ADI iterations for

general elliptic problems

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July 2013
OVERVIEW

Classical alternating direction implicit (ADI) iteration
– Essentially optimal in simple domains
– Applies to narrow class of special elliptic equations
– Useful for variable-coefficient and nonlinear problems

Generalize ADI to arbitrary elliptic systems
– Eliminate symmetry, commutativity, separability, . . .
– Solve Laplace, Helmholtz, Stokes, . . . , with single code

Convert elliptic problems to first-order overdetermined form
– Control computational error via residuals
– Illuminate solvability of boundary value problem
CLASSICAL ALTERNATING DIRECTION IMPLICIT

Separable second-order equations in rectangles

\[-\Delta u = Au + Bu = -\partial_1^2 u - \partial_2^2 u = f\]

efficiently solved by essentially optimal ADI iteration

\[(s + A)(s + B)u^{m+1} = (s - A)(s - B)u^m + 2sf\]

when \(A\) and \(B\) are commuting positive Hermitian operators

Fast damping over geometric range

\[a \geq 0 \quad \rightarrow \quad \left| \frac{s - a}{s + a} \right| \leq 1\]

\[\frac{1}{2} \leq \frac{b}{s} \leq 2 \quad \rightarrow \quad \left| \frac{s - b}{s + b} \right| \leq \frac{1}{3}\]

implies \(O(\epsilon)\) error reduction in \(O(\log N \log \epsilon)\) sweeps
FIRST AND SECOND ORDER ELLIPTIC PROBLEMS

Cauchy-Riemann

\[ \partial_x u = \partial_y v \quad \partial_y u = -\partial_x v \]

Low-frequency time-harmonic Maxwell

\[ \nabla \times E = -\frac{i\omega}{c} H \quad \nabla \cdot E = 4\pi \rho \]
\[ \nabla \times H = \frac{i\omega}{c} E + \frac{4\pi}{c} j \quad \nabla \cdot H = 0 \]

Linear elasticity

\[ \partial_i \sigma_{ij} + F_j = 0 \quad \sigma_{ij} - \frac{1}{2} C_{ijkl} \left( \partial_k u_l + \partial_l u_k \right) = 0 \]

Laplace/Poisson/Helmholtz/Yukawa/ . . .

\[ \Delta u + \lambda u = f \]

Stokes

\[ -\Delta u + \nabla p = f \quad \nabla \cdot u = 0 \]
CONVERTING TO FIRST-ORDER SYSTEMS

Higher-order system of partial differential equations

\[ \cdots + \sum_{ijl} a_{ijkl} \partial_i \partial_j v_l + \sum_{jll} b_{jkl} \partial_j v_l + \sum_{l} c_{kl} v_l = f_k \quad \text{in } \Omega \]

\[ \sum_{l} \alpha_{kl} v_l + \sum_{jl} \beta_{kjl} \partial_j v_l + \cdots = g_k \quad \text{on } \Gamma = \partial\Omega \]

Seek new solution vector \( u = (v, \partial_1 v, \ldots, \partial_d v, \ldots)^T \)

Vector \( u \) satisfies first-order system

\[ Au = \sum_j A_j \partial_j u + A_0 u = f \quad \text{in } \Omega \]

\[ Bu = g \quad \text{on } \Gamma \]

Sparse matrices \( A_j, A_0, B \) localize algebraic structure
PITFALL OF CONVERSION

Robin boundary value problem for 2D Poisson equation

\[ \Delta v + \lambda v = f \quad \text{in} \ \Omega \]
\[ \alpha v + \beta \partial_n v = g \quad \text{on} \ \Gamma \]

3 × 3 square system

\[
Au = \begin{bmatrix}
\partial_1 & -1 & 0 \\
\partial_2 & 0 & -1 \\
\lambda & \partial_1 & \partial_2 \\
\end{bmatrix}
\begin{bmatrix}
v \\
\partial_1 v \\
\partial_2 v \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
f \\
\end{bmatrix}
\]

\[
Bu = \begin{bmatrix}
\alpha & \beta n_1 & \beta n_2 \\
\end{bmatrix}
\begin{bmatrix}
v \\
\partial_1 v \\
\partial_2 v \\
\end{bmatrix} = g
\]

System not elliptic (in sense of Protter): principal part

\[
\sum_j k_j A_j = \begin{bmatrix}
k_1 & 0 & 0 \\
k_2 & 0 & 0 \\
0 & k_1 & k_2 \\
\end{bmatrix} \quad \text{singular for all} \ k!
\]
\[ \Delta v + \lambda v = f \quad \text{in} \ \Omega \]

