ON THE COHOMOLOGICAL BASE CHANGE FOR UNITARY SIMILITUDE GROUPS (APPENDIX TO WUSHI GOLDRING'S PAPER)

SUG WOO SHIN

1. Appendix

[1] This appendix is devoted to the proof of Theorem 1.1 on the automorphic base change for unitary similitude groups. The relationship with other results in the literature is explained between Theorem 1.1 and Remark 1.2. We wish to thank Wushi Goldring and Sophie Morel for their valuable comments on this appendix.

Let F be a CM field and F^+ its maximal totally real subfield so that $[F:F^+]=2$. Let $n\geq 1$. Let G^1 be a unitary group over F^+ associated to a hermitian form on n-dimensional F-vector space. Let G be the associated unitary similitude group over $\mathbb{Q}$ with multipliers in $\mathbb{Q}^\times$ so that the multiplier map $G\to\mathbb{G}_m$ has kernel $\mathrm{Res}_{F^+/\mathbb{Q}}G^1$. We assume that

- F contains an imaginary quadratic subfield E (so that $F=EF^+$).

However we do not assume that G is quasi-split at all finite places, nor do we impose any condition on $G(\mathbb{R})$. Let (ξ,V) be an irreducible algebraic representation of G over $\mathbb{C}$. Let π be a discrete automorphic representation of $G(\mathbb{A})$ such that π_∞ is ξ-cohomological. The latter means that there exists $j\geq 0$ such that

$$H^j(\mathfrak{g},K,\pi_\infty\otimes\xi)\neq 0. \tag{1.1}$$

Let S_{ram} be the set of finite places v of $\mathbb{Q}$ such that either G or π is ramified at v. Let $\mathbf{G}:=\mathrm{Res}_{E/\mathbb{Q}}G$. There is an L-embedding

$$BC: {}^LG=\widehat{G}\rtimes\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to {}^L\mathbf{G}\simeq(\widehat{G}\times\widehat{G})\rtimes\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \tag{1.2}$$

given by $g\rtimes\sigma\mapsto(g,g)\rtimes\sigma$. The corresponding functoriality is usually referred to as (automorphic) base change. Although the global base change is expected to exist unconditionally, [Lab, Cor 5.3] and [Mor10, Prop 8.5.3] seem to be the best results available so far. Our modest goal is to make a small improvement on their work so that Goldring's result applies without unnecessary restriction.

The global base change is believed to be compatible with local base change, which can be constructed explicitly and unconditionally at almost all places. There are two cases to consider.

- At finite places outside S_{ram}: According to the unramified Langlands correspondence, (1.2) induces
$$BC^{S_{\mathrm{ram}},\infty}:\mathrm{Irr}^{\mathrm{ur}}(G(\mathbb{A}^{S_{\mathrm{ram}},\infty}))\to\mathrm{Irr}^{\mathrm{ur}}(\mathbf{G}(\mathbb{A}^{S_{\mathrm{ram}},\infty}))$$
as well as a $\mathbb{C}$-algebra morphism $BC^*:\mathscr{H}^{\mathrm{ur}}(\mathbf{G}(\mathbb{A}^{S_{\mathrm{ram}},\infty}))\to\mathscr{H}^{\mathrm{ur}}(G(\mathbb{A}^{S_{\mathrm{ram}},\infty}))$ such that
$$\mathrm{tr}\,\pi(BC^*\phi)=\mathrm{tr}\,BC^{S_{\mathrm{ram}},\infty}(\pi)(\phi),\qquad\forall\pi\in\mathrm{Irr}^{\mathrm{ur}}(G(\mathbb{A}^{S_{\mathrm{ram}},\infty})),\ \phi\in\mathscr{H}^{\mathrm{ur}}(\mathbf{G}(\mathbb{A}^{S_{\mathrm{ram}},\infty})). \tag{1.3}$$
- At a finite place v split in E: Using the isomorphism $\mathbf{G}(\mathbb{Q}_v)\simeq G(\mathbb{Q}_v)\times G(\mathbb{Q}_v)$, define
$$BC_v:\mathrm{Irr}(G(\mathbb{Q}_v))\to\mathrm{Irr}(\mathbf{G}(\mathbb{Q}_v))$$
by $BC_v(\pi):=\pi\otimes\pi$. There is a corresponding algebra morphism $BC^*:\mathscr{H}(\mathbf{G}(\mathbb{Q}_v))\to\mathscr{H}(G(\mathbb{Q}_v))$ such that
$$\mathrm{tr}\,\pi(BC^*\phi)=\mathrm{tr}\,BC^{S_{\mathrm{ram}},\infty}(\pi)(\phi),\qquad\forall\pi\in\mathrm{Irr}(G(\mathbb{Q}_v)),\ \phi\in\mathscr{H}(\mathbf{G}(\mathbb{Q}_v))). \tag{1.4}$$

