Moduli spaces of Coble hypersurfaces
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Theme: Revisit classical algebraic geometry problems with computer algebra

1. Heisenberg groups

For simplicity, work over complex numbers $\mathbf{C}$.

- Let X be a g-dimensional abelian variety, (i.e., a projective variety with group structure.)
- Given $x \in X$, let $t_x : X \to X$ be translation by x.
- Let $\mathscr{L}$ be an ample line bundle on X and $K(\mathscr{L}) = \{x \in X \mid t_x^* \mathscr{L} \cong \mathscr{L}\}$.

Basic facts:

- There exist $d_1 | d_2 | \cdots | d_g$ such that $K(\mathscr{L}) = \mathbf{Z}/D \times \mathbf{Z}/D$ ($\mathbf{Z}/D = \mathbf{Z}/d_1 \times \cdots \times \mathbf{Z}/d_g$).
- There exists basis $(x_{i_1,\dots,i_g})_{i_j \in \mathbf{Z}/d_j}$ of $\mathrm{H}^0(X; \mathscr{L})$ such that $K(\mathscr{L})$ is generated by translations

$$\sigma_{\mathbf{v}} : x_{i_1,\dots,i_g} \mapsto x_{i_1+v_1,\dots,i_g+v_g}$$

and dilations

$$\tau_{\mathbf{v}} : x_{i_1,\dots,i_g} \mapsto \zeta_{d_1}^{v_1} \cdots \zeta_{d_g}^{v_g} x_{i_1,\dots,i_g}$$

(ζ_d is a primitive dth root of unity).

$K(\mathscr{L})$ is the **Heisenberg group** for $(X, \mathscr{L})$. Denote it by $H(D)$. This basis is the **Heisenberg normal form** for $(X, \mathscr{L})$ (note it depends only on D, so we can compare two different X in the same space).

We will assume that $(X, \mathscr{L})$ is indecomposable, i.e., not a direct product of lower-dimensional polarized abelian varieties.

Examples:

(1) $g = 2$ and $D = (3,3)$. Then $(X, \mathscr{L}) = (\mathrm{Jac}(C), 3\Theta)$ for a smooth genus 2 curve C.
$\mathscr{L}$ gives embedding $X \subset \mathbf{P}^8$.
Let $\mathscr{X}_2$ be the moduli space of such $(X, \mathscr{L})$.

(2) $g = 3$ and $D = (2,2,2)$. Then $(X, \mathscr{L}) = (\mathrm{Jac}(C), 2\Theta)$ for a smooth genus 3 curve C.
$\mathscr{L}$ gives embedding $K(X) \subset \mathbf{P}^7$ where $K(X) = X/\{x \simeq -x\}$.
We will assume C is not hyperelliptic (i.e., C is a plane quartic); this is equivalent to $K(X) \subset \mathbf{P}^7$ being projectively normal.
Let $\mathscr{X}_3$ be the moduli space of such $(X, \mathscr{L})$.

Theorem 1.1 (Beauville, Coble)**.**
In case (1), there is a unique cubic hypersurface in $\mathbf{P}^8$ whose singular locus is X.
In case (2), there is a unique quartic hypersurface in $\mathbf{P}^7$ whose singular locus is $K(X)$.

We will only consider these two cases for this talk.

2. Moduli spaces

(The data of $(X,\mathscr{L})$ can be encoded by a single equation, the Coble hypersurface.)

Get embeddings
$$\mathscr{X}_2 \subset \mathbf{P}(\mathrm{Sym}^3(\mathbf{C}^9)) \qquad \mathscr{X}_3 \subset \mathbf{P}(\mathrm{Sym}^4(\mathbf{C}^8)).$$
(Can do better: by uniqueness, Coble hypersurface is invariant under Heisenberg group — **update last line instead of rewriting**):
$$\mathscr{X}_2 \subset \mathbf{P}(\mathrm{Sym}^3(\mathbf{C}^9))^{H(3,3)}, \qquad \mathscr{X}_3 \subset \mathbf{P}(\mathrm{Sym}^4(\mathbf{C}^8))^{H(2,2,2)}.$$
A straightforward calculation shows that
$$\dim \mathbf{P}(\mathrm{Sym}^3(\mathbf{C}^9))^{H(3,3)} = 4, \qquad \dim \mathbf{P}(\mathrm{Sym}^4(\mathbf{C}^8))^{H(2,2,2)} = 14.$$
Also,
$$\dim \mathscr{X}_2 = 3, \qquad \dim \mathscr{X}_3 = 6.$$
So $\overline{\mathscr{X}_2} \subset \mathbf{P}^4$ is a hypersurface (Burkhardt quartic).

