The Caldero–Chapoton formula for cluster algebras

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Introduction.

In this article, we state the necessary background for cluster algebras and quiver representations to formulate and prove a result of Caldero and Chapoton which gives a nice formula connecting the two subjects. In our exposition, we mostly follow the paper [CC], but we have tried to minimize the reliance of citing outside sources as much as possible. In particular, we avoid having to introduce cluster categories and quantized universal enveloping algebras. Of course, a completely self-contained account would take up far too many pages, so we have taken the liberty of assuming some results from cluster algebras and quiver representations, and a little bit of Auslander–Reiten theory. All of the necessary definitions and results are given in the first section.

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1 Background.

1.1 Quiver representations.

Throughout, we assume that Q is a finite acyclic quiver. We let Q_0 and Q_1 denote its vertices and arrows, respectively, and KQ denotes the path algebra of Q over the field K. We will identify representations of Q with modules over KQ.

Given $\alpha, \beta \in \mathbf{Z}^{Q_0}$, we define the **Euler form** by

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{(i \to j) \in Q_1} \alpha_i \beta_j. \tag{1.1}$$

Alternatively, given two representations M and N, we can define

$$\langle \underline{\dim}(M), \underline{\dim}(N) \rangle = \dim_K \operatorname{Hom}_Q(M, N) - \dim_K \operatorname{Ext}^1_Q(M, N), \tag{1.2}$$

where $\underline{\dim}(M)_i = \dim_K(M_i)$, and one can show that this agrees with (1.1), and hence only depends on the dimension vectors of the representations involved.

The projective indecomposable modules of Q are parametrized by the vertices. For each $i \in Q_0$, we define P_i as follows: for each vertex $j \in Q_0$, we have a copy of our field K if there is a path from i to j (including the trivial path), and 0 otherwise. All arrows going between two copies of K are isomorphisms in P_i. From this description, it is immediate that P_j is a submodule of P_i if and only if there is a path from i to j. We also have that $KQ = \bigoplus_{i \in Q_0} P_i$, which shows that these are all of the projectives.

We shall focus our attention on **simply-laced Dynkin quivers**: by this, we mean that the underlying graph of Q is a Dynkin diagram of type ADE. In this case, Q has an associated root system Φ, and we can label the vertices of Q so that the Euler form agrees with the Cartan form of Φ. We will need the following theorem which establishes that Q has finite representation type.

Theorem 1.1 (Gabriel)**.** *Let Q be a simply-laced Dynkin quiver. Then the map $M \mapsto \sum_i \dim(M_i)\varepsilon_i$ establishes a bijection between the indecomposable representations of Q and the positive roots of Φ, and thus Q has finitely many indecomposable representations up to isomorphism.*

1.2 Cluster algebras.

In this section, we give the relevant definitions and results for coefficient-free cluster algebras.

Fix an integer n. An integer matrix $B = (b_{i,j})$ is **skew-symmetrizable** if there exists a diagonal matrix D with nonnegative integer entries such that DB is skew-symmetric. In this case, we define the **mutation** of B at index w to be the matrix $\mu_w(B) = B'$ given by

$$b'_{y,z} = \begin{cases} -b_{y,z} & \text{if } y = w \text{ or } z = w, \\ b_{y,z} + \frac{1}{2}(|b_{y,w}|b_{w,z} + b_{y,w}|b_{w,z}|) & \text{otherwise.} \end{cases} \tag{1.3}$$

We say that B and B' are **mutation equivalent**. Note that B' is still skew-symmetrizable since DB' is also skew-symmetric.

Now let $\underline{x} = (x_1, \dots, x_n)$ be an algebraically independent generating set of $\mathbf{Q}(u_1, \dots, u_n)$. We call the pair $(\underline{x}, B)$ a **seed**, and for an index w, we define x'_w by

$$x_w x'_w = \prod_{b_{y,w}>0} x_y^{b_{y,w}} + \prod_{b_{y,w}<0} x_y^{-b_{y,w}}, \tag{1.4}$$

and $\underline{x'} = (x_1, \dots, x'_w, \dots, x_n)$. The convention for the above equation is that products taken over empty sets are equal to 1. The pair $(x', \mu_w(B))$ is the **mutation** of the seed (x, B). Define $\underline{u} = (u_1, \dots, u_n)$. The possible tuples obtained by $(\underline{u}, B)$ via successive mutations are called **clusters**, and their elements are called **cluster variables**. The variables $u_1, \dots, u_n$ are the **initial variables**.