The main theorem of appendix is the following. Let $\chi_{(\cdot)}$ signify the central character of a representation.

Theorem 1.1. *For π and S_{ram} as above, there exists an automorphic representation $\Pi=\psi\otimes\Pi^1$ of $\mathbf{G}(\mathbb{A})\simeq GL_1(\mathbb{A}_E)\times GL_n(\mathbb{A}_F)$ such that*

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(i) $\Pi^{S_{\mathrm{ram}},\infty} \simeq BC^{S_{\mathrm{ram}},\infty}(\pi^{S_{\mathrm{ram}},\infty})$,
(ii) $\Pi_v \simeq BC_v(\pi_v)$ *for any place* $v \in S_{\mathrm{ram}}$ *which splits in* E,
(iii) The infinitesimal character of Π_∞ *is the same as that of* $(\xi \otimes \xi)^\vee$ *of* $\mathbf{G}(\mathbb{C}) \simeq G(\mathbb{C}) \times G(\mathbb{C})$.
(iv) $\chi_{\Pi^1}|_{\mathbb{A}_E^\times} = \psi^c/\psi$ *and* $(\Pi^1)^\vee \simeq \Pi^1 \circ c$.
(v) Π^1 *is isomorphic to an isobaric sum* $\Pi_1 \boxplus \cdots \boxplus \Pi_r$ *for some* $r \geq 1$ *and discrete automorphic representations* Π_i *such that* $\Pi_i^\vee \simeq \Pi_i \circ c$.

The theorem is due to Labesse ([Lab, Cor 5.3]) if the following two conditions hold.

- ξ has regular highest weight or $G(\mathbb{R})$ is compact modulo center,
- $[F^+ : \mathbb{Q}] \geq 2$.

(Labesse makes it clear in his footnote 1 that the second condition can be removed with additional work. He works with unitary groups rather than their similitude groups but it should not be difficult to carry over his results.) In the other cases his method does not apply as his condition (*) in the corollary 5.3 is hardly satisfied. The failure of (*) causes the trouble that the coefficients in a certain sum are no longer nonnegative and have alternating signs. His argument relies on the nonvanishing of that sum, which is not obvious when there are alternating signs.

The purity of weight for intersection cohomology is what enables us to get around the above difficulty coming from alternating signs. This strategy (already used by [CL99, Thm A.4.2, Prop A.4.3], based on a result of Kottwitz for a simpler Shimura variety) was adopted in [Mor10, Cor 8.5.3], which led to the proof of Theorem 1.1 modulo the fact that it proves (i) and (ii) outside an unspecified finite set of finite primes unless G is quasi-split over $\mathbb{Q}$. (Strictly speaking, Morel works in the setting $F^+ = \mathbb{Q}$. However her method yields a similar result without that assumption. On the other hand, see [Mor10, Rem 8.5.4] for a case when the unspecified set can be specified.) Thus our contribution may be seen as getting rid of the unspecified set from the picture. For this it could have sufficed to supply necessary changes and complements to Morel's proof. However, for the reader's convenience and completeness of the argument, we decided to rewrite the proof, sketchy as it may sometimes be. No originality is claimed on our part.

