Equivariant Ehrhart Theory of the Permutahedron

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Ehrhart Theory: “Counting Lattice Points in Polytopes”

Given: A $d$-dimensional polytope $P \subset \mathbb{R}^n$ with integer vertices

- Lattice point enumerator $L_P(t) := |tP \cap \mathbb{Z}^n|$ for positive integers $t$
- Ehrhart’s Theorem: $L_P(t)$ is a polynomial of degree $d$
- Ehrhart series:

$$
\sum_{t \geq 0} L_P(t)z^t = \frac{h_P^*(z)}{(1 - z)^{d+1}}
$$

- $h_P^*(z)$ is a polynomial of degree at most $d$
Equivariant Ehrhart Theory [Stapledon 2011]

Main Idea

Generalize Ehrhart theory to also record information about the symmetries of the polytope.

Given: $d$-dim lattice polytope $P \subset \mathbb{R}^n$ with symmetry group $G$

- For each $g \in G$, get a rational fixed polytope $P^g$
- Find the Ehrhart quasipolynomial of $P^g$:

$$L_{P^g}(t) = \begin{cases} f_0(t), & t \equiv 0 \mod p \\ f_1(t), & t \equiv 1 \mod p \\ \vdots \\ f_{p-1}(t), & t \equiv p - 1 \mod p \end{cases}$$

- Find $h^*_{P^g}$ as a function of $g$
The Permutahedron

Definition

The \( n \)-permutahedron \( \Pi_n \) is the convex hull of all permutations of the coordinates of \((1, 2, \ldots, n) \in \mathbb{R}^n\).

\[
\Pi_n = \sum_{1 \leq i < j \leq n} [e_i, e_j] + \sum_{1 \leq i \leq n} e_i
\]

The permutahedron is a zonotope, i.e. a Minkowski sum of line segments:
Fixed Polytopes of the Permutahedron

- $S_n$ acts on $\Pi_n$ by permuting coordinates of points
- Each $\sigma \in S_n$ gives a fixed polytope $\Pi^\sigma_n$

**Figure:** $\Pi_4^{(12)}$ sitting inside of $\Pi_4$
Theorem [Ardila–Schindler–Vindas-Meléndez 2018]

Let $\sigma \in S_n$ be a permutation with cycles $\sigma_1, \ldots, \sigma_m$ of lengths $\ell_1, \ldots, \ell_m$. Then $\Pi^\sigma_n$ has the following zonotope description:

$$
\Pi^\sigma_n = \sum_{1 \leq j < k \leq m} [\ell_j e_{\sigma_k}, \ell_k e_{\sigma_j}] + \sum_{k=1}^{m} \frac{\ell_k + 1}{2} e_{\sigma_k}
$$

In particular, $\Pi^\sigma_n$ is half-integral and is integral exactly when all cycles of $\sigma$ have odd length.

Example: $\sigma = (12)(3)(4) \in S_4$

$$
\Pi_4^{(12)} = [2e_3, e_{12}] + [2e_4, e_{12}] + [e_4, e_3] + \frac{3}{2} e_{12} + e_3 + e_4
$$
Equivariant Ehrhart Theory of the Permutahedron

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Ehrhart Theory for Lattice Zonotopes

Zonotopes have a natural tiling by half-open parallelotopes.

Stanley’s Theorem

Let $Z$ be a zonotope generated by the integer vectors $U = \{u_1, \ldots, u_k\}$. Then the Ehrhart polynomial of $Z$ is

$$L_Z(t) = \sum_{S \subseteq U \text{ lin. indep.}} \text{Vol}(\square S) \cdot t^{|S|}$$

where $\square S$ is the parallelotope generated by $S$. 
Ehrhart Theory for Rational Zonotopes

What if $Z$ is a rational translate of a lattice zonotope?

$Z$ still decomposes into parallelotopes, but now not all of them contain lattice points.
Ehrhart Quasipolynomials of Fixed Polytopes

Let $\sigma \in S_n$ have cycle type $\lambda = (\ell_1, \ldots, \ell_m)$, $\ell_1 \geq \cdots \geq \ell_m$

- $\Pi_\sigma^n$ is the zonotope of the vectors
  \[
  \{\ell_i e_{\sigma_j} - \ell_j e_{\sigma_i} : 1 \leq i < j \leq m\}
  \]

- Linearly independent subsets of these are in bijection with forests on the vertex set $[m]$

- Group forests by the underlying set partition $\pi \models [m]$ given by their connected components.
If all $\ell_i$ are odd, then $\Pi_n^\sigma$ is integral and we have

$$L_{\Pi_n^\sigma}(t) = \sum_{\text{Forests } F \text{ on } [m]} \text{Vol}(\Box_F) \cdot t^{|E(F)|}$$

$$= \sum_{\pi \models [m]} v_\pi \cdot t^{m-|\pi|}$$

($v_\pi$ is the sum of $\text{Vol}(\Box_F)$ for all forests giving rise to $\pi$)

If some $\ell_i$ are even, then $\Pi_n^\sigma$ is half-integral.

