Integrals of Nonlinear Equations of Evolution and Solitary Waves*

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Abstract

In Section 1 we present a general principle for associating nonlinear equations of evolutions with linear operators so that the eigenvalues of the linear operator are integrals of the nonlinear equation. A striking instance of such a procedure is the discovery by Gardner, Miura and Kruskal that the eigenvalues of the Schrödinger operator are integrals of the Korteweg-de Vries equation.

In Section 2 we prove the simplest case of a conjecture of Kruskal and Zabusky concerning the existence of double wave solutions of the Korteweg-de Vries equation, i.e., of solutions which for \(|t|\) large behave as the superposition of two solitary waves travelling at different speeds. The main tool used is the first of a remarkable series of integrals discovered by Kruskal and Zabusky.

§1. In this paper we study the equation

\[(1.1) \quad u_t + uu_x + u_{xxx} = 0\]

introduced by Korteweg and de Vries in their approximate theory of water waves, [3]; we shall refer to it as the KdV equation. Subsequently the KdV equation was found to be relevant for the description of hydromagnetic waves, [2], and in the description of acoustic waves in an anharmonic crystal, [8]. Equation (1.1) is a special instance of a nonlinear evolution equation of the form

\[(1.2) \quad u_t = K(u) .\]

We shall study \(C^\infty\) solutions of (1.1) defined for all \(x\) in \((-\infty, \infty)\), which tend to zero as \(x \to \pm \infty\), together with all their \(x\) derivatives. It is easy to show that such solutions are uniquely determined by their initial values. Let \(v\) be another solution of (1.1):

\[(1.1)_v \quad v_t + vv_x + v_{xxx} = 0 .\]

Subtracting this from (1.1) and denoting \(u - v\) by \(w\), we obtain the linear equation

\[w_t + uw_x + vw_x + w_{xxx} = 0 .\]

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for \( w \). Multiplying by \( w \) and integrating with respect to \( x \) over \((-\infty, \infty)\), we obtain, after integration by parts, the relation

\[
(1.3) \quad \frac{d}{dt} \frac{1}{2} \int w^2 \, dx + \int (v_w - \frac{1}{2} w_u) w^2 \, dx = 0 .
\]

Here we have used the fact that \( w \) and \( w_u \) tend to zero as \( x \to \pm \infty \). Denoting \( \frac{1}{2} \int w^2 \, dx \) by \( E(t) \) and \( \max |2v_w - u_e| \) by \( m \) we obtain from (1.3) the inequality

\[
\frac{d}{dt} E(t) \leq mE(t) .
\]

This differential inequality yields

(1.4) \quad \[ E(t) \leq E(0)e^{mt} \]

which implies, in particular, that if \( E(0) \) is zero then so is \( E(t) \)—and thereby \( w \)—for all \( t \). Furthermore, Sjöberg has shown in [6] that (1.1) has a solution with arbitrarily prescribed initial value \( f(x) \) provided that \( f \) is smooth enough and tends to zero with sufficient rapidity as \( |x| \) tends to infinity.

Equation (1.1) has travelling wave solutions, i.e., solutions of the form \( u(x, t) = s(x - ct) \), \( c \) being the speed of the wave. To see this we substitute into (1.1):

(1.5) \quad \[-cs + \frac{1}{3}s^3 + s_{xx} = 0 .\]

Integrating with respect to \( x \) and imposing the boundary condition that \( s \) and its derivatives vanish at \( x = \pm \infty \), we get

(1.6) \quad \[-cs + \frac{1}{3}s^3 + s_{xx} = 0 .\]

Multiplication by \( 2s_x \) and one more integration gives

(1.7) \quad \[-cs^2 + \frac{1}{3}s^3 + s_x^2 = 0 .\]

From this relation and the assumption \( s_{xx}(0) = 0 \), \( s \) can be determined explicitly:

(1.8) \quad s(x) = 3c \operatorname{sech}^2 \left( \frac{1}{2}x\sqrt{c} \right) .

Thus we see that for every positive speed \( c \), (1.5) has a solution vanishing at \( x = \pm \infty \) uniquely determined except for a shift which can be so chosen that the maximum of \( s \) occurs at \( x = 0 \). We denote this normalized \( s \), explicitly described by (1.8), by \( s(x, c) \); \( s(x, c) \) is symmetric in \( x \), decays exponentially as \( |x| \to \infty \), and \( s(0, c) = 3c \).

On account of its shape, \( s \) is called a solitary wave.

For linear equations it often happens that all solutions are superpositions of special solutions; e.g., all solutions of linear equations with constant coefficients are superpositions of exponential solutions. For nonlinear equations one cannot in general form new solutions out of old ones and hence special families of solutions are not expected to play any special role in the description of all solutions. It was therefore very surprising when Kruskal and Zabusky observed, by analyzing numerically computed solutions, that all solutions of the KdV equation have
hidden in them solitary waves. A precise formulation of this observation is as follows:

Let \( u \) be any solution of (1.1) which is defined for all \( x \) and \( t \) and which vanishes at \( x = \pm \infty \). Then there exist a discrete set of positive numbers \( \epsilon_1, \cdots, \epsilon_N \)—called the eigenspeeds of \( u \)—and sets of phase shifts \( \theta_j \) such that

\[
\lim_{t \to \pm \infty} u(x + ct, t) = \begin{cases} 
    s(x - \theta_j^+, \epsilon_j) & \text{if } c = \epsilon_j, \\
    0 & \text{if } c \neq \epsilon_j.
\end{cases}
\]

(1.9)

It seems likely that the method described in [1] can be used to prove this conjecture.

Note. Consider any equation of evolution (1.2) which does not involve \( x \) and \( t \) explicitly, i.e., whose set of solutions is invariant under translation with respect to \( x \) and \( t \). Suppose that \( u(x, t) \) is a solution of such an equation, and that for a certain value of \( c \)

\[
\lim_{t \to \pm \infty} u(x + ct, t)
\]

exists uniformly on compact sets in \( x \)-space. Clearly the limit is a travelling wave solution; one could then define the eigenspeeds of a solution \( u \) as those values of \( c \) for which the above limit exists and is different from zero. It is far from obvious whether—as is the case for the KdV equation—the eigenspeeds which appear in the limit as \( t \to \infty \) and \( t \to -\infty \) are the same.

The eigenspeeds \( \epsilon_j \) are unequivocally determined by the solution \( u \) under consideration and thus can be regarded as functionals of solutions. It is clear from the definition that if \( u \) is translated by the amounts \( a \) and \( b \) in the \( x \) and \( t \) direction, the eigenspeeds remain the same while the phase shifts change by the amount \( a - cb \).

We saw earlier that solutions are uniquely determined by their initial values; therefore, the eigenspeeds also can be regarded as functionals of the initial values. It follows from translation invariance that the eigenspeeds are invariant functionals (also called integrals); that is, if \( f \) and \( f' \) denote the value of \( u \) at two different times, then \( \epsilon_j(f) = \epsilon_j(f') \). It follows similarly that the difference \( \theta_j^+ - \theta_j^- \) is an integral.