Alternatively, in the case that B is skew-symmetric, one can think of mutations as follows: Define a quiver $Q(B)$ by letting $Q(B)_0$ be a set indexing the rows of B, and draw b_{ij} arrows from j to i if $b_{ij} > 0$. Now label the vertex i with the variable u_i. A mutation μ_k of B gives a corresponding mutation μ_k of $Q(B)$: first, for every pair of arrows $i \to k \to j$, add an arrow $i \to j$ and then delete all resulting directed 2-cycles, and second, all arrows incident with k are reversed. This gives a quiver $Q(B')$, whose labels are the same as $Q(B)$ except that u_k is replaced by u'_k. This is compatible with the matrix mutation defined above and our correspondence between quivers and skew-symmetric matrices. We shall use this idea of mutation in the sequel, although we shall only make use of mutations at sinks, so that we do not have to worry about the first mutation rule.

Now we define the (reduced) **cluster algebra** $\mathcal{A}(B)$ associated to B to be the subalgebra of $\mathbf{Q}(u_1, \dots, u_n)$ generated by the cluster variables. The first remarkable result about cluster algebras is the so-called Laurent phenomena:

Theorem 1.2 (Fomin–Zelevinsky)**.** *Let B be an skew-symmetrizable matrix. Then $\mathcal{A}(B)$ is a subalgebra of $\mathbf{Q}[u_1^{\pm}, \dots, u_n^{\pm}]$.*

Proof. See [FZ1, Theorem 3.1]. ☐

A cluster algebra is said to be of **finite type** if the number of cluster variables is finite. In this case, there is a complete classification of such cluster algebras. First, given a skew-symmetrizable matrix B, define is **Cartan counterpart** $A(B) = (a_{i,j})$ by

$$a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -|b_{i,j}| & \text{if } i \neq j. \end{cases}$$

Theorem 1.3 (Fomin–Zelevinsky). *A cluster algebra $\mathcal{A}(B)$ is of finite type if and only if B is mutation equivalent to a Cartan matrix B' of finite type. In this case, all cluster variables can be written in the form $P(u)/u^\alpha$ where $P(u)$ is a polynomial with positive integer coefficients. The number of non-initial cluster variables is the same as the number of positive roots of the corresponding root system of B'. Furthermore, B' is unique up to simultaneous reordering of columns and rows.*

Proof. See [FZ2, Theorem 1.4] for the first statement, and [FZ2, Theorem 1.10] for the second statement. □

We can refine this statement. Given a cluster algebra of finite type, it has an associated finite type Cartan matrix, and hence a finite root system Φ. Let Δ denote a choice of basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ for Φ, where we are indexing them to be compatible with the action of simple roots $s_i(\varepsilon_j) = \varepsilon_j - a_{i,j}\varepsilon_i$. Every non-initial cluster variable can be written in the form

$$x[\alpha] = \frac{P_\alpha(u)}{u_1^{\alpha_1} \cdots u_n^{\alpha_n}}.$$

Then the map $x[\alpha] \mapsto \sum_i \alpha_i \varepsilon_i$ establishes a bijection between the non-initial cluster variables of $\mathcal{A}(B)$ and the positive roots of Φ (see [FZ2, Theorem 1.9]).

In light of this, it may be reasonable to ask whether the coefficients of $P_\alpha(u)$ count something. This question is answered by the Caldero–Chapoton formula (Theorem 2.1).

1.3 Auslander–Reiten theory.

In this section, we give the basic properties of almost split sequences and Auslander–Reiten translates. A more detailed account can be found in [Wey]. Let Q be a quiver. Given a representation M, write $M^* = \mathrm{Hom}_K(M, K)$, and define the **Auslander–Reiten translate** (AR translate)

$$\tau^-(M) = \mathrm{Ext}^1_Q(M^*, KQ). \tag{1.5}$$

If M is injective, then $\tau^-(M) = 0$. The converse is also true. We will also need the following duality statement:

Theorem 1.4 (Auslander–Reiten duality). *There are natural isomorphisms*

$$\mathrm{Hom}_Q(\tau^-(M), N) \cong \mathrm{Ext}^1_Q(N, M)^* \tag{1.6}$$
$$\mathrm{Ext}^1_Q(\tau^-(M), N) \cong \mathrm{Hom}_Q(N, M)^* \tag{1.7}$$