Which forests $F$ correspond to half-open parallelotopes containing lattice points?
Let $\lambda = (\ell_1, \ldots, \ell_m)$ be the cycle type of $\sigma$.

**Definition**

A set partition $\pi \models [m]$ is $\lambda$-compatible if for all blocks $B \in \pi$, one of the following conditions holds:

- $\ell_j$ is odd for some $j \in B$, or
- the minimum 2-valuation among $\{\ell_j : j \in B\}$ is attained at least twice.

**Lemma [Ardila–S.–Vindas-Meléndez 2019+]**

The half-open parallelotope $\square_F$ contains $\text{Vol}(\square_F)$ lattice points if the underlying set partition $\pi$ of $F$ is $\lambda$-compatible. Otherwise, $\square_F$ contains no lattice points.
**Theorem [Ardila–S.–Vindas-Meléndez 2019+]**

Let $\sigma \in S_n$ have cycle type $\lambda$. Then

\[
L_{\prod_n^\sigma}(t) = \begin{cases} 
\sum_{\pi \vdash [m]} v_\pi \cdot t^{m-|\pi|}, & \text{if } t \text{ even} \\
\sum_{\pi \vdash [m], \lambda-\text{compatible}} v_\pi \cdot t^{m-|\pi|}, & \text{if } t \text{ odd}
\end{cases}
\]

where for $\pi = \{B_1, \ldots, B_k\}$, the sum of volumes $v_\pi$ is

\[
v_\pi = \prod_{i=1}^k \left( \gcd(\ell_j : j \in B_i) \cdot \left( \sum_{j \in B_i} \ell_j \right)^{|B_i|-2} \right).
\]
Corollary [Ardila–S.–Vindas-Meléndez 2019+]

Let \( \sigma \in S_n \) have cycle type \( \lambda \). The Ehrhart series of \( \Pi^\sigma_n \) is

\[
\sum_{\pi \models [m] \text{ \( \lambda \)-compatible}} v_\pi \cdot A_{m-|\pi|}(z) \frac{1}{(1 - z)^{m-|\pi|+1}} + \sum_{\pi \models [m] \text{ not \( \lambda \)-compatible}} 2^{m-|\pi|} \cdot v_\pi \cdot A_{m-|\pi|}(z^2) \frac{1}{(1 - z^2)^{m-|\pi|+1}}
\]

where \( A_{m-|\pi|} \) is an Eulerian polynomial.
Connection to Representation Theory

Let $G$ be the symmetry group of the polytope $P$
- $\rho$: $d$-dim representation of $G$ induced by action on $P$
- $\chi_{tP}$: Permutation character given by the action of $G$ on the set of lattice points $tP \cap \mathbb{Z}^n$
- Equivariant Ehrhart series:

$$
\sum_{t \geq 0} \chi_{tP} z^t = \frac{\varphi_P[z]}{(1 - z) \det(I - \rho z)}
$$

- Evaluating at $g \in G$ gives the Ehrhart series of the subpolytope of $P$ fixed by $g$
- $\varphi_P[z] = \sum_{i \geq 0} \varphi_i z^i$ is a series whose coefficients are virtual characters
Stapledon’s Conjecture

**Definition**

The series \( \sum_{i \geq 0} \varphi_i z^i \) is **effective** if all the \( \varphi_i \)'s are characters.

**Stapledon’s Conjecture (2010)**

The following are equivalent:

1. \( \varphi_P[z] \) is a polynomial
2. \( \varphi_P[z] \) is effective
3. The toric variety of \( P \) admits a \( G \)-invariant non-degenerate hypersurface
The Function $\varphi_{\Pi_n}[z]$

For $\Pi_n$, the representation $\rho$ is the standard representation of $S_n$, and we get

$$(1 - z) \det (I - \rho(\sigma) \cdot z) = \prod_{i=1}^{m} (1 - z^{\ell_i}).$$

Combining this with the Ehrhart series of $\Pi^\sigma_n$ (from before) gives us an expression for $\varphi$ evaluated at $\sigma$:

$$\varphi_{\Pi_n}[z](\sigma) = \prod_{i=1}^{m} (1 - z^{\ell_i}) \cdot \sum_{t \geq 0} \chi_{t\Pi_n(\sigma)} z^t$$

$$= \prod_{i=1}^{m} (1 - z^{\ell_i}) \cdot \sum_{t \geq 0} L_{\Pi^\sigma_n}(t) z^t$$
When is $\varphi_{\Pi_n}$ a Polynomial?

- Check: Is $\varphi_{\Pi_n}[z](\sigma)$ a polynomial $\forall \sigma \in S_n$?
- In other words, do the zeros of $\prod_{i=1}^{m}(1 - z^{\ell_i})$ cancel with the poles of the Ehrhart series?