**Overdetermined** 4 \times 3 elliptic system

\[
Au = \begin{bmatrix}
\partial_1 & -1 & 0 \\
\partial_2 & 0 & -1 \\
0 & -\partial_2 & \partial_1 \\
\lambda & \partial_1 & \partial_2
\end{bmatrix}
\begin{bmatrix}
v \\
\partial_1 v \\
\partial_2 v
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
f
\end{bmatrix}
\]

Compatibility conditions \Rightarrow overdetermined but elliptic

\[
\sum_j k_j A_j = \begin{bmatrix}
k_1 & 0 & 0 \\
k_2 & 0 & 0 \\
0 & -k_2 & k_1 \\
0 & k_1 & k_2
\end{bmatrix}
\]

full-rank \Rightarrow injective for \( k \neq 0 \)

Analytical benefit: controls derivatives \( \partial_j u \) in terms of \( u \) and \( f \)

Computational advantage: controls error via residuals
**CONVERSION OF 2D STOKES**

Stokes for 2-vector $u$ and scalar $p$

$$-\Delta u + \nabla p = f \quad \nabla \cdot u = 0$$

Converts to $13 \times 9$ sparse system with

$$u = \begin{bmatrix} u_1 \\ u_{1,1} \\ u_{1,2} \\ u_2 \\ u_{2,1} \\ u_{2,2} \\ p \\ p_1 \\ p_2 \end{bmatrix} \quad \sum_j k_j A_j = \begin{bmatrix} k_1 \\ k_2 \\ k_2 \\ -k_1 \\ k_1 \\ k_2 \\ -k_1 \\ k_1 \\ k_2 \end{bmatrix}$$

**Elliptic** (in sense of Protter) with Laplace equation $\Delta p = \nabla \cdot f$
BOUNDARY CONDITIONS FOR DERIVATIVES?

Example: Dirichlet \( v = g \rightarrow Bu = g \) with rank-1 matrix \( B \)

Intuitively, the system \( Au = f \) includes compatibility conditions

Analytically, Fourier transform in half space \( \rightarrow \) well-posed

Computational methods enforce compatibility conditions up to boundary \( \rightarrow \) stable

Overdetermined interior \( \leftrightarrow \) underdetermined boundary
LOCAL SOLVABILITY FOR NORMAL DERIVATIVE

Ellipticity of first-order system

\[ Au = \sum_j A_j \partial_j u + A_0 u = f \quad \text{in } \Omega \]

implies any normal part \( A_n = \sum_j n_j A_j \) is left-invertible

\[ A_n^\dagger = (A_n^* A_n)^{-1} A_n^* \quad \rightarrow \quad A_n^\dagger A_n = I \]

Determines any directional derivative

\[ \partial_n u = \sum_i n_i \partial_i u = A_n^\dagger (f - A_T \partial_T u - A_0 u) \]

in terms of tangential derivatives

\[ A_T \partial_T u = \sum_j A_j \partial_j u - A_n \partial_n u = \sum_{ij} A_i \left( \delta_{ij} - n_i n_j \right) \partial_j u \]

and zero-order data
SOLVE BOUNDARY VALUE PROBLEM

With full tangential data plus elliptic system can integrate

$$
\partial_n u = A_n^\dagger \left( f - A_T \partial_T u - A_0 u \right)
$$

inward to solve boundary value problem

Boundary conditions

$$
Bu = \begin{bmatrix}
\alpha & \beta n_1 & \beta n_2 \\
\partial_1 v & \partial_2 v
\end{bmatrix}
\begin{bmatrix}
v \\
\partial_1 v \\
\partial_2 v
\end{bmatrix} = g \quad BB^* = I
$$

determine local projection $B^*Bu = B^*g$ on the boundary

Boundary value problem constrains $(I - B^*B)u$ on boundary

Contrast: hyperbolic systems blind $\perp$ characteristics
GENERALIZED ADI FOR POISSON SYSTEM

Choose arbitrary sweep direction $n$ and normalize

$$\sum_j A_j \partial_j u = \sum_{ij} A_i (n_i n_j + \delta_{ij} - n_i n_j) \partial_j u = A_n \partial_n u + A_T \partial_T u$$

Left-invert $A_n$ by ellipticity and damp on scale $1/s$

$$su^{m+1} + \partial_n u^{m+1} + B_0 u^{m+1} = su^m - B_T \partial_T u^m + A_n^\dagger f$$