Proof. Let $0 \to M \to I_0 \to I_1 \to 0$ be an injective resolution for M. The sequence

$$0 \to \mathrm{Hom}_Q(M^*, KQ) \to \mathrm{Hom}_Q(I_0^*, KQ) \to \mathrm{Hom}_Q(I_1^*, KQ) \to \tau^-(M) \to 0$$

is exact since τ^- kills injective modules. So we get an exact sequence

$$0 \to \mathrm{Hom}_Q(\tau^-(M), N) \to \mathrm{Hom}_Q(\mathrm{Hom}_Q(I_1^*, KQ), N) \to \mathrm{Hom}_Q(\mathrm{Hom}_Q(I_0^*, KQ), N).$$

One also has an exact sequence

$$0 \to \mathrm{Ext}^1_Q(N, M)^* \to \mathrm{Hom}_Q(N, I_1)^* \to \mathrm{Hom}_Q(N, I_0)^*,$$

so it is enough to show that there is a natural isomorphism

$$\mathrm{Hom}_Q(\mathrm{Hom}_Q(I^*, KQ), N) = \mathrm{Hom}_Q(N, I)^*$$

whenever I is injective, and which respects the above sequences. Let $P = \mathrm{Hom}_Q(I^*, KQ)$, which is projective. Note that the composition map

$$\mathrm{Hom}_Q(P, KQ) \otimes_{KQ} N \cong \mathrm{Hom}_Q(P, KQ) \otimes_{KQ} \mathrm{Hom}_Q(KQ, N) \to \mathrm{Hom}_Q(P, N)$$

is an isomorphism: this is true for $P = KQ$, so is true for its direct summands. Then

$$\mathrm{Hom}_Q(P, N) = \mathrm{Hom}_Q(P, KQ) \otimes_{KQ} N = \mathrm{Hom}_Q(N, \mathrm{Hom}_Q(P, KQ)^*)^* = \mathrm{Hom}_Q(N, I)^*,$$

where the last equality follows from the fact that $\mathrm{Hom}_Q(-, KQ)$ gives an equivalence between (left) projective and (right) projectives, and I^* is a right projective.

This establishes (1.6); (1.7) is similar. □

The translate τ^- will allow us to do inductive arguments via the following result.

Theorem 1.5. *Let Q be a simply-laced Dynkin quiver. Then every indecomposable representation M of Q is of the form $(\tau^-)^r(P_i)$ for some $r \geq 0$ and P_i a projective indecomposable.*

We introduce some terminology. Let L and M be KQ-modules. A map $f\colon L \to M$ is **right minimal** if for every $h\colon L \to L$, $fh = f$ implies that h is an automorphism. We say that f is **right almost split** if f is not a retraction, and for every map $g\colon N \to M$ which is not a retraction, there exists a map $\widetilde{g}\colon N \to L$ such that $f\widetilde{g} = g$. Dually, we can define **left minimal** and **left almost split**. A short exact sequence

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is **almost split** if f is left minimal almost split, and g is right minimal almost split. Furthermore, almost split sequences are unique in the sense that M is determined by L and N up to isomorphism.

The AR translate and almost split sequences are linked together in the following theorem.

Theorem 1.6. *Let P_i be a projective indecomposable KQ-module and pick $r > 0$ such that $(\tau^-)^r(P_i) \neq 0$. Then there exists an almost split sequence of the form*

$$0 \to (\tau^-)^{r-1}(P_i) \to \bigoplus_{j \to i} (\tau^-)^{r-1}(P_j) \oplus \bigoplus_{i \to k} (\tau^-)^r(P_k) \to (\tau^-)^r(P_i) \to 0.$$

