Twisted commutative algebras and related structures

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Fix vector spaces V and W and let $X = V \otimes W$.
For $r \geq 0$, let X_r be the set of matrices of rank $\leq r$.

Some facts:

- The ideal $I(X_r)$ of X_r is generated by equations of degree $r+1$.
- X_r is the rth secant variety of X_1
- In char. 0, the minimal free resolution $I(X_r)$ is completely known (Lascoux)
- Similar results for $\bigwedge^2 V$ and $\mathrm{Sym}^2(V)$ (free resolution due to Jozefiak–Pragacz–Weyman)

Why free resolutions?

$I(X_r)$ is generated by all minors of size $r+1$; next term in resolution tells you what relations they satisfy (all come from Laplace expansion). Either duplicate a row and expand:

$$0 = \det \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \end{pmatrix}$$

$$= x_{1,1} \det \begin{pmatrix} x_{1,2} & x_{1,3} \\ x_{2,2} & x_{2,3} \end{pmatrix} - x_{1,2} \det \begin{pmatrix} x_{1,1} & x_{1,3} \\ x_{2,1} & x_{2,3} \end{pmatrix} + x_{1,3} \det \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$$

or compare column and row expansion (example omitted).

The rest encodes higher-order relations amongst determinants, so are basic properties of linear algebra.

Unpredicted fact: via Koszul duality, free resolutions are representations of classical Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{pe}(n)$ (Akin–Weyman, Pragacz–Weyman, Sam)

X_1 is an example of cone over $G/P \cong \mathbf{P}^n \times \mathbf{P}^m$ embedded in irrep V_λ ($G = \mathbf{SL} \times \mathbf{SL}$). The other examples were second Veronese of $\mathbf{P}^n$ and $\mathbf{Gr}(2, n)$.

So what about general G/P?

Have uniform result: $I(G/P)$ always generated by quadrics.
But seems to be unreasonable to hope for complete calculation of free resolution.

Analogs of X_r? What are the degrees of equations of secant varieties of G/P? Based on matrix example, might conjecture minimal equations have degree $r+1$.

[Wrong: Landsberg–Weyman show ideal of second secant of spinor variety of $\mathbf{Spin}(14)$ is generated by quartics.]

Secants of Segre products of projective space

Fix vector spaces $V_1, \dots, V_n$.
Segre embedding: $\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n) \subset \mathbf{P}(V_1 \otimes \cdots \otimes V_n)$

Want to understand defining equations of rth secant variety as $\dim(V_i)$ vary and as d and r vary. Some known results:

- $r = 1$ (classical): the Segre embedding is cut out by equations of degree 2
- $r = 2$ (Raicu 2010): defined by equations of degree 3 (char. 0)
- $r = 3$ (Qi 2013): defined set-theoretically by equations of degree 4 (in char. 0)
- $r = 4$ (Strassen 1983): need equations of degree 9
- ...

May be impossible to get a complete picture.

Theorem (Draisma–Kuttler 2011)

Fix r. There is a constant $d(r)$ so that the rth secant variety of a Segre embedding of projective spaces is set-theoretically defined by equations of degree $\leq d(r)$.

"Bounded rank tensors are defined in bounded degree"

The main point: it doesn't matter how many projective spaces we take or what their dimensions are – we can find $d(r)$ that works for all of them!

Questions for rest of the talk:

- Is there a scheme/ideal-theoretic version of this result?
- Bounds on degree of Gröbner bases?
- What about a statement for higher syzygies in free resolution?
- Other families of G/P? Like Veronese, or Grassmannians, or spinor varieties...

First step: given r, it suffices to work with projective spaces of some dim. $n-1$ (depending only on r).
Fix vector space V of dim. n and functional $\varphi\colon V \to \mathbf{k}$. Define

$$V^{\otimes p} \to V^{\otimes (p-1)}$$
$$v_1 \otimes \cdots \otimes v_p \mapsto \varphi(v_p) v_1 \otimes \cdots \otimes v_{p-1}.$$

Set $V^{\otimes \infty} = \varprojlim_p V^{\otimes p}$.