Since the number of eigenspeeds appears to be unbounded, this analysis shows that if solutions of the KdV equation behave, for large \( t \), as indicated in (1.9), then the KdV equation has an infinity of integrals. Indeed, Kruskal, Gardner and Miura [5] succeeded in constructing explicitly an infinite sequence of integrals; the simplest of these will be described and used in Section 2. We present now a general method for constructing an infinite set of integrals for equations of evolution (1.2).

Let \( \mathcal{B} \) be some space of functions chosen so that, for each \( t \), \( u(t) \) lies in \( \mathcal{B} \). Suppose that to each function \( u \) in \( \mathcal{B} \) we can associate a selfadjoint operator
\[ L = L_u \] over some Hilbert space,
\[ u \rightarrow L_u , \]
with the following property: if \( u \) changes with \( t \) subject to the equation
\[ u_t = K(u) , \]
the operators \( L(t) \), which also change with \( t \), remain unitarily equivalent. If this is the case, then the eigenvalues of \( L_u \) constitute a set of integrals for the equation under consideration.

The unitary equivalence of the operators \( L(t) \) means that there is a one-parameter family of unitary operators \( U(t) \) such that
\[ U(t)^{-1}L(t)U(t) \]
is independent of \( t \). This fact can be expressed by setting the \( t \) derivative of \( (1.11) \) equal to zero:
\[ -U^{-1}U_t U^{-1}L U + U^{-1}L_t U + U^{-1}LU_t = 0 . \]
A one-parameter family of unitary operators satisfies a differential equation of the form
\[ U_t = BU , \]
where \( B(t) \) is an antisymmetric operator. Conversely, every solution of \( (1.13) \) with \( B^* = -B \) is a one-parameter family of unitary operators. Substituting \( (1.13) \) into \( (1.12) \) we get, after multiplication by \( U \) on the left and by \( U^{-1} \) on the right,
\[ -BL + L_t + LB = 0 \]
which is the same as
\[ L_t = BL - LB = [B, L] . \]
If \( u \) satisfies the equation \( u_t = K(u) \), then \( L_t \) can be expressed in terms of \( u \), and all that remains to verify is that equation \( (1.14) \) has an antisymmetric solution \( B \).

The drawback of this method is that it requires one to guess correctly the relation \( (1.10) \) between the function \( u \) and the operator \( L \). Now Gardner, Kruskal and Miura have made the remarkable discovery, [1], that the eigenvalues of the Schrödinger operator
\[ L = D^2 + \frac{1}{2}u \]
are invariant if \( u \) varies according to the KdV equation. We shall presently verify this fact with the aid of the linear operator equation \( (1.14) \); more generally, we shall use equation \( (1.14) \) to find a class of differential equations under which the operators \( (1.15) \) are unitarily equivalent for all \( t \).

With the choice \( (1.15) \) the operator \( L_t \) reduces to multiplication by \( \frac{1}{2}u_t \) so that according to \( (1.14) \) we have to find an antisymmetric operator \( B \) whose commutator with \( L \) is multiplication. An obvious choice is
\[ B_u = D . \]
Indeed, an easy calculation gives
\[ [B_0, L] = \frac{1}{4} u_x. \]
This shows that if \( u \) varies according to the equation
\begin{equation}
(1.17) \quad u_t = u_x,
\end{equation}
the operators (1.15) are unitarily equivalent. Alas, this is a trivial fact since changing \( u \) according to (1.17) amounts to replacing the potential \( u \) by a translate of \( u \), which obviously results in an equivalent operator. For a less trivial result we try a third order antisymmetric operator
\begin{equation}
(1.16)_1 \quad B_1 = D^3 + bD + Db,
\end{equation}
the coefficient \( b \) to be chosen. A brief calculation yields the following value for the commutator:
\[ [B_1, D] = \frac{1}{2} u_x D^2 + \frac{1}{2} u_{xx} D + \frac{1}{2} u_{xxx} - 4 b_x D^3 - 4 b_{xx} D - b_{xxx} + \frac{1}{2} b u_x. \]
Clearly, to eliminate all but the zero order terms we have to choose
\[ b = \frac{1}{8} u. \]
With this choice, \([B_1, L]\) is multiplication by
\[ \frac{1}{2} [u_{xxx} + uu_x]. \]
Setting \( B = 24 B_1 \), we verify that
\[ [B, D] = K(u), \]
where \( K(u) = u_t \) is the KdV equation (1.1).

Clearly this process can be generalized; we could choose \( B_q \) as a skew symmetric differential operator of any odd order \( 2q + 1 \):
\[ B_q = D^{2q+1} + \sum_{j=1}^{q} (b_j D^{2j-1} + D^{2j-1} b_j). \]
Since \([B_q, L]\) is symmetric, the requirement that it be of degree zero imposes \( q \) conditions; these uniquely determine the \( q \) coefficients \( b_j \), and the zero order term of \([B_q, L] = K_q \) determines a higher order KdV equation
\[ u_t = K_q(u), \]
which shares with the KdV equation the property that the eigenvalues of the Schrödinger equation with \( u \) as potential are its integrals.

Gardner has discovered\(^1\) an interesting relation between the higher order KdV equations and the explicit sequence of invariants mentioned earlier; this relation will be described in Section 2.

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\(^1\) Personal communication.
The process presented above can be generalized in a straightforward way to yield the following theorem.

**Theorem 1.1.** Suppose that $L$ is a selfadjoint operator depending on $u$ in the following fashion:

$$L_u = L_0 + M_u,$$

where $L_0$ is independent of $u$ and $M$ depends linearly on $u$. Suppose that there exists an antisymmetric operator $B = B_u$ such that

$$[B, L_u] = M_{K(u)}.$$

Then the eigenvalues of $L_u$ are integrals of

$$u_i = K(u).$$

As an example of this procedure we take $u$ to be a $p \times p$ symmetric matrix variable and take $L$ to be the matrix operator $L = D^2 + \frac{1}{3} u$. If we choose $B$ to be a third order matrix operator we obtain the matrix KdV equation

$$u_t + \frac{1}{3}(uu_x + u_x u) + u_{xxx} = 0.$$ 

Other choices for the operator $L_u$ should lead to other classes of equations.

Having shown that the eigenvalues $\lambda_1(u), \ldots, \hat{\lambda}_N(u)$ of the Schrödinger operator (1.15) are integrals for the KdV equation, one asks how these integrals are related to the eigenspeeds $c_1(u), \ldots, c_N(u)$ which appear in the asymptotic description (1.9). Gardner and Kruskal have found the answer:

$$c_j(u) = 4\lambda_j(u).$$

We give here a derivation of this result which makes use of a general relation involving integrals of nonlinear equations.

