11, (4.18)] (exception: $\eta$ is used instead of $\bar{\eta}$ to conform to the notation of [Mor10]), we have commutative diagrams

$$\mathcal{H}^\text{ur}(G(\mathbb{A}^{S,\infty})) \xrightarrow{\mathbb{C}^*} \mathcal{H}^\text{ur}(H(\mathbb{A}^{S,\infty})) \quad \text{Irr}^\text{ur}(G(\mathbb{A}^{S,\infty})) \xrightarrow{\mathbb{C}^*} \text{Irr}^\text{ur}(H(\mathbb{A}^{S,\infty})) \quad (1.6)$$

and similarly over $\mathbb{A}^{S,p,\infty}$. Choose any $\phi^{S,p,\infty} \in \mathcal{H}^\text{ur}(G(\mathbb{A}^{S,p,\infty}))$. Put $(\phi^H)^{S,p,\infty} := \mathbb{C}^*(\phi^{S,p,\infty}), f^{S,p,\infty} := BC^* (\phi^{S,p,\infty})$ and $(f^H)^{S,p,\infty} := \eta^*(f^{S,p,\infty})$. Take $\phi_p, \phi^H_p, f_p$ and $f^H_p$ to be the unit elements in the corresponding unramified Hecke algebras. At $S'$, choose $f_S$ and let $f_S^H$ be its transfer. Make a hypothesis, depending on $f_S$, that there exists $\phi_S$ (resp. $\phi^H_S$) whose BC-transfer is $f_S$ (resp. $f^H_S$). (This assumption will be satisfied by our later choice of $f_S$.) Since $p$ splits in $E$, one can find a function $\phi^{H,j}_p$ such that $f_p^H(j)$ and $\phi_p^{H,j} (j)$ are associated in the sense of Labesse. At infinity, by construction ([Kot90, 3.7], see also [Mor10, 6.2]) $f^H_\xi$ is a finite linear combination of Euler-Poincaré functions. Hence there exists $\phi^H_\xi$ such that $f^H_\xi$ and $\phi^H_\xi$ are associated (Mor10, Cor 8.1.11).

Applying (1.3) at finite places away from $S$ one obtains

$$\text{tr} \left( \Phi_p^j f^\infty | W_\lambda \right) = \text{tr} \left( \Phi_{p}^j f_S^H \phi^{S,\infty} | BC^{S,\infty}(W_\lambda) \right).$$

On the other hand the spectral expansion of $T^H(\phi^H)$ can be put in the form (cf. [Mor10, Prop 8.2.3] or [Art88, Thm 7.1])

$$T^H(\phi^H) = \sum_{\Pi_H} q_H^\Pi (f_S, \xi) \text{tr} \Pi^{S,p,\infty}((\phi^H)^{S,p,\infty}) \quad (1.7)$$
where \( \Pi_\mathbf{H} \) runs over automorphic representations of \( \mathbf{H}(\mathbb{A}) \) which are \( \theta \)-stable and \( \theta \)-discrete (but not necessarily discrete). Here we wrote \( a^\mathbf{H}_{\Pi_\mathbf{H}}(f_S ; j , \xi) \) for
\[
a_{\text{disc}}(\Pi_\mathbf{H}) \cdot \text{tr} ( \Pi_{\mathbf{H}, p} (\phi^\mathbf{H}_p)(j) ) A_{\Pi_{\mathbf{H}, p}} \cdot \text{tr} ( \Pi_{\mathbf{H}, S} (\phi^\mathbf{H}_S) A_{\Pi_{\mathbf{H}, S}} ) \cdot \text{tr} ( \Pi_{\mathbf{H}, \infty} (\phi^\mathbf{H}_\infty) A_{\Pi_{\mathbf{H}, \infty}} ) .
\]
(1.8)

Note that an intertwining operator for \( \theta \) is not included in the expression \( \text{tr} \Pi_{\mathbf{H}}^{S, \infty} \) of (1.7) because it does not matter for unramified representations up to sign (due to a normalization of the intertwining operator). (See the paragraph above (4.5) in [Shi11].)