Remark 1.2. Eventually the above theorem should be a consequence of the most general base change result for G, which would follow from a full stabilization of the (twisted) invariant trace formula for G and $\mathbf{G}$ and their endoscopic groups, with all the complicated terms. As such a general result would have to await some years to come,[2] we find it reasonable to prove here a simple case, namely Theorem 1.1, especially when it has an immediate arithmetic application.

1.1. **Proof of Theorem 1.1.** We will freely adopt the notation and terminology of [Mor10]. (The reader may refer to the index at the end of that book.) Occasionally we adopt a few things from [Shi11] as well. The symbols ξ and V will be used interchangeably. (The former is used in [Shi11] while the latter in [Mor10].) One notable difference from [Mor10] is our selection of notation for groups, which is as follows:

- G is a unitary similitude group as above, and H denotes its elliptic endoscopic group.
- $\mathbf{G} := \mathrm{Res}_{E/\mathbb{Q}} G \times_{\mathbb{Q}} E$, $\mathbf{H} := \mathrm{Res}_{E/\mathbb{Q}} H \times_{\mathbb{Q}} E$.

(Compare this with the two different uses of $\mathbf{G}$ and $\mathbf{H}$ in [Mor10]. For instance see §2.3 and §8.4 in that book.)

Choose a Hecke character $\omega : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$ whose restriction to $\mathbb{A}^\times/\mathbb{Q}^\times$ is the quadratic character associated to the extension $E/\mathbb{Q}$ via class field theory. Let $\mathrm{Ram}(\omega)$ be the set of finite primes v such that ω is ramified at a place dividing v. We may and will arrange that every prime v in $\mathrm{Ram}(\omega)$ splits in E. For each elliptic endoscopic group H of G, one uses ω to fix an L-embedding $\eta : {}^L H \to {}^L G$ as in [Shi11, §3.2]. Then η is unramified outside $S \cup \mathrm{Ram}(\omega)$.

Let p be a prime outside $S_{\mathrm{ram}} \cup \mathrm{Ram}(\omega)$ and $\wp$ a prime of F_0 dividing p. Put $S := S_{\mathrm{ram}} \cup \mathrm{Ram}(\omega) \cup \{p\}$. As G is unramified at p, it has a smooth reductive integral model over $\mathbb{Z}_p$. Choose a place λ not divided by p. For $i \geq 0$, define a λ-adic vector space

$$H^i(\mathrm{Sh}, V) := \varinjlim_K H^i(M^K(G, \mathcal{X})^*_{\overline{\mathbb{Q}}}, IC^K V_{\overline{\mathbb{Q}}}),$$

[2]At the time of press Chung Pang Mok released a paper extending Arthur's endoscopic classification for automorphic representations to quasi-split unitary groups. This represents a significant step.

where $K=K^pG(\mathbb{Z}_p)$, and K^p runs over all sufficiently small open compact subgroups K^p of $G(\mathbb{A}^{p,\infty})$. Define

$$W_\lambda^+ := \sum_{2|i} H^i(\mathrm{Sh},V), \quad W_\lambda^- := \sum_{2\nmid i} H^i(\mathrm{Sh},V), \quad W_\lambda = W_\lambda^+ - W_\lambda^-,$$

which are considered in the Grothendieck group of $\mathcal{H}(G(\mathbb{A}^{p,\infty}))\times\mathcal{H}(G(\mathbb{Q}_p),G(\mathbb{Z}_p))\times\mathrm{Frob}_\wp^{\mathbb{Z}}$-modules (cf. [Mor10, Rem 6.3.3]). In view of Zucker's conjecture (proved by Looijenga, Saper-Stern and Looijenga-Rapoport) and the Matsushima-Borel-Casselman's formula, (1.1) implies that $H^j(\mathrm{Sh},V)\neq 0$. In particular,

$$W_\lambda^+\neq 0 \quad \text{or} \quad W_\lambda^-\neq 0. \tag{1.5}$$

It is primarily due to Beilinson, Deligne and Pink that $IC^K V_{\overline{\mathbb{Q}}}$ is pure of some weight and the following is satisfied (cf. [Mor10, pp.112-113]; $IC^K V_{\overline{\mathbb{Q}}}$ is pure of weight 0 there due to the assumption of the assumption that V is pure of weight 0, but we do not impose it here).

