1 Math 221, Fall 2013: Problem Set 02

Please hand in detailed solutions to the following 3 problems.

Exercise 1 Prove the following inequalities and for each give an example of a nonzero \( n \)-vector or \( m \times n \) matrix for which equality is achieved:

(a) \( \|x\|_\infty \leq \|x\|_2 \)
(b) \( \|x\|_2 \leq \sqrt{n} \|x\|_\infty \)
(c) \( \|A\|_\infty \leq \sqrt{n} \|A\|_2 \)
(d) \( \|A\|_2 \leq \sqrt{m} \|A\|_\infty \)

Exercise 2 Let \( A \) be an \( n \times n \) matrix and \( B \) be the \( (n-p) \times (n-q) \) matrix obtained by deleting the last \( p \) rows and \( q \) columns of \( A \). (a) Find matrices \( P \) and \( Q \) that carry out the deletion via the matrix multiplications \( B = PAQ \). (b) Show that \( \|B\| \leq \|A\| \) for any matrix \( p \)-norm.

Exercise 3 Let \( A \) be an \( m \times n \) matrix with full rank \( r = n \leq m \). (a) Show that the linear system

\[
\begin{bmatrix}
I + AA^* & A \\
A^* & A^*A
\end{bmatrix}
\begin{bmatrix}
\mathbf{r} \\
\mathbf{x}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{b} \\
A^* \mathbf{b}
\end{bmatrix}
\]

has a solution \( (\mathbf{r}, \mathbf{x})^T \) for any \( \mathbf{b} \). (b) Show that \( y = \mathbf{x} \) minimizes \( \|Ay - \mathbf{b}\|_2 \) over all vectors \( y \). (c) Find explicit formulas for the inverse of

\[
\begin{bmatrix}
I + AA^* & A \\
A^* & A^*A
\end{bmatrix}
\]