This relation is an extension of the following well-known fact about quadratic integrals $Q$ for linear equations: the bilinear functional $Q(u, v)$ is independent of $t$ for any pair of solutions $u$ and $v$. This result can be deduced from the invariance of $Q$ for the solutions $u + v$ and $u - v$. The corresponding result for integrals of a nonlinear equation

$$u_t = K(u)$$

is derived by considering one-parameter families $u_\epsilon$ of solutions; these can be constructed e.g., by making the initial value of $u_\epsilon(t)$ a function of $\epsilon$:

$$u_\epsilon(0) = u_0 + \epsilon f.$$

We assume that the nonlinear operator $K$ depends differentiably on $u$, i.e., that

$$\frac{d}{d\epsilon} K(u + \epsilon v) |_{\epsilon=0} = V(u)v$$

(1.20)
exists and is a linear function of \( \nu \). We call the linear operator \( V(\nu) \) the variation of \( K \). Differentiating the equation (1.19) with respect to \( \varepsilon \), we obtain the variational equation

\[
(1.21) \quad \nu_t = V(\nu) \nu
\]

for the quantity

\[
(1.22) \quad \nu = \frac{du_\varepsilon}{d\varepsilon} \bigg|_{\varepsilon=0}.
\]

Let \( I(\nu) \) be an integral of the equation under consideration; we assume that \( I(\nu) \) is differentiable in the Frechet sense, i.e., that

\[
\frac{d}{d\varepsilon} I(\nu + \varepsilon \nu) \bigg|_{\varepsilon=0}
\]

exists and is a linear functional of \( \nu \). This linear functional can be represented as

\[
(1.23) \quad \frac{d}{d\varepsilon} I(\nu + \varepsilon \nu) = (G(\nu), \nu),
\]

where \( (\ , \ ) \) is some convenient bilinear functional; \( G(\nu) \) is called the gradient of \( I \).

Let \( u_\varepsilon(t) \) be the one-parameter family of solutions considered before; then

\[
I(u_\varepsilon(t))
\]

is independent of \( t \), for every value of \( \varepsilon \). Differentiating with respect to \( \varepsilon \), we see that \( (G(\nu), \nu) \) is independent of \( t \). We formulate this result as

**Lemma 1.1.** Let \( u(t) \) be any solution of the equation (1.19), \( \nu(t) \) any solution of the corresponding linear variational equation (1.21). Let \( I \) be any integral for (1.19), \( G \) its gradient; then

\[
(1.24) \quad (G(u(t)), \nu(t))
\]

is independent of \( t \).

We suppose now that equation (1.19) is translation invariant and that it has a solitary wave solution \( s(x - ct) \). We also assume that the bilinear functional \( (\ , \ ) \) is symmetric as well as translation invariant.

Let \( \nu \) be any solution of (1.21); according to (1.24),

\[
(G(s(x - ct)), \nu(x, t))
\]

is independent of \( t \). Using the translation independence of \( (\ , \ ) \) and introducing the abbreviation

\[
(1.25) \quad \nu(x + ct, t) = w(x, t),
\]

we see that

\[
(G(s(x)), w(x, t))
\]
is equal to the above quantity and therefore also independent of $t$. Differentiating with respect to $t$, we obtain that 

\[(1.26) \quad (G(s), w_t) = 0.\]

From (1.25), 

\[w_t = cv_x + v_t;\]

using equation (1.21) and the fact that $V$ commutes with translation, we get 

\[w_t = [cD + V(s)]w,\]

where $D$ denotes $\partial/\partial x$. Substituting this into (1.26) we have 

\[(1.27) \quad (G(s), [cD + V(s)]w) = 0.\]

Denote the adjoint of $V$ by $V^*$. Since $(\ , \ )$ is translation invariant, $D^* = -D$; hence we see from (1.27) that 

\[([ -cD + V^*(s)]G(s), w) = 0.\]

The value of $w$ at any particular time, say $t = 0$, can be prescribed arbitrarily; therefore, it follows from the above relation that in fact 

\[(1.28) \quad [ -cD + V^*(s)]G(s) = 0.\]

Next we make our final assumption about the differential equation (1.19); we assume namely that it is energy preserving, i.e., that for a solution $u$ of (1.19) 

\[(u(t), u(t))\]

is independent of $t$. Differentiating with respect to $t$ and using (1.19), we see that this amounts to requiring that, for all $u$,

\[0 = 2(u, u_t) = 2(u, K(u)).\]

The initial value of $u$ is arbitrary. Putting $u_\epsilon$ in place of $u$, we see after differentiating with respect to $\epsilon$ that 

\[(v, K(u)) + (u, V(u)v) = 0.\]

Using the adjoint of $V$ this can be written as 

\[(u, K(u) + V^*(u)u) = 0.\]

Since $v$ is arbitrary, this implies that 

\[(1.29) \quad K(u) + V^*(u)u = 0.\]

We turn now to the equation for solitary waves: 

\[0 = cs_x + K(s).\]

Expressing $K$ from (1.29) we can rewrite this as 

\[[cD - V^*(s)]s = 0;\]
i.e., the solitary wave \( s \) belongs to the nullspace of the linear operator \( cD - V(s) \).

We can state the following lemma.

**Lemma 1.2.** Suppose the equation of evolution (1.19) has the following properties:

1. \( K(u) \) depends differentiably on \( u \); denote its variation by \( V(u) \).
2. (1.19) is translation invariant and preserves a positive translation invariant quadratic functional, which we call energy.
3. (1.19) has a solitary wave solution \( s \).
4. The only function annihilated by \( cD - V(s) \) and vanishing at \( \pm \infty \) is a multiple of \( s \).

Let \( I(u) \) be an integral for (1.19) such that

(1) \( I(u) \) is differentiable,
(2) \( G(s) \) vanishes at \( \pm \infty \), where \( G(u) \) is the gradient of \( I(u) \) with respect to energy.

Then

\[
(1.30) \quad G(s) = \kappa s,
\]

where \( \kappa \) depends on \( I \) and \( c \), i.e., every solitary wave is an eigenfunction of the gradient of an integral.

It is easy to verify that the KdV equation is energy preserving with respect to the \( L_2 \) scalar product. In that case, \( K(u) = -uu_x - u_{xxx} \) so that

\[
V(u)v = -(uv)_x - v_{xxx},
\]

and thus

\[
V(s) = uD + D^3.
\]

It is not hard to show that every function annihilated by

\[
(C - s)D - D^3
\]

and vanishing at \( \pm \infty \) is a constant multiple of \( s \).

We apply now the foregoing to the integral \( I(u) = \lambda(u) \), where \( \lambda \) is an eigenvalue of \( L = D^3 + \frac{1}{6} u \):

\[
(1.31) \quad Lw = \lambda w.
\]

To compute the gradient of \( \lambda \) we replace \( u \) by \( u + \epsilon v \) and differentiate with respect to \( \epsilon \). Denoting \( d/d\epsilon \) by a dot and using the fact that \( \dot{L} = \frac{1}{6} \dot{u} = \frac{1}{6} v \), we get

\[
\dot{L}w + \frac{1}{6} \dot{vw} = \lambda w + \dot{\lambda} w.
\]