Lemma 1.3. *There exists some integer $a\in\mathbb{Z}$ such that for every $i\geq 0$, every eigenvalue α of $\mathrm{Frob}_\wp$ on $H^i(\mathrm{Sh},V)$ is a Weil $i+a$-number.*

Corollary 6.3.2, Remark 6.3.3 and Proposition 8.3.1 of [Mor10] state that

Proposition 1.4. *Let $f^\infty = f^{p,\infty}\mathbf{1}_{G(\mathbb{Z}_p)}$ with $f^{p,\infty}\in\mathcal{H}(G(\mathbb{A}^{p,\infty}))$.*

(i) One can construct a function $f^H=(f^H)^{p,\infty}f^{H,(j)}f^H_\xi\in C_c^\infty(H(\mathbb{A}),\xi_H^{-1})$ for each elliptic endoscopic triple (H,s,η_0) such that for every sufficiently large integer $j>0$,

$$\mathrm{tr}\,(\Phi_\wp^j f^\infty|W_\lambda) = \sum_{(H,s,\eta_0)\in\mathcal{E}(G)} \iota(G,H)ST^H(f^{H,(j)}).$$

(ii) Suppose that $f^H\in C_c^\infty(H(\mathbb{A}),\xi_H^{-1})$ and $\phi^{\mathbf{H}}\in C_c^\infty(\mathbf{H}^0(\mathbb{A}),\xi_{\mathbf{H}}^{-1})$ are associated in the sense of [Lab99, 3.2] *and that f^H_∞ and $\phi^{\mathbf{H}}_\infty$ are as in* [Mor10, Prop 8.3.1]. *Then there is a constant $c\in\mathbb{R}^\times$ (independent of $\phi^{\mathbf{H}}$ and f^H) such that*

$$T^{\mathbf{H}}(\phi^{\mathbf{H}}) = c\cdot ST^H(f^H).$$

Now we are ready to start the proof. In the notation of diagram of [Shi11, (4.18)] (exception: η is used instead of $\widetilde{\eta}$ to conform to the notation of [Mor10]), we have commutative diagrams

$$\begin{array}{ccc} \mathscr{H}^{\mathrm{ur}}(\mathbf{G}(\mathbb{A}^{S,\infty})) & \xrightarrow{\widetilde{\zeta}^*} & \mathscr{H}^{\mathrm{ur}}(\mathbf{H}(\mathbb{A}^{S,\infty})) \\ {\scriptstyle BC^*}\downarrow & & \downarrow{\scriptstyle BC^*} \\ \mathscr{H}^{\mathrm{ur}}(G(\mathbb{A}^{S,\infty})) & \xrightarrow[\eta^*]{} & \mathscr{H}^{\mathrm{ur}}(H(\mathbb{A}^{S,\infty})) \end{array} \qquad \begin{array}{ccc} \mathrm{Irr}^{\mathrm{ur}}(\mathbf{G}(\mathbb{A}^{S,\infty})) & \xleftarrow{\widetilde{\zeta}_*} & \mathrm{Irr}^{\mathrm{ur}}(\mathbf{H}(\mathbb{A}^{S,\infty})) \\ {\scriptstyle BC}\uparrow & & \uparrow{\scriptstyle BC} \\ \mathrm{Irr}^{\mathrm{ur}}(G(\mathbb{A}^{S,\infty})) & \xleftarrow[\eta_*]{} & \mathrm{Irr}^{\mathrm{ur}}(H(\mathbb{A}^{S,\infty})) \end{array} \tag{1.6}$$