Coble gave a set of cubic equations which set-theoretically define $\overline{\mathscr{X}_3} \subset \mathbf{P}^{14}$.

Theorem 2.1 (Ren–S.–Schrader–Sturmfels)**.** *The prime ideal of $\overline{\mathscr{X}_3} \subset \mathbf{P}^{14}$ is generated by* 35 *cubics and* 35 *quartics and its homogeneous coordinate ring is Gorenstein.*

Proof. Use Macaulay2:
- We have a parametrization of $\mathscr{X}_3$ (to be explained) and use this to find the quartics.
- Can show the result is Cohen–Macaulay by calculating Hilbert series before and after cutting it by 7 random linear forms.
- Then use Jacobian criterion (i.e., show that there is a smooth point – can randomly generate a point using parametrization – by calculating Jacobian matrix) to show it is prime.
- Can read off Gorenstein from Hilbert series: Stanley showed that a Cohen–Macaulay graded domain is Gorenstein if and only if the numerator of its Hilbert series is palindromic. □

3. Hyperplane arrangements

Lift $H(2,2,2) \subset \mathbf{PGL}_7$ to $\widetilde{H}(2,2,2) \subset \mathbf{GL}_8$. Then
$$N(\widetilde{H}(2,2,2))/\widetilde{H}(2,2,2) \cong \mathbf{Sp}_6(\mathbf{F}_2).$$
(N denotes "normalizer", **Sp** denotes symplectic group). This is almost a reflection group:
$$\mathbf{Sp}_6(\mathbf{F}_2) \times \mathbf{Z}/2 \cong W(\mathrm{E}_7)$$
(Weyl group of type E_7).

There is a configuration of 63 hyperplanes (details omitted) in a hermitian space $\mathfrak{h} = \mathbf{C}^7$ whose reflections generate the finite group $W(\mathrm{E}_7)$.

For every collection of 7 mutually orthogonal hyperplanes, take product of their linear forms. The subspace spanned by all of them is 15-dimensional, and is $W(\mathrm{E}_7)$-equivariantly isomorphic to $\mathrm{Sym}^4(\mathbf{C}^8)^{\widetilde{H}(2,2,2)}$. Get rational map
$$\mathbf{P}(\mathfrak{h}) = \mathbf{P}^6 \dashrightarrow \mathbf{P}^{14} = \mathbf{P}(\mathrm{Sym}^4(\mathbf{C}^8)^{\widetilde{H}(2,2,2)})$$

whose image is $\mathscr{X}_3$. (proof omitted)

This gives desired parametrization, and can use it to find equations vanishing on $\mathscr{X}_3$. (Alternatively, can think of $\mathbf{P}^{14}$ as embedded in larger space whose coordinates represent all products of orthogonal linear forms and represent equations as binomials.)

4. Further directions

There is a similar situation for $\mathscr{X}_2$. The relevant rational map is

$$\mathbf{P}^3 \dashrightarrow \mathbf{P}^4 = \mathbf{P}(\mathrm{Sym}^3(\mathbf{C}^9)^{\widetilde{H}(3,3)}).$$

These have a common origin: slices of coregular representations, i.e., an action of a complex Lie group on a representation with a polynomial ring of invariants.

This common origin suggests many more instances of Coble hypersurfaces for embeddings of abelian varieties (into products of projective spaces and quadric hypersurfaces).

References

[B] Arnaud Beauville, The Coble hypersurfaces, arXiv:math/0306097v1.

[RSSS] Qingchun Ren, Steven V Sam, Gus Schrader, Bernd Sturmfels, The universal Kummer threefold, arXiv:1208.1229v3.