We take the \( L_2 \) scalar product with \( w \); using the fact that \( L \) is symmetric and that (1.31) is satisfied, we get rid of \( \dot{w} \) and obtain

\[
(\omega, w) = \lambda(w, w).
\]
Assuming that \( w \) is normalized so that \((w, w) = 1\), we get
\[
\hat{\lambda} = (\frac{1}{2} w w, w) = \int \frac{1}{2} w^2 \, dx = (\frac{1}{2} w^2, v).
\]
Since \( \hat{\lambda} = (d/dv) I = (G(u), v) \), this shows that the gradient \( G \) of \( \lambda \) is given by
(1.32)
\[
G(u) = \frac{1}{2} w^2.
\]
Since eigenfunctions \( w \) vanish at \( \pm \infty \), the hypothesis preceding (1.30) is fulfilled; therefore we conclude from (1.30) that \( G(s) \) is a constant multiple of \( s \):
\[
G(s) = \frac{1}{2} w^2 = \kappa s,
\]
i.e., that the eigenfunction \( w \) of
\[
L = D^2 + \frac{1}{2} s
\]
is
(1.33)
\[
w = \text{const. } s^{1/2}.
\]
This relation is easily verified by an explicit calculation. Taking the constant to be 1 we get
\[
w_x = \frac{1}{2} s_x s^{-1/2},
\]
\[
w_{xx} = \frac{1}{2} s_{xx} s^{-1/2} - \frac{1}{4} s_y s^{3/2}.
\]
Using relations (1.6) and (1.7) for the solitary wave we have
\[
w_{xx} = \frac{1}{2} (cs - \frac{1}{2} s^2) s^{-1/2} + \frac{1}{4} (\frac{1}{2} s^3 - cs^2).
\]
Substituting this into the eigenvalue equation
\[
Lw = w_{xx} + \frac{1}{2} s w,
\]
we obtain after a brief calculation
(1.34)
\[
Lw = L s^{1/2} = \frac{1}{2} cs s^{1/2}.
\]
This proves that
\[
e(s) = 4 \lambda(s).
\]

Let \( u \) be any solution of the KdV equation which contains a solitary wave travelling with speed \( c \), i.e., such that, given any positive \( \varepsilon \) and \( X \), there exists a \( T \) such that
(1.35)
\[
|u(x + cT, T) - s(x - \theta)| < \varepsilon
\]
for all \( |x| < X \). We claim that then the operator \( L_T = D^2 + u(T) \) has \( \frac{1}{4} c \) as an approximate eigenvalue and
\[
w_T(x) = s^{1/2}(x - cT - \theta)
\]
as approximate eigenfunction, in the sense that
(1.36)
\[
\|L_T w - \frac{1}{4} c w\| \leq \delta \|w\|,
\]
where \( \delta \) tends to zero as \( \varepsilon \to 0 \) and \( X \to \infty \). To see this we use the fact that, according to (1.35), in the interval

\[
(1.37) \quad cT - X < x < cT + X
\]

\( u \) differs by \( \varepsilon \) from \( s(x - cT - \theta) \); therefore, using (1.34) we conclude that in the interval (1.37)

\[
(1.38) \quad |L_T w_T - \frac{1}{2} \varepsilon w_T| < \varepsilon w_T.
\]

On the other hand, it follows from formula (1.8) that, outside of the interval (1.37), \( w_T \) and its second derivative are bounded by \( \exp \{-\text{const.} \cdot |X - x|\} \). Denoting by \( M \) the supremum\(^*\) \( u(x, t) \), it follows that

\[
(1.38)' \quad |L_T w_T - \frac{1}{2} \varepsilon w_T| < (\text{const.} + M) e^{-\text{const.} \cdot |X - x|},
\]

outside the interval (1.37). Combining (1.38) and (1.38)' we deduce (1.36).

According to spectral theory, inequality (1.36) implies that \( \frac{1}{2} e \) lies within \( \delta \) of a point of the spectrum of \( L_T \); since we have seen earlier that the spectrum of \( L_T \) is independent of \( T \), it follows that \( \frac{1}{2} e \) is an eigenvalue of \( L \).

This proves one half of (1.18); the second half—that if \( \lambda \) is an eigenvalue of \( L \), then \( 4 \lambda \) is an eigenspeed of \( u \)—has not yet been demonstrated.

\( \S 2. \) In [9] Kruskal and Zabusky have studied the interaction of solitary waves; in particular they posed the following problem:

Let \( d \) be a solution of KdV which for \( t \) large negative represents two solitary waves travelling with speed \( c_1 \), respectively \( c_2 \), approaching each other; what is the asymptotic behaviour of \( d(x, t) \) for large positive values of \( t \)?

They solved this problem by computing \( d \); for \( -T \) large negative they set

\[
(2.1) \quad d(x, -T) = s(x, c_1) + s(x - X, c_2),
\]

where the separation distance \( X \) was chosen so large that the two solitary waves overlap only by a negligible amount. Since solitary waves die down exponentially, even a moderate value of \( X \) accomplishes this. The speeds were chosen so that \( c_1 > c_2 \), and \( X \) was taken to be positive so that at \( t = -T \) the faster wave lies to the left of the slower one.

If the equation governing the motion were linear, the solitary waves would not interact at all and so after the lapse of \( 2X/(c_1 - c_2) \) time the relative position of the two would be merely interchanged. Numerical calculation of the solution of the KdV equation with initial values (2.1) showed that for \( S > 2X/(c_1 - c_2) \)

\[
d(x, -T + S) = s(x - c_1 S - \theta_1, c_1) + s(x - X - c_2 S - \theta_0, c_2),
\]

except for deviations that could be accounted for by truncation error, i.e., the same as would be obtained in a linear case except for phase shifts \( \theta_1 \) and \( \theta_2 \).

The actual process of interaction, i.e., the behaviour of \( d(x, t) \) around the time \(-T + X/(c_1 - c_2)\) is far from being a mere superposition. In fact, Kruskal and

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* In Section 2 we shall demonstrate the uniform boundedness of solutions of the KdV equation.
Zabusky observed that in cases when $c_1 \gg c_2$, i.e., when the first wave was much higher (and therefore faster) than the second one, the big wave swallows up the small one during the interaction, and reemits it later. In cases where $c_1$ and $c_2$ were comparable, the two waves interact as follows: As soon as the big wave comes reasonably close to the smaller one in front of it, the big wave begins to shrink and the smaller one begins to grow, until the two waves interchange their roles; thereafter they separate.

In this section we present a rigorous proof of the fact that the KdV equation indeed has solutions which behave like two solitary waves approaching, interacting, and then separating; furthermore we give a precise estimate for the ratio of speeds which lead to the two different kinds of interaction described above. It turns out that there is yet a third manner, intermediate between the other two, in which the interaction can take place.

We shall call these special solutions double waves, and denote them by $d(x, t; c_1, c_2)$. Note that for fixed $c_1$ and $c_2$ there is a two-parameter family of double waves, the parameters being the phases of each solitary wave at $t = -\infty$.

In [8], Kruskal and Zabusky derived an ordinary differential equation with respect to $x$ which $d(x, t; c_1 c_2)$ satisfies for each fixed $t$. The argument in [8] is formal; here we present an (almost) rigorous derivation.