2 The Caldero–Chapoton formula.

For this section, our treatment follows [CC]. Let Q be a simply-laced Dynkin quiver, and let M be a representation of Q over a field K. For a dimension vector $\underline{e}$, we let $\mathbf{Gr}(\underline{e}, M)$ be set of submodules of M with dimension $\underline{e}$. This is naturally a subset of the product of Grassmannians $\prod_{i \in Q_0} \mathbf{Gr}(\underline{e}_i, M_i)$, and is defined in terms of incidence relations, which are closed conditions. In particular, given $\varphi\colon M_i \to M_j$, a subrepresentation $N \subseteq M$ must satisfy $\varphi(N_i) \subseteq N_j$, and this is equivalent to asking for the vanishing of the $(\underline{e}_j + 1) \times (\underline{e}_j + 1)$ minors of the matrix whose rows are the φ applied to the basis vectors for N_i, together with basis vectors for N_j (relative to some fixed basis of M). So $\mathbf{Gr}(\underline{e}, M)$ is a projective subvariety, which we call the **quiver Grassmannian**. For $K = \mathbf{C}$, we set $\chi(\mathbf{Gr}(\underline{e}, M))$ to be the Euler characteristic of $\mathbf{Gr}(\underline{e}, M)$ thought of as a complex analytic space (in general, this variety need not be smooth). Letting $\underline{m} = \underline{\dim}(M)$, we define $X_M \in \mathbf{Q}(u_1, \dots, u_n)$ by

$$X_M = \sum_{\underline{e}} \chi(\mathbf{Gr}(\underline{e}, M)) \prod_{i=1}^{n} u_i^{-\langle \underline{e}, \varepsilon_i \rangle - \langle \varepsilon_i, \underline{m} - \underline{e} \rangle}. \tag{2.1}$$

Let E_Q be the $\mathbf{Q}[u_1, \dots, u_n]$-submodule of $\mathbf{Q}(u_1, \dots, u_n)$ generated by the X_M, as M ranges over all isomorphism classes of representations of Q. Then we have the following theorem:

Theorem 2.1 (Caldero–Chapoton)**.** *With the notation above, E_Q is a subalgebra of $\mathbf{Q}(u_1, \dots, u_n)$, and furthermore, it is equal to the cluster algebra of B_Q. The non-initial cluster variables are given by X_M where M ranges over the indecomposable representations of Q.*

The strategy of the proof is as follows. First, we show that E_Q is a subalgebra generated by the u_i and X_M for M indecomposable. In particular, we show that $X_{M \oplus N} = X_M X_N$ for all representations M and N. This is the content of Proposition 2.4. Second, we establish relations among the variables X_M for M indecomposable in Propositions 2.5 and 2.6. Finally, we show by induction with the help of the Auslander–Reiten translate τ^- and the established relations that each X_M is a cluster variable. Since each X_M is distinct, and they give the expected number of generators for $\mathcal{A}(B)$ by Theorems 1.3 and 1.1, we conclude that $\mathcal{A}(B) = E_Q$.

In order to get a handle on the Euler characteristic, we will need the following result.

Lemma 2.2. *Let X be a variety defined over $\mathbf{Z}$, and suppose that there exists a polynomial $P(t)$ such that for some prime p, $P(p^r) = \#X(\mathbf{F}_{p^r})$ for all $r \geq 1$. Then $P(t)$ has integer coefficients, and the Euler characteristic of $X(\mathbf{C})$ (as a complex analytic space and with compact cohomology) is $P(1)$.*

Proof. Fix a prime p for which there exists a polynomial $P(t) = c_d t^d + \cdots + c_1 t + c_0$ such that $P(p^r) = \#X(\mathbf{F}_{p^r})$. Let Φ be the Frobenius map $x \mapsto x^p$ on $\overline{X} = X \times \overline{\mathbf{F}_p}$ where $\overline{\mathbf{F}_p}$ denotes an algebraic closure of $\mathbf{F}_p$. For each i, let $\lambda_{i,1}, \dots, \lambda_{i,n_i}$ be the eigenvalues of Φ on the ith ℓ-adic cohomology group (with compact support) $\mathrm{H}^i_c(\overline{X}; \overline{\mathbf{Q}_\ell})$ where ℓ is some prime different from p. Since the fixed points of Φ^r are precisely $X(\mathbf{F}_{p^r})$, we have by the Grothendieck–Lefschetz trace formula (see [Del, Théorème 3.2]):

$$\sum_{m=0}^{d} c_m p^{rm} = P(p^r) = \#X(\mathbf{F}_{p^r}) = \sum_{i \geq 0} (-1)^i \sum_{j=1}^{n_i} \lambda_{i,j}^r.$$