Two subvarieties of interest in $V^{\otimes p}$:
$X_p^{\leq r}$: rth secant variety of Segre
$Y_p^{\leq r}$: flattening variety — tensors such that all flattenings (into matrices) have rank $\leq r$

Set $X_\infty^{\leq r} = \varprojlim_p X_p^{\leq r}$ and $Y_\infty^{\leq r} = \varprojlim_p Y_p^{\leq r}$.
Then $X_\infty^{\leq r} \subseteq Y_\infty^{\leq r}$.

$G_p := S_p \ltimes \mathbf{GL}(V)^p$ acts on $V^{\otimes p}$ and preserves $X_p^{\leq r}$ and $Y_p^{\leq r}$.

There are embeddings $G_p \subset G_{p+1}$ compatible with our inverse system, so $G_\infty = \bigcup_p G_p$ acts on $V^{\otimes \infty}$ and preserves $X_\infty^{\leq r}$ and $Y_\infty^{\leq r}$.

Theorem (Draisma–Kuttler)

$Y_\infty^{\leq r} \subset V^{\otimes \infty}$ is cut out by finitely many G_∞-orbits of polynomials (certain determinants of size $r+1$).

Let Π be a monoid acting on a topological space X.
X is Π-**Noetherian** if every descending chain of Π-stable closed subsets $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ eventually stabilizes.

If X is a Π-Noetherian variety, then every Π-stable subvariety is cut out set-theoretically by finitely many Π-orbits of equations.

Theorem (Draisma–Kuttler)

$Y_\infty^{\leq r}$ is G_∞-Noetherian.

Corollary

$X_\infty^{\leq r} \subset V^{\otimes\infty}$ is cut out set-theoretically by finitely many G_∞-orbits of equations. In particular, there is a finite bound $d(r)$ on their degrees.

Pick v with $\varphi(v) = 1$; gives embedding $V^{\otimes p} \subset V^{\otimes\infty}$. Use this to transfer result on bound $d(r)$ to $X_p^{\leq r}$.

The groups G_p and G_∞ are not large enough to give ideal-theoretic statements, even for the Segre!

Intuitively, the problem is that the number of G_p-representations in the degree 2 part of the ideal of the Segre increases with p:

$I_2 \subset \mathrm{Sym}^2(V_1 \otimes \cdots \otimes V_n)$ is sum of $F_1(V_1) \otimes \cdots \otimes F_n(V_n)$ where

- each F_i is either $\bigwedge^2$ or Sym^2,
- at least one F_i is $\bigwedge^2$,
- $\bigwedge^2$ appears an even number of times.

This isn't a proof, but Draisma–Kuttler give a rigorous argument.

How to find larger group/structure?
Look to Segre embeddings for motivation.

The simplest Segre embedding:

$$\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$$

cut out by single determinant equation $x_{11}x_{22} - x_{12}x_{21}$.

More convenient to write this as

$$\mathbf{P}(V_1) \times \mathbf{P}(V_2) \subset \mathbf{P}(V_1 \otimes V_2)$$

where $\dim(V_1) = \dim(V_2) = 2$.

Two ways to get equations from smaller Segres:

- Flattening (reduce number of projective spaces):

$$\begin{aligned}\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n) \times \mathbf{P}(V_{n+1}) &\subset \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n \otimes V_{n+1}) \\ &\subset \mathbf{P}(V_1 \otimes \cdots \otimes (V_n \otimes V_{n+1}))\end{aligned}$$

  (can be done in many ways)

- Functoriality (reduce dimensions of projective spaces). Given $V_i \to V_i'$, have

$$\begin{tikzcd}
\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n) \arrow[r] \arrow[d, dashed] & \mathbf{P}(V_1 \otimes \cdots \otimes V_n) \arrow[d, dashed] \\
\mathbf{P}(V_1') \times \cdots \times \mathbf{P}(V_n') \arrow[r] & \mathbf{P}(V_1' \otimes \cdots \otimes V_n')
\end{tikzcd}$$

In both cases, can pullback equations.
All equations can be generated from the 2×2 determinant using these operations.