Our starting point is Lemma 1.1, the constancy of the functional (1.24) for pairs of solutions $u, v$ of the nonlinear equation and of the variational equation, respectively. Taking $u$ to be $d$ we see that, for any solution $v$ of (1.21),

$$\langle G(d), v \rangle,$$

where $G$ is the gradient of some integral, is independent of $t$. By definition, for $t$ large negative

$$d(x, t; c_1, c_2) = s(x - c_1 t - \theta_1, c_1) + s(x - c_2 t - \theta_2, c_2) + \text{error}(t).$$

The error term in (2.3) is due to the interaction of the tails of the solitary waves into which the double wave $d$ decomposes. Since the tails of solitary waves decay exponentially, and since the separation of the two solitary waves is proportional to $t$, we expect the error term in (2.3) to decay exponentially in $t$. Suppose that the gradient $G$ is a local operator; it follows from (2.3) that

$$(2.4) \quad G(d) = G(s_1) + G(s_2) + \text{error},$$

where the error in (2.4) also tends to zero exponentially as $t \to -\infty$.

Suppose that $G$ satisfies

$$(2.5) \quad G(s_1) = 0, \quad G(s_2) = 0,$$

then it follows from (2.4) that $V(d)$ tends to zero exponentially as $t \to -\infty$.

We turn now to the functions $v$; since these satisfy a linear equation, it can be shown, by an argument similar to the one which led to inequality (1.4), that any $v$ increases at most exponentially. In fact, since exponentially increasing solutions
of the variational equation usually indicate an instability, and since on the other hand numerical evidence indicates that double waves are stable, it is reasonable to expect that all solutions \( v \) of the variational equation grow at a rate slower than exponential.

We return now to the functional (2.2); we have shown that as \( t \to -\infty \) the first factor \( G(d) \) tends to zero exponentially, and that the second factor \( v \) tends to infinity slower than exponentially. It follows then that \( (G(d), v) \) tends to zero as \( t \to -\infty \); on the other hand, according to (2.2), this functional is independent of \( t \). Therefore it follows that \( (G(d), v) \) is zero for all \( t \).

At any particular time \( t_0 \) the initial values of \( v \) may be prescribed arbitrarily; therefore it follows that

\[
G(d) = 0
\]

at time \( t_0 \), that is for any time.

There remains the task of constructing an integral \( I \) whose gradient \( G \) is local and satisfies (2.5), i.e., annihilates both solitary waves \( s_1 \) and \( s_2 \). Here we rely on Lemma 1.2, relation (1.30), according to which solitary waves are eigenfunctions of the gradient of every integral:

\[
G(s_1) = \kappa(G)s_1, \quad G(s_2) = \kappa(G)s_2.
\]

It follows that, given three independent integrals, an appropriate linear combination of them will have a gradient which annihilates any two given solitary waves.

We turn now to the task of finding three independent integrals whose gradients are local operators. We have already noted in Section 1 that the energy

\[
I_1(u) = \int \frac{1}{2} u^2 \, dx = \frac{1}{2} (u, u)
\]

is an integral for the KdV equation. Another integral was found by Whitham [7]:

\[
I_2(u) = \int (\frac{1}{3} u^3 - u_x^2) \, dx.
\]

A third one was discovered by Kruskal and Zabusky:

\[
I_3(u) = \int (\frac{1}{4} u^4 - 3uu_x^2 + 3u_x^4) \, dx.
\]

Subsequently, Kruskal and Zabusky found two more explicit integrals and Miura four more; an infinite sequence \(^8\) of such integrals was constructed in [5].

\(^8\) Denote by \( I_n \) the \( n \)-th integral of this sequence and by \( G_n \) its gradient. The relation discovered by Gardner between \( I_n \) and the \( n \)-th generalized KdV operator \( K_n \) described in Section 1 is

\[
K_n = DG_n.
\]
For the discussion of the double wave we need only the first three integrals. The gradients of these are

\begin{align}
(2.8)_1 & \quad G_1(u) = u , \\
(2.8)_2 & \quad G_2(u) = u^2 + 2u_x , \\
(2.8)_3 & \quad G_3(u) = u^3 + 3u_x^2 + 6u_xu_{xx} + \frac{1}{8}u_{xxxx} .
\end{align}

Note that these are indeed local operators. The proof that $I_1, I_2, I_3$ are integrals of KdV consists in verifying that the product of $G_i(u)$ with $K(u) = uu_x + u_{xxx}$ is a perfect $x$ derivative. Indeed an explicit calculation gives

\begin{align}
(2.9)_1 & \quad G_1(u)K(u) = (uu_{xx} - \frac{1}{2}u_x^2 + \frac{3}{4}u^3)_x = H_1(u)_x , \\
(2.9)_2 & \quad G_2(u)K(u) = (u_x^2 + uu_{xx} + \frac{1}{4}u^4)_x = H_2(u)_x , \\
(2.9)_3 & \quad G_3(u)K(u) = (\frac{3}{8}u_x^2 + \frac{1}{4}u_x^2u_x + \frac{3}{8}u_{xxx}u_x + \frac{3}{8}u_{xxxx})_x = H_3(u)_x .
\end{align}

We know that solitary waves are eigenfunctions of $G_n$; a calculation gives the eigenvalues $\kappa(G_n)$ as follows:

\begin{align}
(2.10)_1 & \quad G_1(s) = s , \\
(2.10)_2 & \quad G_2(s) = 2cs , \\
(2.10)_3 & \quad G_3(s) = \frac{1}{8}c^2s .
\end{align}

We form now the linear combination

\begin{equation}
I = I_3 + AI_2 + BI_1
\end{equation}

whose gradient is

\begin{equation}
G = G_3 + AG_2 + BG_1 .
\end{equation}

In view of (2.10),

\begin{equation}
G(s) = (\frac{1}{8}c^2 + 2Ac + B)s .
\end{equation}

Thus $G$ annihilates $s_1$ and $s_2$ if $c_1$ and $c_2$ satisfy the equation

\begin{equation}
\kappa(c) = \frac{1}{8}c^2 + 2Ac + B = 0 .
\end{equation}

In view of the relation between coefficients and roots of a quadratic equation this means that

\begin{align}
(2.14)_A & \quad A = -\frac{9}{8}(c_1 + c_2) , \\
(2.14)_B & \quad B = \frac{1}{8}c_1c_2 .
\end{align}

In view of (2.6) we conclude: Let $d$ be a double wave with speeds $c_1$ and $c_2$, and define the constants $A, B$ by (2.14); then for each $t, d$ satisfies

\begin{equation}
G(d) = G_3(d) + AG_2(d) + BG_1(d) = 0 .
\end{equation}
This is a nonlinear ordinary differential equation of fourth order\(^4\); therefore its solutions form a 4-parameter family of functions. On the other hand, the double waves form a 2-parameter family. We shall accordingly deduce from (2.15) a second order equation satisfied by all double waves. To do this we make use of the earlier observation (see (2.9)) that the invariance of \(I(u)\) for solutions of KdV is equivalent with the fact that \(G(u)K(u)\) is a perfect \(x\) derivative. Multiplying (2.15) by \(K(u)\) and using (2.9) we see that
\[
(H_3 + AH_4 + BH_5)_x = 0 .
\]
Integrating this and using the fact that \(d\) and all its derivatives are zero at \(x = \pm \infty\), we deduce
\[
H_3(d) + AH_4(d) + BH_5(d) = 0 .
\]