Let A_r be the left hand side, and B_r be the right hand side. In general, sequences defined by $\sum_i a_i \gamma^i$ satisfy linear recurrences, and conversely, sequences which satisfy linear recurrences are given by such expressions. The correspondence is unique: any one piece of data determines the other, so since $A_r = B_r$ define the same sequences, we have an equality of sets $\{p^m\} = \{\lambda_{i,j}\}$. So we know that for each i and j, $\lambda_{i,j} = q^m$ for some m, and $c_m = \sum_{i \geq 0} (-1)^i \#\{j \mid \lambda_{i,j} = q^m\}$. Now plugging in $t = 1$ into $P(t)$ gives

$$P(1) = \sum_{m=0}^{d} c_m = \sum_{m=0}^{d} \sum_{i \geq 0} (-1)^i \#\{j \mid \lambda_{i,j} = q^m\} = \sum_{i \geq 0} (-1)^i \dim \mathrm{H}^i_c(\overline{X}; \overline{\mathbf{Q}_\ell}),$$

and the last term agrees with the Euler characteristic of $X(\mathbf{C})$ with compact cohomology (see [SGA4, Exposé XVI, Théorème 4.1]). ∎

The Plücker ideal of a Grassmannian is defined over $\mathbf{Z}$, as are the incidence relations for quiver Grassmannians, since the isomorphism types of the representations of Q can be defined over $\mathbf{Z}$. Hence $\mathbf{Gr}(\underline{e}, M)$ has a $\mathbf{Z}$-form. Since quiver Grassmannians are projective varieties, the condition "with compact cohomology" can be omitted. So to apply the above lemma, we need to count the number of submodules of M when $K = \mathbf{F}_q$ where q is a prime power.

Proposition 2.3. *Let Q be an acyclic quiver and M a representation of Q defined over $\mathbf{Z}$. Let $p_{\underline{e}}(M, q)$ denote the number of submodules of M of dimension $\underline{e}$ after base changing to $\mathbf{F}_q$. Then $p_{\underline{e}}(M, q)$ is a polynomial in q.*

Proof. Without loss of generality, we may assume that $M_i \neq 0$ for all $i \in Q_0$ by replacing Q with a smaller quiver if necessary. Let $s \in Q_0$ be a sink, and pick a line $L \in M_s$. Then L is a submodule of M, and every other submodule of M falls into two categories: those that contain L and those that do not. Define $\underline{e}'_j = \underline{e}_j$ if $j \neq i$, and $\underline{e}'_i = \underline{e}_i - 1$. The submodules of dimension $\underline{e}$ which contain L are in bijection with the submodules of dimension $\underline{e}'$ in M/L, and hence is counted by $p_{\underline{e}'}(M/L, q)$.

The submodules N of dimension $\underline{e}$ which do not contain L give rise to submodules $(N + L)/L$ of dimension $\underline{e}$ in M/L. The number of N' such that $N' + L = N + L$ is $q^{\underline{e}_i}$, which can be seen by picking a basis for N_i and N'_i. Of course, not every submodule of dimension $\underline{e}$ in M/L has a preimage of the form $N + L$ where $\underline{\dim}(N) = \underline{e}$. The problem lies with lines $L' \in M_j$ for $(j \to i) \in Q_1$ which map isomorphically to L. So we have to subtract off their contributions (which count submodules in $M/(L + L')$), and adjust for overcounting, etc. This process will terminate after a finite number of steps because Q is finite and acyclic. We omit the exact details, but the point is that

$$p_{\underline{e}}(M, q) = p_{\underline{e}'}(M/L, q) + q^{\underline{e}_i} p_{\underline{e}}(M/L, q) - \cdots$$

where each term is a polynomial in q by induction on $\underline{\dim}(M)$. $\square$

The first statement of Theorem 2.1 follows from the following proposition.

Proposition 2.4. *For representations M and N of Q, we have $X_M X_N = X_{M \oplus N}$. Hence E_Q is a subalgebra of $\mathbf{Q}(u_1, \dots, u_n)$ generated by $\{u_1, \dots, u_n\} \cup \{X_M \mid M \text{ indecomposable}\}$.*

Proof. By the bilinearity of the Euler form, it is enough to show that

$$\chi(\mathbf{Gr}(\underline{g}, M \oplus N)) = \sum_{\underline{e}+\underline{f}=\underline{g}} \chi(\mathbf{Gr}(\underline{e}, M))\chi(\mathbf{Gr}(\underline{f}, N)). \tag{2.2}$$

