Math 228a, Fall 2012: Problem Set 01

Exercise 1 Convert the second-order equation
\[ y'' + 2y' + y = f(t) \]
into a 2 × 2 first-order system y' = Ay, where A is a 2 × 2 matrix. Evaluate the matrix exponential \( e^{tA} \) analytically. (Note that A may not be diagonalizable; use Jordan normal form, the Cayley-Hamilton theorem, or a recurrence formula if necessary.) Find explicit formulas for the solutions of the initial value problem with \( y(0) = y'(0) = 1 \) and the boundary value problem with \( y(0) = y(1) = 1 \). (Let \( f(t) = te^{-t} \) for both problems.)

Exercise 2 Write the midpoint rule (with uniform mesh size \( h = (b-a)/N \))
\[ \frac{u_{j+1} - u_j}{h} + Q_{j+1/2} \left( \frac{u_{j+1} + u_j}{2} \right) = f_{j+1/2}, \quad Au_0 + Bu_N = g \]
for the two-point boundary value problem
\[ y' + Qy = f, \quad Ay(a) + By(b) = g \]
as a \( (N + 1) \times (N + 1) \) linear system \( L_h u = F \). Write a program (in your favorite programming language) which sets up and solves the linear system and compares the solution values \( u_j \) with exact solution values \( y(x_j) \) in the maximum norm. Choose random nonzero 3 × 3 matrices Q, A and B and test your program on the resulting problem. Choose data \( f \) and \( g \) so that the exact solution is \( y(x) = (\cos(x), \cos(2x), \cos(4x))^T \) on the interval \([a, b] = [0, 1]\). Verify second-order \( O(h^2) \) convergence over the range \( 10 \leq N \leq 10240 \). Verify the stability inequality
\[ \| L_h^{-1} \|_\infty \leq S, \]
for a constant \( S \) independent of \( h \) over the range of mesh sizes employed. If your computer takes too long, stop when you see a firm trend and explain it.