Next we make use of the translation invariance of the integrals \(I\) under consideration, i.e., of the fact that \(I(u_\epsilon)\) is independent of \(\epsilon\), where \(u_\epsilon\) is the translate of \(u\) by \(\epsilon\) in the \(x\) direction. Differentiating with respect to \(\epsilon\), we get
\[
0 = \frac{d}{d\epsilon} I(u_\epsilon)|_{\epsilon=0} = \int G(u)u_\epsilon dx .
\]
This implies that \(G(u)u_\epsilon\) is a perfect \(x\) derivative. Indeed, an explicit calculation gives
\[
(2.17)_1 \quad G_1(u)u_\epsilon = (\frac{1}{2}u^2)_x = \frac{1}{2}J_1(u)_x ,
\]
\[
(2.17)_2 \quad G_2(u)u_\epsilon = (\frac{1}{3}u^3)_x = \frac{1}{3}J_2(u)_x ,
\]
\[
(2.17)_3 \quad G_3(u)u_\epsilon = (\frac{4}{5}u^4)_x = \frac{4}{5}J_3(u)_x .
\]

Thus multiplying (2.15) by \(u_\epsilon\) we obtain, after integration and use of the fact that \(d\) and its derivatives vanish at \(\infty\),
\[
(2.18) \quad J_3(d) + AJ_5(d) + BJ_3(d) = 0 .
\]

Both (2.16) and (2.18) are third order differential equations. Expressing \(d_{xxx}\) from (2.18), substituting into (2.16) and multiplying by \(d_x^2\), we get an equation of second order and fourth degree in \(d_{xx}\) which we write symbolically as
\[
(2.19) \quad Q(d_{xx}, d_x, d) = 0 .
\]
From this equation \(d_{xx}\) can be expressed as a 4-valued function of \(d\) and \(d_x\); since \(x\) does not appear explicitly in these equations, this second order equation is equivalent to a first order autonomous system of equations for \(d\) and \(d_x\). By studying carefully the geometry of all four branches of the corresponding vector-field one can show that (2.19) indeed has solutions which tend to zero as \(x \to \pm \infty\) and has the shape of a double wave, i.e., has two maxima and one minimum.

\(^4\) This equation appears in [8].
I shall not present the details because
(i) I did not carry them out completely,
(ii) the formulas are horribly complicated,
(iii) an explicit formula for double waves (indeed N-tuple waves) was derived recently in [1].

We show now how to use equation (2.19) to study the time history of double waves. For this purpose we consider the maximum value of \( d(x, t) \) with respect to \( x \), or rather the relative maxima of \( d \) as functions of time. Denote by \( m = m(t) \) the value of a relative maximum of \( d(x, t) \); at the point \( y \) where the relative maximum occurs
\[
d_y = 0 .
\]

It follows from (2.20) by the implicit function theorem that if \( d_{xx} < 0 \) at \( y \), then \( y \) is a differentiable function of \( t \). Hence \( m = d(y, t) \) also is differentiable and satisfies
\[
m_t = d_x y_t + d_t = d_t .
\]

Since \( d \) satisfies the KdV equation, we have, in view of (2.20),
\[
m_t = -K(d) = -d_{xx} .
\]

We proceed now to determine \( d_{xxx} \) at a local maximum. At a point where \( d_x = 0 \), equation (2.16) simplifies considerably; using formulas (2.9) and denoting the value of \( d \) by \( m \), we get
\[
\frac{2}{9}d_{xxx}^2 + \frac{2}{9}d_x^2m + d_{xx}m^3 +\frac{1}{3}m^6 + A(d_{xx}^2 + m^2d_{xx} + \frac{1}{4}m^4) + B(md_{xx} + \frac{1}{2}m^3) = 0 .
\]

Similarly, at a point where \( d_x = 0 \), equation (2.18) becomes
\[
d_{xx}^2 = \frac{5}{9}(\frac{1}{4}m^4 + A\frac{1}{4}m^3 + B\frac{1}{4}m^2) = P(m) .
\]

From (2.23) we deduce that
\[
d_{xx} = -P^{1/2}(m) ,
\]
the negative sign being chosen since the second derivative is nonpositive at a local maximum point. Substituting (2.24) into (2.22) we get
\[
d_{xxx} = R(m) ,
\]
where
\[
R(m) = -a(m) + b(m) P^{1/2}(m) ,
\]
with
\[
a(m) = \frac{2}{9}mP(m) + \frac{2}{9}m^5 + \frac{2}{9}AP(m) + \frac{5}{9}Am^4 + \frac{5}{9}Bm^3
\]
and
\[
b(m) = \frac{2}{9}m^3 + \frac{5}{9}Am^2 + \frac{5}{9}Bm .
\]
Combining (2.21) and (2.25) we have

\( m_t = \pm R(m)^{1/2} \);

the sign to be taken as positive when \( m \) increases, negative when \( m \) decreases. Thus \( m \) as function of \( t \) is governed by equation (2.28).

To study the behaviour of \( m \) we have to know something about the function \( R(m) \). First of all we investigate the sign of \( P(m) \); since \( P(m) \) is \( \frac{1}{2}m^2 \) times the quadratic polynomial

\[ \frac{1}{4}m^2 + \frac{1}{4}Am + \frac{1}{4}B, \]

it will be positive if the discriminant of (2.29) is:

\[ \text{discr} = \frac{1}{4}B - \frac{1}{4}A^2. \]

Using formulas (2.14) for \( A \) and \( B \) we have

\[ \text{discr} = \frac{9}{8}(3c_1c_2 - c_1^2 - c_2^2). \]

This quadratic form is positive if and only if

\[ \frac{c_1}{c_2} \leq \frac{3 + \sqrt{5}}{2} = 2.62. \]

Thus we have proved

**Lemma 2.1.** If \( c_1 \) and \( c_2 \) satisfy (2.30), \( P(m) \) is positive for all real values of \( m \). If (2.30) is violated, \( P(m) \) is negative in the interval \( (n_1, n_2) \),

\[ n_{1,2} = \frac{M}{2}[m_1 + m_2 \pm (m_1^2 + m_2^2 - 3m, m_2)^{1/2}] . \]

It is easy to verify that the interval \( (n_1, n_2) \) lies inside \( (m, m) \).

We turn now to \( R(m) \); a lengthy calculation, the gist of which is described in [10], yields the following

**Lemma 2.2.** \( R(m) \) has a double zero at \( m_1 \) and at \( m_2 \) and \( d^2R/dm^2 \) is positive at these points.

**Lemma 2.3.** (a) In the range

\[ \frac{c_1}{c_2} < \frac{3 + \sqrt{5}}{2} , \]

the function \( R(m) \) is positive in \( (m_1, m_2) \).