Let $\pi \colon M \oplus N \to N$ be the projection on the second factor, and define a map

$$\begin{aligned} \zeta \colon \mathbf{Gr}(\underline{g}, M \oplus N) &\to \coprod_{\underline{e}+\underline{f}=\underline{g}} (\mathbf{Gr}(\underline{e}, M) \times \mathbf{Gr}(\underline{f}, N)) \\ L &\mapsto (L \cap M, \pi(L)) \end{aligned}$$

This map is surjective since $\zeta(A \oplus B) = (A, B)$. Now base change to $\mathbf{F}_q$ so that we can count the sizes of the fibers. Define a map

$$\begin{aligned} F \colon \operatorname{Hom}_Q(B, M/A) &\to \zeta^{-1}(A, B) \\ f &\mapsto L_f = \{(m, b) \in M \oplus B \mid f(b) = m + A\} \end{aligned}$$

First, this map is well-defined because $\pi(L_f) = B$ by construction, and $L_f \cap (M, 0) = A$ since $m + A = 0$ if and only if $m \in A$. We can define an inverse of F as follows. For $L \in \zeta^{-1}(A, B)$, define a map $f_L\colon B \to M/A$ by $f_L(b) = m + A$ where $m \in M$ satisfies $(m, b) \in L$. Since $L \cap (M, 0) = A$, this is of course independent of the choice of such an m, so f_L is well-defined.

Since F is a bijection, we conclude that the fiber over (A, B) contains $q^{d(B,M/A)}$ points, where $d(B, M/A) = \dim_{\mathbf{F}_q}(\mathrm{Hom}_Q(B, M/A))$. This number d depends only on the isomorphism classes of M/A and B. Let $\varphi(X, M)$ be the number of submodules of A of M such that $M/A \cong X$, and let $\psi(Y, N)$ be the number of submodules of N isomorphic to Y. These functions turn out to be polynomials in q, but we can avoid using this fact. We know from the above discussion that

$$p_{\underline{g}}(M \oplus N, q) = \sum_{X,Y} q^{d(Y,X)} \varphi(X, M)\psi(Y, N).$$

Without the $q^{d(Y,X)}$, the right hand side counts the number of points of $\coprod_{\underline{e}+\underline{f}=\underline{g}}(\mathbf{Gr}(\underline{e}, M) \times \mathbf{Gr}(\underline{f}, N))$. So substituting $q = 1$ gives the desired equality (2.2) by Lemma 2.2. $\square$

By definition, the u_i are cluster variables. The remaining work is to show that the variables X_M for M indecomposable can be obtained from the u_i via the exchange relations (1.4). The plan is as follows: first, we show that if $M = P_i$ is a projective indecomposable, then X_{P_i} is a cluster variable. Second, we show that if X_M is a cluster variable with M indecomposable, then $X_{\tau^-(M)}$ is also a cluster variable assuming that $\tau^-(M) \neq 0$, i.e., M is not injective. Since all indecomposables are of the form $(\tau^-)^r(P_i)$ for some i and r by Theorem 1.5, this will finish the proof of the theorem.

First we prove some relations among the variables X_M.

Proposition 2.5. *For $1 \leq i \leq n$,*

$$u_i X_{P_i} = 1 + \Big(\prod_{i \to j} X_{P_j}\Big)\Big(\prod_{k \to i} u_k\Big) \tag{2.3}$$

Proof. Fix a value of i. Let $\underline{d} = \underline{\dim}(P_i)$ and $R = \mathrm{rad}(P_i)$ be the radical of P_i, which in this case is the unique maximal submodule $\bigoplus_{i \to j} P_j$. So every proper submodule M of P_i is contained in R, and we have

$$X_{P_i} = u_i^{-1} + \sum_{\underline{e}} \chi(\mathbf{Gr}(\underline{e}, R)) \prod_{k=1}^{n} u_k^{-\langle \underline{e}, \varepsilon_k \rangle - \langle \varepsilon_k, \underline{d} - \underline{e} \rangle}, \tag{2.4}$$

where the u_i^{-1} term comes from $P_i \subseteq P_i$.

