show that there is a unique solution \( y \) for any set of data \( f \) and \( g \) if and only if there is a fundamental matrix \( Y \) such that \( D = AY(a) - BY(b) \) is invertible.

**Exercise 2** Let \( Q_{ij} = \delta_{n+1-i,j} \) be the matrix elements of the matrix \( Q \) which reverses a vector: \( y = Qx \) has elements \( (y_1, y_2, \ldots, y_n) = (x_n, x_{n-1}, \ldots, x_1) \).

(a) Find a fundamental matrix \( Y_0(t) \) for the system \( y' + Q_0 y = 0 \) where \( \lambda \) is a positive constant and \( Q_0 = \lambda Q \).

(b) Given \( a \) and \( b \) with \( a < b \), determine \( \lambda \) so that the system \( y' + Q_0 y = f \) with periodic boundary conditions \( y(a) - y(b) = g \) is solvable for all data \( f \) and \( g \).

(c) Find the Green function \( G_0 \) for (b).

(d) Use \( Q_0 \) as a background problem to derive an integral equation

\[ \sigma(t) + A(t) \int_a^t \sigma(s)ds + B(t) \int_t^b \sigma(s)ds = r(t) \]

that makes \( Y_0(t)^{-1}(y'(t) + Q_0 y(t)) = \sigma(t) \) solve a given periodic boundary value problem

\[ y' + Q(t)y = f(t) \]

with

\[ y(a) - y(b) = g. \]

(e) Test the integral equation with \( p \)-point Gaussian quadrature using \( p = 2^3 \) points on the BVP \( y' + Q(t)y = f(t) \) on \( 0 \leq t \leq \pi \), \( y(0) - y(\pi) = g \), where

\[ Q(t) = \begin{bmatrix} 1 - 9 \cos 2t & 1 + 9 \sin 2t \\ -1 + 9 \sin 2t & 1 + 9 \cos 2t \end{bmatrix} \]

and \( f \) and \( g \) are chosen to make \( y(t) = (\cos(5t^2), 2 + e^{-10t})^T \) the exact solution. For each precision \( \epsilon = 10^{-6} \) estimate which grid size \( p \) delivers a solution with max-norm error less than \( \epsilon \) at roughly the least cost in CPU time.

**Exercise 3** Write a program to solve

\[ K_1 \sigma(t) = \sigma(t) + A(t) \int_a^t \sigma(s)ds + B(t) \int_t^b \sigma(s)ds = r(t) \]

where \( A(t) \) and \( B(t) \) are given \( n \times n \) matrix functions of \( t \in [a,b] \) and \( r(t) \) is a given \( n \)-vector function. Divide the interval \( [a,b] = C_0 \) into \( 2^L \) subintervals \( C_j = [a_j, b_j] \), solve the integral equations \( K_j \beta = r \), \( K_j \alpha = A \) and \( K_j \beta = B \) on each \( C_j \) directly by \( p \)-point Gaussian quadrature and Gaussian elimination or
QR factorization of the resulting $np \times np$ linear systems, and piece the resulting solutions together recursively on levels $L - 1$ through 0 to give $\sigma$ on the union of $2^L$ level-$L$ intervals. Test on $\sigma = Y_0^{-1}(y' + Q_0 y)$ where $y$ is the exact solution of problem 2(e).