Twisted commutative algebras
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1. Motivating example: equations of secant varieties

Fix vector spaces $V_1, \dots, V_n$.

Segre embedding: $\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n) \subset \mathbf{P}(V_1 \otimes \cdots \otimes V_n)$

For $r \geq 1$, let $\Sigma(r, \mathbf{V})$ be the rth secant variety, i.e., the closure of the union of lines through r distinct points of $\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n)$.

Want to understand defining equations as $\dim(V_i)$ vary and as d and r vary. Some known results:

- $r = 1$ (classical): the Segre embedding is cut out by equations of degree 2
- $r = 2$ (Raicu 2011): defined by equations of degree 3 (in char. 0)
- $r = 3$ (Qi 2013): defined set-theoretically by equations of degree 4 (in char. 0)
- $r = 4$ (Strassen 1983): need equations of degree 9
- ...

May be impossible to get a complete picture given Strassen's equations above. But here is a workaround result:

Theorem 1.1 (Draisma–Kuttler (2011))**.** *For each r, there is a constant $d(r)$ so that $\Sigma(r, \mathbf{V})$ is set-theoretically defined by equations of degree at most $d(r)$.*

$d(r)$ is independent of n and $\dim(V_i)$, but may depend on the characteristic of the field.

Obvious conjecture:

Conjecture 1.2. *For each r, there is a constant $d(r)$ so that $\Sigma(r, \mathbf{V})$ is ideal-theoretically defined by equations of degree at most $d(r)$.*

$d(r)$ is independent of n and $\dim(V_i)$ and the characteristic of the field.

Draisma–Kuttler's methods were "topological": the key idea is to show that a certain G-space (one that contains the limit of $\Sigma(r, \mathbf{V})$ as $n \to \infty$ and $\dim(V_i) \to \infty$) is topologically noetherian when taking into account the G-action.

Need machinery to take into account scheme structures.

2. Δ-modules

Snowden introduced Δ-modules. This is a sequence of Σ_n-equivariant functors $\mathcal{F}_n \colon \mathrm{Vec}^{\times n} \to \mathrm{Vec}$ with transition maps

$$f_n \colon \mathcal{F}_n(V_1, \dots, V_{n-1}, V_n \otimes V_{n+1}) \to \mathcal{F}_{n+1}(V_1, \dots, V_n, V_{n+1}).$$

satisfying some compatibility conditions. This is a huge structure and Δ-modules form an abelian category. There is a notion of **finite generation**: $\mathcal{F}$ is finitely generated if it has finitely many elements such that the smallest Δ-submodule containing them is $\mathcal{F}$.

Translation: using the functoriality (i.e., linear maps $V_i \to V_i'$ plus the Σ_n actions plus the transition maps, every element can be built from a finite collection of elements).

For fixed r, the assignment $(V_1, \dots, V_n) \mapsto \mathbf{k}[\Sigma(r, \mathbf{V})]$ is a Δ-module where the transition maps come from the natural inclusions

$$\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n) \times \mathbf{P}(V_{n+1}) \subset \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n \otimes V_{n+1}).$$

So for each i, $(V_1, \dots, V_n) \mapsto \mathrm{Tor}_i^A(\mathbf{k}[\Sigma(r, \mathbf{V})], \mathbf{k})$ is a Δ-module where $A = \mathbf{k}[\mathbf{P}(V_1 \otimes \cdots \otimes V_n)]$.

Tor has a natural grading, and we have a result:

Theorem 2.1 (Sam–Snowden 2014)**.** *Fix i, j, r. The Δ-module $(V_1, \dots, V_n) \mapsto \mathrm{Tor}_i^A(\mathbf{k}[\Sigma(r, \mathbf{V})], \mathbf{k})_j$ is finitely presented.*

When $i = 1$, this is the space of minimal equations of degree j. So to prove the result, want to show that this vanishes for $j \gg 0$.