The dimension vector of R is $\underline{d} - \varepsilon_i$ because $P_i/R = S_i$ is the simple module concentrated at i. Hence we also have the formula

$$\begin{aligned} X_R &= \sum_{\underline{e}} \chi(\mathbf{Gr}(\underline{e}, R)) \prod_{k=1}^{n} u_k^{-\langle \underline{e}, \varepsilon_k \rangle - \langle \varepsilon_k, \underline{d} - \varepsilon_i - \underline{e} \rangle} \\ &= \left(u_i \prod_{k \to i} u_k^{-1} \right) \sum_{\underline{e}} \chi(\mathbf{Gr}(\underline{e}, R)) \left(\prod_{k=1}^{n} u_k^{-\langle \underline{e}, \varepsilon_k \rangle - \langle \varepsilon_k, \underline{d} - \underline{e} \rangle} \right), \end{aligned} \tag{2.5}$$

where the second equality follows from the fact that $\langle \varepsilon_k, \varepsilon_i \rangle$ is 1 if $k = i$, is -1 if $(k \to i) \in Q_1$, and is 0 otherwise. Combining (2.4) and (2.5) gives

$$X_{P_i} = u_i^{-1} + \left(u_i^{-1} \prod_{k \to i} u_k \right) X_R.$$

Now multiply by u_i and use that $X_R = \prod_{i \to j} X_{P_j}$, which follows from Proposition 2.4. $\square$

Proposition 2.6. *For M a non-injective indecomposable representation of Q, we have $X_M X_{\tau^-(M)} = X_B + 1$ where B fits into an almost split exact sequence*

$$0 \to M \xrightarrow{\iota} B \xrightarrow{\pi} \tau^-(M) \to 0. \tag{2.6}$$

Proof. Write $N = \tau^-(M)$, and set $\underline{m} = \underline{\dim}(M)$ and $\underline{n} = \underline{\dim}(N)$. Let $K = \mathbf{F}_q$. By Lemma 2.2, we have

$$X_{M\oplus N} = \left(\sum_{L \subseteq M\oplus N} \prod_{i=1}^{n} u_i^{-\langle \underline{\dim}(L), \varepsilon_i \rangle - \langle \varepsilon_i, \underline{m}+\underline{n}-\underline{\dim}(L)\rangle} \right)\Bigg|_{q=1}$$

where we interpret the term in parentheses as a polynomial in q. By Theorem 1.4, we have $\langle \underline{n}, \varepsilon_i\rangle + \langle \varepsilon_i, \underline{m}\rangle = 0$, so we can separate the term $L = 0 \oplus N$ from the above sum to get

$$X_{M\oplus N} = 1 + \left(\sum_{\substack{L \subseteq M\oplus N \\ L \neq 0\oplus N}} \prod_{i=1}^{n} u_i^{-\langle \underline{\dim}(L), \varepsilon_i \rangle - \langle \varepsilon_i, \underline{m}+\underline{n}-\underline{\dim}(L)\rangle} \right)\Bigg|_{q=1}. \tag{2.7}$$

Now fix a dimension vector $\underline{g}$ and define a map

$$\begin{aligned} \zeta\colon \mathbf{Gr}(\underline{g}, B) &\to \coprod_{\underline{e}+\underline{f}=\underline{g}} \mathbf{Gr}(\underline{e}, M) \times \mathbf{Gr}(\underline{f}, N) \\ L &\mapsto (\iota^{-1}(L), \pi(L)) \end{aligned}$$

We claim that $\zeta^{-1}(0, N)$ is empty, and otherwise, the fibers are affine spaces.

For the first case, suppose $L \in \zeta^{-1}(0, N)$. Then $\pi(L) = N$ and $\iota^{-1}(L) = 0$, which means that $L \subset B$ maps isomorphically to N, which contradicts that the sequence (2.6) is not split.

So pick $(A, C) \neq (0, N)$. If $C \neq N$, then the inclusion $C \to N$ is not a retraction since N is indecomposable. So the fact that π is right almost split means that there is a section $C \to B$ of π. Then we have a split exact sequence

$$0 \to A \to B \to C \to 0,$$

and the proof of Proposition 2.4 shows that the fiber $\zeta^{-1}(A, C)$ is in bijection with $\mathrm{Hom}_Q(C, M/A)$.