Δ-modules (Snowden 2010)

Δ-modules axiomatize these two operations (flattening and functoriality). We define them shortly.

The previous discussion can be summarized as: the equations of the Segre embedding is a principal Δ-module (i.e., generated by 1 element)

Δ-modules apply to other situations:

- Any family of varieties closed under flattening and functoriality (Δ-varieties) fit in the framework. The set of Δ-varieties is closed under join and taking tangents (so includes higher secants and tangential varieties of Segres)
- They extend beyond equations to arbitrary order syzygies (e.g., higher Tor groups of Segre embeddings)

Fix a field $\mathbf{k}$. Let Vec^Δ be the following category:

- Objects are pairs $(I, \{V_i\}_{i\in I})$, I a finite set and V_i finite-dimensional vector spaces.
- A morphism $(I, \{V_i\}) \to (J, \{W_j\})$ is a surjection $f\colon J \to I$ together with linear maps $V_i \to \bigotimes_{j\in f^{-1}(i)} W_j$.

Intuition: the surjection f encodes **flattenings** and the linear maps encode **functoriality**.

Definition

A Δ-**module** is a *polynomial* functor $\mathrm{Vec}^\Delta \to \mathrm{Vec}$.

(Polynomial means that morphisms transform like polynomial functions.)

Δ-modules form an abelian category, all operations calculated pointwise

- Tautological example: $(I, \{V_i\}) \mapsto \bigotimes_{i \in I} V_i$
- Ambient space: $R\colon (I, \{V_i\}) \mapsto \mathrm{Sym}^{\bullet}(\bigotimes_{i \in I} V_i)$
- Segre: $S\colon (I, \{V_i\}) \mapsto \bigoplus_{d \geq 0}(\bigotimes_{i \in I} \mathrm{Sym}^d(V_i))$
- Tor modules: $T_i\colon (I, \{V_i\}) \mapsto \mathrm{Tor}_i^{R(I,\{V_i\})}(\mathbf{k}, S(I, \{V_i\}))$.
- coordinate rings of secants, tangents, ...
- Tor modules of secants, tangents, ...

A Δ-**module** is a *polynomial* functor $F\colon \mathrm{Vec}^{\Delta} \to \mathrm{Vec}$.

An **element** of F is $x \in F(I, \{V_i\})$ for some $(I, \{V_i\}) \in \mathrm{Vec}^{\Delta}$.

The submodule of F generated by a collection of elements is the smallest submodule containing all of them.

F is **finitely generated** if generated by finitely many elements.

Theorem (Sam–Snowden)

Tor modules of Segre embeddings are finitely generated Δ-modules.

Originally proven by Snowden in characteristic 0.

F is **Noetherian** if every submodule is finitely generated.

Theorem (Sam–Snowden 2014)

Finitely generated Δ-modules are Noetherian. In particular, they have resolutions by finitely generated projective Δ-modules.

The first part was partially proven by Snowden in characteristic 0.

So the following result is new, even in characteristic 0:

Theorem (Sam–Snowden)

Tor modules of Segre embeddings are finitely presented Δ-modules.

Idea: introduce Gröbner basis ideas and prove submodules of finitely generated "free" Δ-modules have finite Gröbner bases.