and similarly over $\mathbb{A}^{S,p,\infty}$. Choose any $\phi^{S,p,\infty}\in\mathcal{H}^{\mathrm{ur}}(\mathbf{G}(\mathbb{A}^{S,p,\infty}))$. Put $(\phi^H)^{S,p,\infty}:=\widetilde{\zeta}^*(\phi^{S,p,\infty})$, $f^{S,p,\infty}:=BC^*(\phi^{S,p,\infty})$ and $(f^H)^{S,p,\infty}:=\eta^*(f^{S,p,\infty})$. Take ϕ_p, ϕ_p^H, f_p and f_p^H to be the unit elements in the corresponding unramified Hecke algebras. At S, choose f_S and let f_S^H be its transfer. Make a hypothesis, depending on f_S, that there exists ϕ_S (resp. $\phi_S^{\mathbf{H}}$) whose BC-transfer is f_S (resp. f_S^H). (This assumption will be satisfied by our later choice of f_S.) Since p splits in E, one can find a function $\phi_p^{\mathbf{H},(j)}$ such that $f_p^{H,(j)}$ and $\phi_p^{\mathbf{H},(j)}$ are associated in the sense of Labesse. At infinity, by construction ([Kot90, §7], see also [Mor10, 6.2]) f^H_ξ is a finite linear combination of Euler-Poincaré functions. Hence there exists $\phi^{\mathbf{H}}_\xi$ such that f^H_ξ and $\phi^{\mathbf{H}}_\xi$ are associated ([Mor10, Cor 8.1.11]).

Applying (1.3) at finite places away from S one obtains

$$\mathrm{tr}\,(\Phi_\wp^j f^\infty|W_\lambda) = \mathrm{tr}\,(\Phi_\wp^j f_S\phi^{S,\infty}|BC^{S,\infty}(W_\lambda)).$$

On the other hand the spectral expansion of $T^{\mathbf{H}}(\phi^{\mathbf{H}})$ can be put in the form (cf. [Mor10, Prop 8.2.3] or [Art88, Thm 7.1])

$$T^{\mathbf{H}}(\phi^{\mathbf{H}}) = \sum_{\Pi_{\mathbf{H}}} a^{\mathbf{H}}_{\Pi_{\mathbf{H}}}(f_S,\xi)\,\mathrm{tr}\,\Pi_{\mathbf{H}}^{S,p,\infty}((\phi^{\mathbf{H}})^{S,p,\infty}) \tag{1.7}$$

where $\Pi_{\mathbf{H}}$ runs over automorphic representations of $\mathbf{H}(\mathbb{A})$ which are θ-stable and θ-discrete (but not necessarily discrete). Here we wrote $a^{\mathbf{H}}_{\Pi_{\mathbf{H}}}(f_S, j, \xi)$ for

$$a_{\mathrm{disc}}(\Pi_{\mathbf{H}}) \cdot \mathrm{tr}\,(\Pi_{\mathbf{H},p}(\phi_p^{\mathbf{H},(j)})A_{\Pi_{\mathbf{H},p}}) \cdot \mathrm{tr}\,(\Pi_{\mathbf{H},S}(\phi_S^{\mathbf{H}})A_{\Pi_{\mathbf{H},S}}) \cdot \mathrm{tr}\,(\Pi_{\mathbf{H},\infty}(\phi_\xi^{\mathbf{H}})A_{\Pi_{\mathbf{H},\infty}}). \tag{1.8}$$

Note that an intertwining operator for θ is not needed in the expression $\mathrm{tr}\,\Pi_{\mathbf{H}}^{S,\infty}((\phi^{\mathbf{H}})^{S,\infty})$ of (1.7) because it does not matter for unramified representations up to sign (due to a normalization of the intertwining operator). (See the paragraph above (4.5) in [Shi11].)