We may use (1.6) to rewrite as
\[
T^\mathbf{H}(\phi^\mathbf{H}) = \sum_{\Pi_\mathbf{H}} a^\mathbf{H}_{\Pi_\mathbf{H}}(f_S ; j , \xi) \cdot \text{tr} (\Pi_{\mathbf{H}}(\phi^\mathbf{H}))(\phi^{S,p,\infty}) .
\]
Hence Proposition 1.4 tells us that \( \text{tr} (\Phi^\mathbf{I}_s, f_S, \phi^{S,\infty} | BC^{S,\infty}(W_\Lambda)) \) equals
\[
\sum_{(H,s,\eta)} \sum_{\Pi_\mathbf{H}} \iota(G,H) a^\mathbf{H}_{\Pi_\mathbf{H}}(f_S ; j , \xi) \cdot \text{tr} (\Pi_{\mathbf{H}}(\phi^\mathbf{H}))(\phi^{S,p,\infty}) .
\]

When the functions at \( S \cup \{ p, \infty \} \) are fixed, there are only finitely many terms contributing to both sides of the formula as the choice of \( \phi^{S,p,\infty} \) varies (and the other functions outside \( S \cup \{ p, \infty \} \) vary accordingly). By using the linear independence of \( \mathcal{H}^m(G(\mathbb{A}^{S,p,\infty})) \)-modules, we deduce
\[
\text{tr} (\Phi^\mathbf{I}_s, f_S | W_\Lambda (\Pi^{S,p,\infty})) = \sum_{(H,s,\eta)} \sum_{\Pi_\mathbf{H}} \iota(G,H) a^\mathbf{H}_{\Pi_\mathbf{H}}(f_S ; j , \xi) \cdot \text{tr} (\Pi_{\mathbf{H}}(\phi^\mathbf{H}))(\phi^{S,p,\infty}) .
\]

**Claim.** The left hand side of (1.9) does not vanish for some \( j \gg 0 \) and \( f_S \). Moreover this holds for \( f_S \) such that the following holds: for every \( H \), any endoscopic transfer \( f^\mathbf{H}_S \) of \( f_S \) is in the image of the BC-transfer from \( \mathbf{H} \) to \( H \). (Namely \( f^\mathbf{H}_S \) is a BC-transfer of some \( \phi^{\mathbf{H}} \)).

**Proof of claim.** For the first assertion it suffices to show that
\[
\text{tr} (f_S | W_\Lambda (\Pi^{S,\infty})) = \text{tr} (f_S | W^+_\Lambda (\Pi^{S,\infty})) - \text{tr} (f_S | W^-_\Lambda (\Pi^{S,\infty})) \in \text{Groth}(\text{Frob}_Z^Z) \otimes_Z \mathbb{C}
\]
is nontrivial. Thanks to purity of weight, it is enough to show that \( \text{tr} (f_S | W^+_\Lambda (\Pi^{S,\infty})) \neq 0 \) for either \( ? = + \) or \( ? = - \). Take \( f_S = 1_{K_S} \) for an open compact subgroup \( K_S \subset G(\mathbb{Q}_S) \). Since \( \pi \) is automorphic and cohomological, Matsushima-type formula for \( L^2 \)-cohomology (see [Art96, §2] for instance) implies that \( H^j(\mathbb{Z}, V) \) contains \( \pi \) as a \( G(\mathbb{A}^{\infty}) \)-submodule where \( j \) is as in (1.1). Hence \( \text{tr} (f_S | W^+_\Lambda (\Pi^{S,\infty})) \neq 0 \) for \( ? = + \) (resp. \( ? = - \)) when \( j \) is even (resp. odd), if \( K_S \) is small enough such that \( \pi_S \) has a nonzero \( K_S \)-fixed vector.

It remains to take care of the second requirement of the claim. This is satisfied if \( K_S \) is sufficiently small by the lemma 8.4.1.(i) of [Mor10].