(b) In the range

\[ \frac{3 + \sqrt{5}}{2} < \frac{c_1}{c_2} < 3 , \]

\( R \) is positive in \( (m_1, n_1) \) and in \( (m_2, n_2) \), where \( n_1, n_2 \) are defined by (2.31).
(c) In the range

\[ 3 < \frac{c_1}{c_2}, \]

\( R \) is positive on \( (m_2, n_2) \) and on \( (m_1, m_1 - m_2) \), and \( R(m_1 - m_2) = 0. \)

Eventually we shall be interested not only in the value of the maximum as function of \( t \) but also in its location \( \gamma(t) \):

\[ d(y(t), t) = m(t). \]

The criterion for a maximum is

\[ d_x(y, t) = 0; \]

differentiating this with respect to \( t \) we get

\[ d_{xx} y_t + d_{xt} = 0, \]

or

\[ y_t = -\frac{d_{xt}}{d_{xx}}. \]
Differentiating the KdV equation we have

\[(2.33) \quad d_{zz} = -d_x^2 - dd_{xx} - d_{xxxx}.\]

Using equation (2.15) we can express \(d_{xxxx}\) as function of \(d, d_x, d_{xx}\), and \(d_{xxx}\). Furthermore, at a local maximum point we have \(d_x = 0\), and \(d_{xx}, d_{xxx}\) can be expressed as in (2.24) and (2.25) as functions of \(m\). Substituting these expressions into (2.33) and (2.32) we get an expression of the form

\[(2.34) \quad y_i = Y(m).\]

If we know \(m\) as function of \(t, y(t)\) can be determined by integration.

We shall not calculate \(Y(m)\) explicitly. It suffices to note that the discussion applies to the solitary waves \(s_1\) and \(s_2\); for these the value of the maximum is \(m_1\), respectively \(m_2\), and the location of the maximum moves with speed \(c_1 = \frac{1}{2}m_1\) and \(c_2 = \frac{1}{2}m_2\), respectively. Therefore (2.34) implies that

\[(2.35) \quad Y(m_1) = c_1, \quad Y(m_2) = c_2.\]

We are now in a position to prove

**THEOREM 2.1.** For any pair of speeds \(c_1\) and \(c_2\) there exists a double wave, i.e., a solution \(d(x, t)\) of the KdV equation such that

\[(2.36) \quad d(x, t) = s(x - c_1 t - \theta_1^\pm; c_1) - s(x - c_2 t - \theta_2^\pm; c_2)\]

tends to zero uniformly as \(t \to \pm \infty\).

Proof: We take as initial value of \(d\) a solution (2.19); according to the existence theorem in [6] the KdV equation has a solution \(d(x, t)\) for all time with these initial data, and \(d\) and all its derivatives are zero at \(x = \pm \infty\). We claim that for all values of \(t\), \(d(x, t)\) satisfies (2.19). To see this we note first that since \(d(x, 0)\) satisfies (2.19), it also satisfies (2.15):

\[(2.37) \quad G(d(0)) = 0.\]

According to Lemma 1.1—equation (1.24)—for any solution \(v\) of the variational equation,

\[(2.38) \quad (G(d), v)\]

is independent of \(t\). Relation (2.37) shows that, at \(t = 0\), (2.38) is zero; therefore (2.38) is zero for all time. Since the value of \(v\) can be prescribed arbitrarily at any particular time, it follows that \(G(d)\) is zero at each time. From this, and from the fact that \(d\) and its derivatives are zero at \(x = \pm \infty\), we deduce as before that \(d\) satisfies equation (2.19) at each \(t\).

We turn now to case (a) of Lemma 2.3 and claim:

For any time \(t\), \(d(x, t)\) has exactly two local maxima.
Proof: In case (a) the number of local maxima is independent of $t$, since at the time of the creation or disappearance of a local maximum $d_{xx} = 0$; on the other hand, according to (2.23), $d_{xx}^2 = P(m)$ at a local maximum, and in case (a), $P(m)$ is positive for all $m$.

The time history of each local maximum is governed by equation (2.33). In case (a), $R$ is positive in $(m_1, m_2)$ and has a double zero at $m_1$ and at $m_2$. It follows from this that each solution of (2.33) whose values lie in $(m_1, m_2)$ goes from $m_1$ to $m_2$ as $t$ goes from $+\infty$ to $-\infty$, or in the other direction, depending on the sign in (2.33). Furthermore, the approach of $m(t)$ to $m_1$ or $m_2$ as $t \to \pm \infty$ is exponential. It follows from this that the function $Y(m)$ occurring in (2.34), (2.35) tends to $c_1$, respectively $c_2$, exponentially as $t \to \pm \infty$. Integrating (2.34) we conclude therefore that

$$y(t) - c_{1,2}t$$

tends to a limiting value as $t \to \pm \infty$.

Next we show that, as $m(t) \to m_1$ or $m_2$, the shape of the curve $d(x, t)$ around the maximum point $y(t)$ tends to the shape of a solitary wave, i.e., that

$$\lim_{t \to \pm \infty} \int_{x - y(t) > \lambda} d^3(x, t) \, dx < e(X),$$

uniformly on bounded $x$ intervals. To prove this we merely note that for each fixed $t$, $s_1(x)$, $s_2(x)$ and $d(x, t)$ all satisfy the fourth order equation (2.15). Furthermore, it follows from relations (2.20), (2.24), (2.35) and (2.25) that the Cauchy data of $d$ at $x = y$ for a fourth order ordinary differential equation, i.e., the values of $d$ and of its first three $x$ derivatives tend as $t \to \pm \infty$ to the Cauchy data of $s_1(x)$, respectively $s_2(x)$, at $x = 0$. Relation (2.40) follows then from the continuous dependence of solutions of ordinary differential equations on their Cauchy data.

From (2.40) we can deduce our assertion about the number of maxima; for, assume that there were more than two of them. Then at least two would tend to the same limit, say $m_1$, as $t \to \infty$. It follows from (2.39) that the separation of these two locations of maxima tends to a constant; but this is incompatible with relation (2.39) which says that $d$ looks like $s_1$ centered around either location.

Likewise it is impossible for $d$ to have only one maximum; since then, $d(x - y(t), t)$ would be monotonically decreasing in $x$ on either side of $y(t)$ and so would tend uniformly to one solitary wave as $t \to -\infty$, the other as $t \to +\infty$. We claim that, for all $t$,

$$\int_{|x - y(t)| > \lambda} d^3(x, t) \, dx < e(X),$$

It is not hard to show that the values of any solution of (2.19) which is zero at $x = \pm \infty$ lie in $(m_1, m_2)$. 

---

\[ \text{Note:} \]
where \( \varepsilon(X) \) tends to zero as \( X \to \infty \). If this were so, we could deduce that

\[
(2.42) \quad \lim_{t \to \pm \infty} \int d^2(x - y(t), t) \, dx = \left\{ \begin{array}{l}
\int s_1^2(x) \, dx , \\
\int s_2^2(x) \, dx .
\end{array} \right.
\]

But this is a contradiction. For, on the one hand, for \( c_1 \neq c_2 \),

\[
\int s_1^2 \, dx \neq \int s_2^2 \, dx ,
\]
on the other hand, \( \int d^2 \, dx \) is invariant under both \( x \) and \( t \) translation, and so the integral on the left side of (2.42) is independent of \( t \). But then the limits as \( t \to + \infty \) and \( - \infty \) cannot have different values.

To prove (2.41) we merely note that \( \int d(x, t) \, dx = I_\theta(d) \) is also an integral for the KdV equation, from which (2.41) follows with

\[
e(X) = s(X) ,
\]
because of the monotonicity of \( d \) on either side of \( y(t) \).