We may use (1.6) to rewrite as

$$T^{\mathbf{H}}(\phi^{\mathbf{H}}) = \sum_{\Pi_{\mathbf{H}}} a^{\mathbf{H}}_{\Pi_{\mathbf{H}}}(f_S, j, \xi) \cdot \mathrm{tr}\,\widetilde{\zeta}_*(\Pi_{\mathbf{H}}^{S,p,\infty})(\phi^{S,p,\infty}).$$

Hence Proposition 1.4 tells us that $\mathrm{tr}\,(\Phi_\wp^j f_S \phi^{S,\infty} | BC^{S,\infty}(W_\lambda))$ equals

$$\sum_{(H,s,\eta_0)} \sum_{\Pi_{\mathbf{H}}} \iota(G,H) a^{\mathbf{H}}_{\Pi_{\mathbf{H}}}(f_S, j, \xi) \cdot \mathrm{tr}\,\widetilde{\zeta}_*(\Pi_{\mathbf{H}}^{S,p,\infty})(\phi^{S,p,\infty}).$$

When the functions at $S \cup \{p,\infty\}$ are fixed, there are only finitely many terms contributing to both sides of the formula as the choice of $\phi^{S,p,\infty}$ varies (and the other functions outside $S \cup \{p,\infty\}$ vary accordingly). By using the linear independence of $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{A}^{S,p,\infty}))$-modules, we deduce

$$\mathrm{tr}\,(\Phi_\wp^j f_S | W_\lambda\{\Pi^{S,p,\infty}\}) = \sum_{(H,s,\eta_0)} \sum_{\substack{\Pi_{\mathbf{H}} \\ \widetilde{\zeta}_*(\Pi_{\mathbf{H}}^{S,p,\infty}) \simeq BC(\pi^{S,p,\infty})}} \iota(G,H) a^{\mathbf{H}}_{\Pi_{\mathbf{H}}}(f_S, j, \xi). \tag{1.9}$$

Claim. The left hand side of (1.9) does not vanish for some $j \gg 0$ and f_S. Moreover this holds for f_S such that the following holds: for every H, any endoscopic transfer f_S^H of f_S is in the image of the BC-transfer from $\mathbf{H}$ to H. (Namely f_S^H is a BC-transfer of some $\phi_S^{\mathbf{H}}$.)

Proof of claim. For the first assertion it suffices to show that

$$\mathrm{tr}\,(f_S | W_\lambda\{\Pi^{S,\infty}\}) = \mathrm{tr}\,(f_S | W_\lambda^+\{\Pi^{S,\infty}\}) - \mathrm{tr}\,(f_S | (W_\lambda^-\{\Pi^{S,\infty}\}) \in \mathrm{Groth}(\mathrm{Frob}_\wp^{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{C}$$

is nontrivial. Thanks to purity of weight, it is enough to show that $\mathrm{tr}\,(f_S | W_\lambda^?\{\Pi^{S,\infty}\}) \neq 0$ for either $? = +$ or $? = -$. Take $f_S = \mathbf{1}_{K_S}$ for an open compact subgroup $K_S \subset G(\mathbb{Q}_S)$. Since π is automorphic and cohomological, Matsushima-type formula for L^2-cohomology (see [Art96, §2] for instance) implies that $H^j(\mathrm{Sh}, V)$ contains π as a $G(\mathbb{A}^\infty)$-submodule where j is as in (1.1). Hence $\mathrm{tr}\,(f_S | W_\lambda^?\{\Pi^{S,\infty}\}) \neq 0$ for $? = +$ (resp. $? = -$) when j is even (resp. odd), if K_S is small enough such that π_S has a nonzero K_S-fixed vector.

It remains to take care of the second requirement of the claim. This is satisfied if K_S is sufficiently small by the lemma 8.4.1.(i) of [Mor10].