New commutative algebra

- **Hilbert series:** Use characters of the general linear group to encode a Δ-module into a Hilbert series. *We can show that they are rational functions.*
- **Krull dimension:** Gabriel defined Krull dimension for objects in any Abelian category $\mathcal{A}$: the zero object has Kdim -1. Let $\mathcal{A}^{\leq d}$ be the subcategory of objects of Kdim $\leq d$. An object has Kdim $\leq d+1$ if its image in $\mathcal{A}/\mathcal{A}^{\leq d}$ has finite length. *Can we compute this for Δ-modules? Is it connected to combinatorial properties of Hilbert series?*
- **Regularity:** Minimal projective resolutions are tricky to define due to the action of general linear groups. 2 ways to fix this:
  - work in char. 0 where representation theory is semisimple,
  - modify the category so that group actions don't appear

  After fixing this, we can define Castelnuovo–Mumford regularity. *Is it finite?*

- Fix r. Are the equations of the rth secant variety of the Segre embedding a finitely generated Δ-module?

  It can be shown that, for fixed d, the degree d equations form a finitely generated Δ-module, so we are asking: is the ideal defined by equations of bounded degree (stronger property than provided by Draisma–Kuttler).

- How about the Tor modules? If we know that the equations of the rth secant variety of the Segre are defined in bounded degree, does this imply the same for all higher Tor modules?

  For the Segre, this follows from the existence of a quadratic Gröbner basis.

- Analogy: Δ-modules are like *vector spaces* and we really want to study Δ-*algebras* like $(I, \{V_i\}) \mapsto \mathrm{Sym}(\bigotimes_{i \in I} V_i)$.

Similar result for Grassmannians:

Theorem (Draisma–Eggermont 2014)

Fix r. There is a constant $d(r)$ so that the rth secant variety of the Plücker embedding of the Grassmannian is set-theoretically defined by equations of degree $\leq d(r)$.

However, we don't have an analogous theory of Δ-modules for Grassmannians.

This is already an issue for the Plücker embedding.

Kasman–Pedings–Reiszl–Shiota consider two types of maps between embedded Grassmannians:

- Functorial: $\bigwedge^p V \to \bigwedge^p V'$ from linear map $V \to V'$
- Dual: $\bigwedge^p V \cong (\bigwedge^{n-p} V)^*$

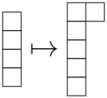
Use these to generate equations.

They show that equation of $\mathbf{Gr}(2,4) \subset \mathbf{P}(\bigwedge^2 \mathbf{k}^4)$ generates enough equations to set-theoretically define all Plücker embeddings.

However, $\mathbf{Gr}(2,4)$ does not generate all Plücker equations!

Problem is that number of representations in $\mathrm{Sym}^2(\bigwedge^p V)$ grows with p: it is a sum of Schur functors $\mathbf{S}_\lambda$ where $\lambda = (2^i, 1^{2p-2i})$ and $i \equiv p \mod 2$.

Functorial maps send $\mathbf{S}_\lambda V$ to $\mathbf{S}_\lambda V'$, while dual maps give complement of shape in (2^n), i.e., $\mathbf{Gr}(2,5) \cong \mathbf{Gr}(3,5)$ does



So $\bigwedge^{4p} \mathbf{k}^{4p} = \mathbf{S}_{1^{4p}}(\mathbf{k}^{4p}) \subset \mathrm{Sym}^2(\bigwedge^{2p} \mathbf{k}^{4p})$ are not generated by anything smaller, but they have a simple formula:

$$f_p := \sum_{S \subset [4p],\ \#S = 2p,\ 1 \in S} (-1)^S x_S x_{[4p] \setminus S}$$

Note that f_1 is the equation of $\mathbf{Gr}(2,4) \subset \mathbf{P}^5$.

Question

Is there another operation on Grassmannians that generates f_p from smaller f_i?

Recall we had an analogy: Δ-modules are like *vector spaces* and we really want to study Δ-*algebras* like $(I, \{V_i\}) \mapsto \mathrm{Sym}(\bigotimes_{i \in I} V_i)$.

Formally, there is a tensor product on Δ-modules (defined pointwise), and Δ-algebras are commutative algebras using this tensor product.

Conjecture

Finitely generated Δ-algebras are Noetherian.

We don't know how to approach this.
Twisted commutative algebras (defined next) are an intermediate structure which seem to be a lot easier.