The claim implies that the right hand side of (1.9) is nonzero. In particular there exists a \( \theta \)-stable and \( \theta \)-discrete automorphic representation \( \Pi_\mathbf{H} \) such that \( \zeta_\Lambda (\Pi^{S,p,\infty}) \simeq BC(\pi^{S,p,\infty}) \). Hence \( \Pi := \zeta_\Lambda (\Pi_\mathbf{H}) \), defined to be a character twist of \( \text{Ind}_{\mathbf{H}}^{\mathbf{A}}(\Pi) \) (see [Shi11, §4.4] for the precise definition), is automorphic and satisfies (iv) of the theorem, which amounts to the \( \theta \)-stable property of \( \Pi \). A fortiori assertion (v) follows easily from the construction of \( \Pi \) and the fact that \( \Pi_\mathbf{H} \) is \( \theta \)-stable and \( \theta \)-discrete. Moreover
\[
\Pi^{S,p,\infty} \simeq BC^{S,p,\infty}(\pi^{S,p,\infty}) .
\]
(1.10)

The character identities at \( v \in S \) obtained from (1.9) have the form
\[
\text{tr} (\phi_S | a_\pi \cdot + \cdots ) = \sum_{H \in I_B} \sum_{I_H} b_I \text{tr} \left( \zeta_\Lambda (\Pi^{\mathbf{H}})(f_S) A_{\zeta_\Lambda (\Pi^{\mathbf{H}})} \right)
\]
where \( a \) and \( b_I \) are nonzero complex numbers and \( I_B \) is a finite index set parametrizing \( I_H \) such that \( BC(\pi^{S,p,\infty}) = (\Pi^{\mathbf{H}})^{S,p,\infty} \) and the summand of (1.9) is nonzero. The base change character identity at split places (cf. [Shi11, 4.2]) shows that there exists \( \Pi^{\mathbf{H}} \) (i.e. on the right hand side of (1.9)) such that \( BC(\pi_S) = \Pi^{\mathbf{H}} \) for every \( v \in S \) split in \( E \). So we could have defined \( \Pi \) by using that \( \Pi^{\mathbf{H}} \). Then condition (ii) holds. Moreover, the coefficient for \( \Pi^{\mathbf{H}} = \Pi^{\mathbf{H}} \) in (1.9) being nonzero implies, in view of (1.8), that \( \Pi_{\mathbf{H}, p} \) is unramified at \( p \), since \( \phi^\mathbf{H}_p(j) \) belongs to the unramified Hecke algebra.
Recall that $S = S_{\text{ram}} \cup \text{Ram}(\omega)$ and every $v \in \text{Ram}(\omega)$ splits in $E$. Hence (1.10) is improved to
\[ \Pi^{S_{\text{ram}},p,\infty} \simeq BC^{S_{\text{ram}},p,\infty}(\pi^{S_{\text{ram}},p,\infty}). \] (1.11)

For (iii), one uses the trace computation of Euler-Poincaré functions and their twisted analogues at infinity. A careful book keeping of their infinitesimal characters yields the result.

It remains to improve on (1.11) to include the place $p$. The key point is that the choice of $p$, made at the start of the proof, was auxiliary. Choose any other prime $p'$ outside $S_{\text{ram}} \cup \text{Ram}(\omega)$ which splits in $E$ and repeat the above argument. Then we obtain $\Pi'$ satisfying $(\Pi')^{S_{\text{ram}},p',\infty} \simeq BC^{S_{\text{ram}},p',\infty}(\pi^{S_{\text{ram}},p',\infty})$ as well as (ii), (iii) and (iv). Applying Jacquet-Shalika’s strong multiplicity one to $\Pi$ and $\Pi'$, we deduce that $\Pi_p$ and $\Pi'_p$ appear as subquotients of the same parabolic induction. On the other hand, $\Pi_p$ and $\Pi'_p$ are both unramified. Indeed, we have seen this for $\Pi_p$ above, and $\Pi'_p \simeq BC(\pi_p)$ is unramified as $\pi_p$ is. Therefore, $\Pi_p \simeq \Pi'_p$ since there exists at most one unramified representation in a parabolic induction. Hence $\Pi_p \simeq BC(\pi_p)$ as desired.

References