Error mode $e^{ik^T x}$ damped by matrix symbol

$$\rho(k) = \prod_s \prod_n (s + ik_n + B_0)^{-1} (s - ik_T B_T)$$

Spectral radius $0.9^S$ with $S = O(\log N)$ sweeps
SPECTRAL RADIUS FOR POISSON SYSTEM

$s = 1$

$s = 4$

$s = 16$

$s = 64$
GENERALIZED ADI FOR YUKAWA SYSTEM

Yukawa as first-order system

\[ \Delta u - \lambda u = f \quad \rightarrow \quad A_j u_j + A_0 u = f \]

introduces nonzero eigenvalues \( \pm \sqrt{\lambda} \) of \( B_0 = A_n^\dagger A_0 \)

Divide by zero when \( s = \sqrt{\lambda} \)

Use matrix sign function \( K = s \operatorname{sgn}(B_0) \)
computed by Schur decomposition \( B_0 = U T U^* \)

\[ K u^{m+1} + \partial_n u^{m+1} + B_0 u^{m+1} = K u^m - B_T \partial_T u^m + A_n^\dagger f \]

Spectral radius \( 0.9^S \) with \( \lambda = 10 \)
SPECTRAL RADIUS FOR YUKAWA WITH $\lambda = 10$
GENERAL STRUCTURE

Given operators $A$ and $B$ with cheap resolvents $(sI - A)^{-1}$ and $(sI - B)^{-1}$ find an efficient scheme for the solution of

$$(A + B)u = f$$

Underlies many computational problems where

- $A$ is sparse and $B$ is low-rank
- $A$ and $B$ are both sparse but in different bases
- fast schemes deliver $A^{-1}$ and $B^{-1}$

Challenging when $A$ and $B$ don’t commute

Solution very unlikely in this generality
**ADI\(^2\) Approach**

1. \(A\) and \(B\) may not be invertible (or even square): square
   \[
   (A + B)^*(A + B)u = (A + B)^*f = g
   \]

2. Solve corresponding heat equation
   \[
   \partial_t u = -(A + B)^*(A + B)u + g
   \]
   to get \(u\) as \(t \to \infty\)

3. Discretize time and split
   \[
   (I + sA^*A)(I + sB^*B)u^{m+1} = (I - s(A^*B + B^*A))u^m + sg
   \]
   to get \(u\) as \(t \to \infty\)

4. Alternate directions for symmetric symbol
   \[
   \rho = (I + sB^*B)^{-1}(I + sA^*A)^{-1}(I - 2s(A^*B + B^*A))(I + sA^*A)^{-1}(I + sB^*B)^{-1}
   \]

Similar with more operators \(A, B, C, D, \ldots\)
ADI² FOR POISSON/YUKAWA/HELMHOLTZ

Second-order equation → overdetermined first-order system

\[ \Delta u + \lambda u = f \rightarrow (A + B + C)u = A_1 \partial_1 u + A_2 \partial_2 u + A_0 u = f \]

with high-frequency zero-order operator \( C = O(\lambda) \)

Fourier mode \((k_1, k_2)\) of error damped with symbol \( \rho \)

\[ \rho = (I + sC^*C)^{-1} (I + sB^*B)^{-1} (I + sA^*A)^{-1} \cdot \]
\[ (I - 2s(A^*B + A^*C + B^*A + B^*C + C^*A + C^*B)) \cdot \]
\[ (I + sA^*A)^{-1} (I + sB^*B)^{-1} (I + sC^*C)^{-1} \]

\[ \hat{\rho} = \frac{1}{(1 + sk_1^2)^2(1 + sk_2^2)^2(1 + s\lambda^2)^2} \begin{bmatrix} 1 & is(\lambda + 1)bk_1 & is(\lambda + 1)bk_2 \\ -is(\lambda + 1)bk_1 & b^2 & 0 \\ -is(\lambda + 1)bk_2 & 0 & b^2 \end{bmatrix} \]

where \( b = (1 + s\lambda^2)/(1 + s) \)

Eigenvalues of \( \hat{\rho} \) bounded by 1, controlled by \( s \) for all \( \lambda \)
SPECTRAL RADIUS FOR HELMHOLTZ WITH $\lambda = 1$

$s = 1/64$

$s = 1/16$

$s = 1/4$

$s = 1$
CONCLUSIONS

Conversion to first-order overdetermined systems yields
– analytical understanding of elliptic structure
– well-posed computational formulations
– fast iterations for numerical solution

Coming attractions:
– fast spectral boundary integral methods
– efficient implicit methods for elliptic moving interfaces
– nonuniform fast Fourier transforms for geometric data
– optimized post-Gaussian quadrature methods