1 Math 221, Fall 2013: Problem Set 07

Please hand in solutions for the following problems.

**Exercise 1** Write a program which finds the eigenvalues of the rank-one update \( D + \rho uu^T \) of a diagonal matrix \( D \) with an arbitrary vector \( u \). Solve the secular equation by bisection, Newton’s method, or the Newton-like method described in Demmel. Test on matrices \( D \) of size \( n = 10 \) to \( 100 \) with both well-separated diagonal entries \( d_j = j \) and clustered diagonal entries \( d_j = 1 + 2^{-j/2} \), using \( \rho = 1 \) and random vectors \( u \). Evaluate eigenvectors \( w_j \) by both the unstable formula \( w_j = (D - \lambda_j)^{-1}u/\|\lambda_j - D\lambda_j^{-1}u\|_2 \) and recomputing \( u \) before applying the unstable formula. Measure the the residual \( \|W^T \Lambda W - T\|_2 \) and loss of orthogonality \( \|W^T W - I\|_2 \) in all four cases and draw conclusions.

**Exercise 2** Generate a \( n \times n \) symmetric tridiagonal matrix \( T \) with \( n = 2^k \), \( k = 2 \) through \( 10 \), and well-separated eigenvalues \( \lambda_j = j \). Find the eigensystems of nonoverlapping \( 1 \times 1 \) diagonal blocks, piece them together into \( 2 \times 2 \) blocks, and repeat to find the eigensystem of \( T \). Measure the relative accuracy of the eigenvalues and the orthogonality of the eigenvectors.

**Exercise 3** Derive and implement a fast butterfly algorithm for the nonuniform Laplace transform

\[
\hat{f}(t_k) = \sum_{j=1}^{N} f_j e^{t_k s_j} \quad 1 \leq k \leq N
\]

where \( f_j, t_k \) and \( s_j \) are \( 3N \) given complex numbers with \( \max_{k,j} |t_k s_j| \leq O(N) \). Use Stirling’s formula to estimate the error in truncating the exponential series after \( m \) terms. Test your fast transform on uniform random data and estimate the breakeven point versus direct evaluation for three-digit, six-digit and twelve-digit absolute accuracy. In order to avoid exponential growth, distribute \( t_k \) over the rectangle \( |\Re z| \leq 1, |\Im z| \leq N \) and \( s_j \) over the rectangle \( |\Im z| \leq 1/N, |\Re z| \leq 1 \).