□

The claim implies that the right hand side of (1.9) is nonzero. In particular there exists a θ-stable and θ-discrete automorphic representation $\Pi_{\mathbf{H}}$ such that $\widetilde{\zeta}_*(\Pi_{\mathbf{H}}^{S,p,\infty}) \simeq BC(\pi^{S,p,\infty})$. Hence $\Pi := \widetilde{\zeta}_*(\Pi_{\mathbf{H}})$, defined to be a character twist of $\text{n-ind}_{\mathbf{H}(\mathbb{A})}^{\mathbf{G}(\mathbb{A})} \Pi_{\mathbf{H}}$ (see [Shi11, §4.4] for the precise definition), is automorphic and satisfies (iv) of the theorem, which amounts to the θ-stable property of Π. A fortiori assertion (v) follows easily from the construction of Π and the fact that Π_H is θ-stable and θ-discrete. Moreover

$$\Pi^{S,p,\infty} \simeq BC^{S,p,\infty}(\pi^{S,p,\infty}). \tag{1.10}$$

The character identities at $v \in S$ obtained from (1.9) have the form

$$\mathrm{tr}\,(\phi_S | a\pi_S + \cdots) = \sum_H \sum_{i \in I_H} b_i \mathrm{tr}\,(\widetilde{\zeta}_*(\Pi_{\mathbf{H}}^i)(f_S) A_{\widetilde{\zeta}_*(\Pi_{\mathbf{H}}^i)})$$

where a and b_i are nonzero complex numbers and I_H is a finite index set parametrizing $\Pi_{\mathbf{H}}^i$ such that $BC(\pi^{S,p,\infty}) = (\Pi_{\mathbf{H}}^i)^{S,p,\infty}$ and the summand of (1.9) is nonzero. The base change character identity at split places (cf. [Shi11, 4.2]) shows that there exists $\Pi_{\mathbf{H}}^i$ (i.e. on the right hand side of (1.9)) such that $BC_v(\pi_v) = \Pi_v$ for every $v \in S$ split in E. So we could have defined Π by using that $\Pi_{\mathbf{H}}^i$. Then condition (ii) holds. Moreover, the coefficient for $\Pi_{\mathbf{H}} = \Pi_{\mathbf{H}}^i$ in (1.9) being nonzero implies, in view of (1.8), that $\Pi_{\mathbf{H},p}$ is unramified at p, since $\phi_p^{\mathbf{H},(j)}$ belongs to the unramified Hecke algebra.

Recall that $S = S_{\mathrm{ram}} \cup \mathrm{Ram}(\varpi)$ and every $v \in \mathrm{Ram}(\varpi)$ splits in E. Hence (1.10) is improved to

$$\Pi^{S_{\mathrm{ram}},p,\infty} \simeq BC^{S_{\mathrm{ram}},p,\infty}(\pi^{S_{\mathrm{ram}},p,\infty}). \tag{1.11}$$

For (iii), one uses the trace computation of Euler-Poincaré functions and their twisted analogues at infinity. A careful book keeping of their infinitesimal characters yields the result.

It remains to improve on (1.11) to include the place p. The key point is that the choice of p, made at the start of the proof, was auxiliary. Choose any other prime p' outside $S_{\mathrm{ram}} \cup \mathrm{Ram}(\omega)$ which splits in E and repeat the above argument. Then we obtain Π' satisfying $(\Pi')^{S_{\mathrm{ram}},p',\infty} \simeq BC^{S_{\mathrm{ram}},p',\infty}(\pi^{S_{\mathrm{ram}},p',\infty})$ as well as (ii), (iii) and (iv). Applying Jacquet-Shalika's strong multiplicity one to Π and Π', we deduce that Π_p and Π'_p appear as subquotients of the same parabolic induction. On the other hand, Π_p and Π'_p are both unramified. Indeed, we have seen this for Π_p above, and $\Pi'_p \simeq BC(\pi_p)$ is unramified as π_p is. Therefore, $\Pi_p \simeq \Pi'_p$ since there exists at most one unramified representation in a parabolic induction. Hence $\Pi_p \simeq BC(\pi_p)$ as desired.

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