1 Math 221, Fall 2013: Problem Set 01

Please hand in detailed solutions to the following 6 problems.

Exercise 1 A square matrix \( R \) is lower triangular if its entries \( r_{ij} \) vanish for \( i < j \). (a) Show that if \( R \) is lower triangular and nonsingular then its inverse \( R^{-1} \) is also lower triangular. (Hint: look at the space spanned by the first \( k \) columns of \( R \) for each \( k \).) (b) Solve the equation \( R + R^T = A \) for a lower triangular matrix \( R \), where \( A \) is a given square symmetric matrix and \( R^T \) is the transpose of \( R \).

Exercise 2 Prove the Pythagorean theorem: if \( x_i \) are \( n \) orthogonal vectors in \( C^n \) then
\[
\left\| \sum_{i=1}^{n} x_i \right\|_2^2 = \sum_{i=1}^{n} \left\| x_i \right\|_2^2.
\]

Exercise 3 Suppose an \( m \times m \) complex matrix \( S \) is skew-Hermitian: \( S^* = -S \). (a) Show that its eigenvalues are pure imaginary. (b) Show that \( I - S \) is nonsingular. (c) Show that \( Q = (I - S)^{-1}(I + S) \) is unitary. (d) Does every unitary matrix come from some \( S \) this way? Is \( S \) unique?

Exercise 4 Let \( u \) and \( v \) be \( m \)-vectors and \( A = I + uv^* \). (a) For what \( u \) and \( v \) is \( A \) invertible? (b) When \( A \) is invertible find a formula for its inverse. (c) Find all the eigenvalues and eigenvectors of \( A \). (d) Generalize these results to the case where \( u \) and \( v \) are \( m \) by \( r \) matrices of rank \( r \) and \( r \) is (much) smaller than \( m \).

Exercise 5 (a) Produce an orthonormal basis for the range of
\[
A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.
\]
(b) Find a matrix \( Q \) such that \( Q^*Q = I \) and \( A = QR \), where \( R \) is upper triangular with positive diagonal elements. (c) Discuss the relationship between (a) and (b).

Exercise 6 Prove the inequality
\[
\|A\|_2^2 \leq \|A\|_1 \|A\|_\infty
\]
for an arbitrary \( n \times n \) matrix \( A \) and the matrix norms induced by the standard 1, 2 and \( \infty \) norms.