Having shown that \( d(x, t) \) has exactly two maxima, it follows easily by the arguments already presented that, as \( t \to \pm \infty \), \( d(x, t) \) tends uniformly to the superposition of two solitary waves, each travelling at its own speed. This completes the proof of Theorem 2.1 in case (a).

The analysis presented above shows that the time history of the two maxima is as follows: as \( t \) goes from \( - \infty \) to \( + \infty \) the height of the larger solitary wave decreases from \( m_1 \) to \( m_2 \), while the height of the smaller one increases from \( m_2 \) to \( m_1 \). Thus in this case the two solitary waves interchange their roles without passing through each other, as observed by Kruskal and Zabusky in their calculations.

We turn now to case (b); here \( R(m) \) is not defined inside \( (n_1, n_2) \), which means (see equation (2.23)) that no relative maximum of \( d(x, t) \) can lie in that interval. We assert that the absolute maximum lies above that interval at all times. For, suppose on the contrary that at some time the absolute maximum is less than \( n_1 \); then, since the absolute maximum is a continuous function of \( t \), and since on the other hand the value of the absolute maximum cannot cross \( (n_1, n_2) \), it follows that \( d(x, t) \) does not exceed \( n_2 \) for any value of \( x \) and \( t \). Let \( m(t) \) denote a local maximum at time \( t \). As we saw earlier, \( m(t) \) satisfies the differential equation (2.33): \( m_1 = \pm R_{\theta/2}(m) \). In case (b), those solutions of this differential equation whose values lie in \( (m_1, n_2) \) behave as follows: depending on the sign in (2.33), \( m(t) \) tends with increasing (decreasing) \( t \) to \( n_2 \):

\[
\lim_{t \to t_0} m(t) = n_2 .
\]

Denote as before the location of the maximum by \( y(t) \), and set

\[
\lim_{t \to t_0} y(t) = y_0 .
\]
Relations (2.20) and (2.24) imply that
\[ d_2(y_0, t_0) = 0, \quad d_{xx}(y_0, t_0) = 0, \]
while relation (2.25) and the fact that \( R(n_2) > 0 \) imply that
\[ d_{xx}(y_0, t_0) > 0. \]
This means that the function \( d(x, t_0) \) does not have a local—much less global—maximum at \( x = y_0 \). Since the value of \( d \) at \( x = y_0 \) is \( n_2 \), it follows that the global maximum of \( d(x, t_0) \) exceeds \( n_2 \); this contradicts our previous assumption, and so proves the assertion.

Denote by \( M(t) \) the absolute maximum of \( d(x, t) \); denote by \( m(t) \) that local maximum which at, say, \( t = 0 \) equals \( M(0) \) and which satisfies the differential equation (2.33) for local maxima. The function \( m(t) \) is either increasing or decreasing. Suppose it is increasing, then, since all local maxima satisfy (2.33), it follows that \( m(t) \) remains the absolute maximum for \( t > 0 \); it further follows from (2.33) + that \( m(t) \) tends to \( m_1 \) as \( t \) tends to +\( \infty \), and that for some finite time \( t_0 \)
\[ \lim_{t \to t_0} m(t) = n_1. \]
When \( m \) reaches the value \( n_1 \), it ceases to be a local maximum, so that
\[ M(t_0) > m(t_0). \]
Denote by \( t_1 \) the infimum of those values of \( t \) for which \( m(t) \) is the absolute maximum; clearly \( t_0 < t < 0 \). At time \( t_1 \) there must be at least two points where the absolute maximum is assumed. Denote by \( n(t) \) that local maximum which is equal to \( m(t) \) at \( t = t_1 \) but is a decreasing function of \( t \); \( n(t) \) satisfies equation (2.33) −, and it follows that, as \( t \to -\infty \), \( n(t) \) tends to \( m_1 \).

Denote by \( y(t) \) the location at time \( t \) of the absolute maximum. As before we deduce that
\[ \lim_{|t| \to \infty} d(x + y(t), t) = s_1(x), \]
uniformly on bounded \( x \) intervals.

The analysis presented in the previous case, when applied to this situation, shows that for \( |t| \) large there can be at most one additional local maximum; suppose it exists and is located at \( z(t) \). Then it follows as above that
\[ \lim_{|t| \to \infty} d(x + z(t), t) = s_2(x), \]
uniformly on bounded \( x \) intervals.

A careful examination of equation (2.33) satisfied by \( d \) shows that \( d \) does indeed have another local maximum for \( |t| \) large; this proves the theorem in case (b).

Case (c) can be analyzed in a similar fashion. The main difference is that the
function \( R(m) \) vanishes at \( m = m_1 - m_2 \); it is easy to show that this causes solutions of \( m_t = \pm R^{1/3}(m) \) in the range \( m_1 - m_2 \leq m(t) < m_1 \) to behave in the following fashion:

There is a value \( t_0 \) such that \( m(t_0) = m_1 - m_2 \), and \( m \) satisfies

\[
m_t = \begin{cases} 
-R^{1/3}(m) & \text{for } t < t_0, \\
R^{1/3}(m) & \text{for } t_0 < t.
\end{cases}
\]

In particular \( m(t) \) tends to \( m_1 \) as \( |t| \) tends to \( \infty \).

As in case (b), one can show that for \( |t| \) large there is exactly one absolute and one local maximum, and that the asymptotic relation (2.36) holds. This completes the proof of Theorem 2.1.

The time history of the local maxima is as follows:

As \( t \) goes from \(-\infty\) to \( \infty \), the amplitude of the smaller solitary wave increases until it reaches the value \( n_2 \), at which point the local maximum disappears. In the meanwhile the amplitude of the larger solitary wave decreases steadily; in case (c), this amplitude reaches its minimum value \( m_1 - m_2 \) at some time \( T \) and some point \( X \). The double wave is symmetric with respect to this occurrence, i.e.,

\[
d(X - x, T - t) = d(x, t).
\]

In case (b), the amplitude of the larger solitary wave decreases until it reaches the value \( n_1 \), at which point the local maximum disappears. Before this happens however another local maximum is created which starts increasing. Denote by \( T \) the time when the two maxima are equal, and denote by \( X \) the midpoint between the two local maxima; the double wave satisfies the symmetry relation (2.43).

Speaking qualitatively we might say that in cases (b) and (c) the big wave first absorbs, then re-emits the small wave, and that in case (b) the absorption of the small wave raises a secondary peak on the big wave.

Up to a certain point one can give a similar analysis of \( N \)-tuple waves, that is, solutions of the KdV equation which as \( t \to \pm \infty \) split apart into a superposition of \( n \)-solitary waves with speeds \( c_1, c_2, \ldots, c_N \). Using the first \( N \) of the sequence of integrals constructed in [5] one can derive an ordinary differential equation of order \( 2N \) with respect to \( x \). An elegant argument of Gardner (personal communication) shows that the generalized KdV operators \( K_n, n = 1, \ldots, N \), are integrating factors for this differential operator. In this fashion we can obtain by elimination a differential equation of order \( N \) for the \( N \)-tuple wave, but the resulting equation is too complicated to yield any useful information. Happily, these \( N \)-tuple waves have been described explicitly in [1].

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Bibliography


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