**Theorem [Ardila–S.–Vindas-Meléndez 2019+]**

The series $\varphi_{\Pi_n}[z]$ is a polynomial if and only if $n \leq 3$.

**Theorem [Ardila–S.–Vindas-Meléndez 2019+]**

Stapledon’s conjecture holds for all permutahedra $\Pi_n$. 
Example: $\varphi_{\Pi_3}[z]$

<table>
<thead>
<tr>
<th>Cycle type of $\sigma \in S_3$</th>
<th>$\chi_{t\Pi_3}(\sigma)$</th>
<th>$\sum_{t \geq 0} \chi_{t\Pi_3}(\sigma)z^t$</th>
<th>$\varphi<a href="%5Csigma">z</a>$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 1, 1)$</td>
<td>$3t^2 + 3t + 1$</td>
<td>$\frac{1 + 4z + z^2}{(1 - z)^3}$</td>
<td>$1 + 4z + z^2$</td>
</tr>
<tr>
<td>$(2, 1)$</td>
<td>$\begin{cases} t + 1 &amp; \text{if } t \text{ is even} \ t &amp; \text{if } t \text{ is odd} \end{cases}$</td>
<td>$\frac{1 + z^2}{(1 - z)(1 - z^2)}$</td>
<td>$1 + z^2$</td>
</tr>
<tr>
<td>$(3)$</td>
<td>$1$</td>
<td>$\frac{1}{1 - z} = \frac{1 + z + z^2}{1 - z^3}$</td>
<td>$1 + z + z^2$</td>
</tr>
</tbody>
</table>

$\varphi_{\Pi_3}[z] = \chi_{triv} + (\chi_{triv} + \chi_{alt} + \chi_{std})z + \chi_{triv}z^2$

$\varphi_{\Pi_3}$ is a polynomial and is effective!
Example: $\varphi_{\Pi_4}[z]$

<table>
<thead>
<tr>
<th>Cycle type of $\sigma \in S_4$</th>
<th>$\chi_{\Pi_4}(\sigma)$</th>
<th>$\sum_{t \geq 0} \chi_{\Pi_4}(\sigma)z^t$</th>
<th>$\varphi<a href="%5Csigma">z</a>$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1, 1, 1)</td>
<td>$16t^3 + 15t^2 + 6t + 1$</td>
<td>$\frac{1 + 34z + 55z^2 + 6z^3}{(1-z)^4}$</td>
<td>$1 + 34z + 55z^2 + 6z^3$</td>
</tr>
<tr>
<td>(2, 1, 1)</td>
<td>$\begin{cases} 4t^2 + 3t + 1 &amp; \text{if } t \text{ is even} \ 4t^2 + 2t &amp; \text{if } t \text{ is odd} \end{cases}$</td>
<td>$\frac{1 + 6z + 20z^2 + 24z^3 + 11z^4 + 2z^5}{(1-z)(1-z)(1-z^2)(1+z)^2}$</td>
<td>$1 + 4z + 11z^2 + 2z^3 + \sum_{i=4}^{\infty} 4(-1)^i z^i$</td>
</tr>
<tr>
<td>(3, 1)</td>
<td>$t + 1$</td>
<td>$\frac{1}{(1-z)^2} = \frac{1 + z + z^2}{(1-z)(1-z^3)}$</td>
<td>$1 + z + z^2$</td>
</tr>
<tr>
<td>(4)</td>
<td>$\begin{cases} 1 &amp; \text{if } t \text{ is even} \ 0 &amp; \text{if } t \text{ is odd} \end{cases}$</td>
<td>$\frac{1}{1-z^2} = \frac{1 + z^2}{1-z^4}$</td>
<td>$1 + z^2$</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>$\begin{cases} 2t + 1 &amp; \text{if } t \text{ is even} \ 2t &amp; \text{if } t \text{ is odd} \end{cases}$</td>
<td>$\frac{1 + 2z + 3z^2 + 2z^3}{(1-z^2)(1-z^2)}$</td>
<td>$1 + 2z + 3z^2 + 2z^3$</td>
</tr>
</tbody>
</table>

$$
\varphi_{\Pi_4}[z] = \chi_{triv} + (3\chi_{triv} + \chi_{alt} + 5\chi_{std} + 3\chi_{\mathbb{Z}_2} + 3\chi_{\mathbb{Z}_2})z \\
+ (6\chi_{triv} + 9\chi_{std} + 4\chi_{\mathbb{Z}_2} + 5\chi_{\mathbb{Z}_2})z^2 \\
+ (\chi_{alt} + \chi_{\mathbb{Z}_2} + \chi_{\mathbb{Z}_2})z^3 + \ldots
$$

$\varphi_{\Pi_4}$ is not a polynomial and is not effective.
References


Thank you!