So now we assume that $A \neq 0$ and $C = N$. Again by Theorem 1.4, we have an isomorphism

$$\mathrm{Ext}^1_Q(N, A) \cong \mathrm{Hom}_Q(A, M)^*,$$

which is nonzero since $A \neq 0$. Let $E \in \mathrm{Ext}^1_Q(N, A)$ be the extension corresponding to the inclusion $A \to M$ in the above isomorphism. In this extension, the surjection $\mu\colon E \to N$ is non-split, so again since π is right almost split, we have a map $\varphi\colon E \to B$ such that $\mu = \pi\varphi$. Hence the following diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & E \arrow[r, "\mu"] \arrow[d, "\varphi"] & N \arrow[r] \arrow[d, "1_N"] & 0 \\ 0 \arrow[r] & M \arrow[r, "\iota"] & B \arrow[r, "\pi"] & N \arrow[r] & 0 \end{tikzcd}$$

commutes. Now if $\varphi(e) = 0$ for $e \in E$, then $\mu(e) = 0$, so $e \in A$. Since $\iota(e) = \varphi(e) = 0$, we conclude that $e = 0$ since ι is injective. Thus φ is injective, so $E \in \zeta^{-1}(A, N)$.

Fixing the above notation, we define a map

$$\begin{aligned} F\colon \mathrm{Hom}_Q(N, M/A) &\to \zeta^{-1}(A, N) \\ f &\mapsto E_f = \{\iota(m) + \varphi(e) \mid m \in M,\ e \in E,\ f\mu(e) = m + A\}. \end{aligned}$$

Since $\pi\iota = 0$ and $\pi\varphi = \mu$, it is immediate that $\pi(E_f) = \mu(E) = N$. Similarly, if $\iota^{-1}(\varphi(e)) \neq \varnothing$, then $\mu(e) = \pi\varphi(e) \in \pi\iota(M) = 0$, so $\iota(m) + \varphi(e) \in E_f$ is in the image of ι if and only if $e = 0$ and $m \in A$. Hence $\iota^{-1}(E_f) = A$, and F is well-defined.

We define an inverse to F as follows: for $D \in \zeta^{-1}(A, N)$, define a function f_D by $f_D(n) = m + A$ for some m such that $\iota(m) + \varphi(e) \in D$ with $\mu(e) = n$. If $\mu(e') = n$ and $\iota(m') + \varphi(e') \in D$, then $e - e' \in A$, and since $\iota(m - m') + \varphi(e - e') \in D$, we conclude that $m - m' \in A$ since $\iota^{-1}(D) = A$ and φ is injective. So $f_D(n)$ is independent of the choice of m made.

The reader can verify that these maps are indeed inverse to one another. As in the proof of Proposition 2.4, we can show using ζ and (2.7) that $1 + X_B = X_{M\oplus N}$, which is equal to $X_M X_N$ by Proposition 2.4. $\square$

Corollary 2.7. *If $M = (\tau^-)^r(P_i) \neq 0$, then X_M is a cluster variable.*

Proof. Without loss of generality, assume that the vertices of Q_0 are ordered in such a way that 1 is a sink of Q, and for $i > 1$, i is a sink of $\mu_{i-1}\cdots\mu_2\mu_1(Q)$. We prove the corollary by induction on r, and then induction on the ordering of the vertices. First suppose $r = 0$.

Mutating at 1, we have an exchange relation

$$u_1 u_1' = 1 + \prod_{k\to 1} u_k.$$

Comparing with (2.3), we see that $u_1' = X_{P_1}$ since 1 is a sink, and we replace u_1 by X_{P_1} in $\mu_1(Q)$. Now for $i > 1$, we have that i is a sink of $Q' = \mu_{i-1}\cdots\mu_1(Q)$. All arrows $k \to i$ in Q' satisfy either $(k \to i) \in Q_1$ or $(i \to k) \in Q_1$, where $k < i$ in the second case since Q is a tree. So by induction, we may suppose that for $k < i$, the vertex k is labeled with X_{P_k}. Then applying the mutation μ_i to Q', we get the exchange relation

$$u_i u_i' = 1 + \Big(\prod_{(i\to j)\in Q_1} X_{P_j}\Big)\Big(\prod_{(k\to i)\in Q_1} u_k\Big),$$

so again by (2.3), we have $u_i' = X_{P_i}$. Note that $Q = \mu_n\mu_{n-1}\cdots\mu_1(Q)$ since we have flipped every arrow exactly twice.

Now suppose that $r > 0$. By induction on r, we may assume that $i \in Q_0$ is labeled with $(\tau^-)^{r-1}(P_i)$. Now the remainder of the proof follows just as above using Proposition 2.6 and Theorem 1.6. $\square$

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