Example 2.2. The simplest example is $i = r = 1$ and $j = 2$ (equations of the Segre embedding):

Pick coordinates $(x_{i_1,\dots,i_n})$; the minimal equations defining this subvariety are of the form

$$x_{i_1,\dots,i_k,j_{k+1},\dots,j_n} x_{i'_1,\dots,i'_k,j'_{k+1},\dots,j'_n} - x_{i_1,\dots,i_k,j'_{k+1},\dots,j'_n} x_{i'_1,\dots,i'_k,j_{k+1},\dots,j_n}$$

if we range over k and allow permutations of the indices. These are obtained "by substitution of indices" from the 2×2 determinant $x_{1,1}x_{2,2} - x_{1,2}x_{2,1}$ which defines the simplest Segre embedding $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$.

So $\mathrm{Tor}_1(\mathbf{k}[\Sigma(1, \mathbf{V})], \mathbf{k})_2$ is generated by any nonzero element of its value on $(\mathbf{k}^2, \mathbf{k}^2)$. $\square$

Since Tor comes from homology of Koszul complex, this result follows from abstract result:

Theorem 2.3 (Sam–Snowden 2014)**.** *The category of Δ-modules is noetherian: subquotients of finitely generated Δ-modules remain finitely generated.*

Δ-modules are like a replacement for linear algebra, but really want to deal with multiplicative structures. There is a tensor product defined pointwise, can define Δ-algebras (commutative, associative). The main example is $(V_1, \dots, V_n) \mapsto \mathbf{k}[\mathbf{P}(V_1 \otimes \cdots \otimes V_n)]$. The following would imply the original conjecture.

Conjecture 2.4. *Finitely generated Δ-algebras are noetherian.*

3. Twisted commutative algebras

At present, conjecture is too hard. But there is an intermediate structure which should offer insight.

Let Vec be the category of vector spaces over $\mathbf{k}$

Intuitively, a **twisted commutative algebra** is a polynomial functor from Vec to commutative $\mathbf{k}$-algebras.

An endofunctor of Vec is **polynomial** if it is a subquotient of a direct sum of functors $V \mapsto V^{\otimes d}$ (category of endofunctors of Vec is Abelian). Let Pol be the category of polynomial functors.

This includes symmetric and exterior powers and Schur functors $\mathbf{S}_\lambda$ (labeled by integer partitions λ):

Pol has a tensor structure: $(\mathcal{F} \otimes \mathcal{G})(V) := \mathcal{F}(V) \otimes \mathcal{G}(V)$.

A **tca** $\mathcal{A}$ is a commutative algebra in $(Pol, \otimes)$, i.e., $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ such that ...

An $\mathcal{A}$-module is $\mathcal{M}$ with $\mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ such that ...

$\mathcal{M}$ is f.g. if it is a quotient of $\mathcal{A} \otimes V$ for some finite length $V \in Pol$.

Example: Fix F. Set $E \mapsto \mathrm{Sym}(E \otimes F)$. Call this tca $\mathrm{Sym}(F\langle 1\rangle)$.

Some definitions. Let $\mathcal{A}$ be a tca.

- $\mathcal{A}$ is **noetherian** if every finitely generated $\mathcal{A}$-module is noetherian.
- $\mathcal{A}$ is **weakly noetherian** if $\mathcal{A}$ is noetherian as an $\mathcal{A}$-module, i.e, every ideal is finitely generated.
- $\mathcal{A}$ is **topologically noetherian** if "$\mathrm{Spec}(\mathcal{A})$" is noetherian, i.e., every increasing chain of radical ideals stabilizes.

$\mathcal{A}$ is f.g. if it is a quotient of $\mathrm{Sym}(\mathcal{F})$ for a finite length functor $\mathcal{F}$. Polynomial functors have a degree, so we can say in which degree $\mathcal{A}$ is generated. $\mathrm{Sym}(F\langle 1\rangle)$ is generated in degree 1.

Conjecture 3.1. *A finitely generated tca is noetherian.*

Some results (in char. 0):

- Snowden 2010: tca's generated in degree ≤ 1 are noetherian.
- Eggermont 2014: tca's generated in degree ≤ 2 are topologically noetherian.
- Nagpal–Sam–Snowden 2015: $\mathrm{Sym}(\mathcal{F})$ is noetherian for $\mathcal{F} \in \{\mathrm{Sym}^2, \wedge^2\}$.

The next two steps should give insight into the general problem:

- Is $\mathrm{Sym}(\wedge^3)$ topologically noetherian?
- Is a tca generated in degree ≤ 2 noetherian?