Twisted commutative algebras

Twisted commutative algebras give a formalism for studying algebras and modules with symmetry.
We will now restrict to characteristic 0.

A **twisted commutative algebra (tca)** is a polynomial functor A from vector spaces to commutative $\mathbf{k}$-algebras.

Example: Fix a vector space U and define $A(V) = \mathrm{Sym}(U \otimes V)$. This tca is denoted $\mathrm{Sym}(U\langle 1 \rangle)$.

- All data of tca is retained if we take $V = \mathbf{k}^\infty$, and consider $A(\mathbf{k}^\infty)$ with the action of $\mathbf{GL}(\infty)$.
- A-modules M are $A(\mathbf{k}^\infty)$-modules with compatible $\mathbf{GL}(\infty)$-action.
- Notions of homomorphisms, injective, surjective, kernels, cokernels, etc. make sense and naive definitions are correct.
- M is **finitely generated** if it has a surjection from $A(\mathbf{k}^\infty) \otimes W$ where W is a finite length representation of $\mathbf{GL}(\infty)$ (i.e., direct sum of finitely many Schur functors).
- M is **Noetherian** if every submodule is finitely generated. A is Noetherian if every finitely generated module is Noetherian.

Proposition (Sam–Snowden 2012)

The category of $\mathrm{Sym}(\mathbf{k}\langle 1\rangle) = \mathrm{Sym}(\mathbf{k}^\infty)$*-modules is equivalent to the category of* **FI-modules** *defined by Church–Ellenberg–Farb.*

FI-modules are algebraic structures used to study sequences of symmetric group representations.
Examples: (co)homology of arithmetic groups, configuration spaces, $\mathcal{M}_{g,n}$, ...

Proposition (Sam–Snowden 2013)

The category of finite length $\mathrm{Sym}(\mathrm{Sym}^2(\mathbf{k}^\infty))$ *(resp.* $\mathrm{Sym}(\bigwedge^2(\mathbf{k}^\infty))$*) is equivalent to the category of finite length representations of the infinite orthogonal (resp. symplectic) group (studied by Olshanski'i, Penkov–Serganova, ...).*

Conjecture

Every finitely generated tca is noetherian.

A priori, there is a weaker notion of noetherian:

- A is **weakly noetherian** if every ideal of A is finitely generated

The previous conjecture is equivalent to

Conjecture

Every finitely generated tca is weakly noetherian.

Proof of equivalence: A is noetherian if and only if $\mathbf{S}_\lambda \otimes A$ is noetherian for all λ, and the latter is implied by $A \otimes \mathrm{Sym}(\mathbf{k}^\infty \oplus \bigwedge^2 \mathbf{k}^\infty)$ being weakly noetherian.

A tca is **bounded** if the number of rows of λ of any Schur functor appearing in its decomposition is bounded.
The study of bounded tca's can be reduced to finitely generated algebras, so they are all noetherian.

The tca's $\mathrm{Sym}(F(\mathbf{k}^\infty))$ where F is a polynomial functor of degree ≤ 1 is bounded, and is unbounded otherwise.

Theorem (Nagpal–Sam–Snowden 2015)

The tca's $\mathrm{Sym}(\mathrm{Sym}^2 \mathbf{k}^\infty)$ *and* $\mathrm{Sym}(\bigwedge^2 \mathbf{k}^\infty)$ *are noetherian.*

Concretely, for any λ, any submodule of $\mathbf{S}_\lambda(\mathbf{k}^\infty) \otimes \mathrm{Sym}(\mathrm{Sym}^2 \mathbf{k}^\infty)$ is generated by finitely many Schur functors.

We don't even know the analogous statement for $\mathrm{Sym}(\mathrm{Sym}^3 \mathbf{k}^\infty)$.
A topological version: is there an infinite descending chain of $\mathbf{GL}(\infty)$-equivariant subvarieties of $\mathrm{Sym}^3 \mathbf{k}^\infty$?