1 Math 128b, Spring 2020: Problem Set 7

Exercise 1 A q-stage Runge-Kutta method

\[ k_i = f \left( u_n + h \sum_{j=1}^{q} S_{ij} k_j \right), \quad 1 \leq i \leq q \]

\[ u_{n+1} = u_n + h \sum_{i=1}^{q} w_i k_i \]

is algebraically stable if

\[ B_{ij} = w_i S_{ij} + w_j S_{ji} - w_i w_j \]

defines a positive semidefinite matrix \( B \), i.e. iff

\[ x^T B x \geq 0 \]

for all \( x \in \mathbb{R}^q \).

(a) Show that algebraic stability implies linear error growth: any solution \( v \) of the perturbed method

\[ \kappa_i = f \left( v_n + h \sum_{j=1}^{q} S_{ij} \kappa_j \right), \quad 1 \leq i \leq q \]

\[ v_{n+1} = v_n + h \sum_{i=1}^{q} w_i \kappa_i + h \tau_n \]

with local truncation error \( \tau_n \) bounded by

\[ \| \tau_n \| \leq \tau \]

for all \( n \) must satisfy

\[ \| u_n - v_n \| \leq T \tau \]

for \( 0 \leq t_n \leq T \).

(b) Show that q-stage Gauss-Runge-Kutta methods are algebraically stable for all \( q \geq 1 \).

(c) Verify linear error growth experimentally for 3-stage Gauss-Runge-Kutta by measuring error after \( N = 1, 2, \ldots, 10 \) orbits of the figure-8 problem

\[ x'' = -x \]

\[ y'' = -4y \]

with initial conditions \( x(0) = 1 \)
\[ x'(0) = 0 \]
\[ y(0) = 0 \]
\[ y'(0) = 2 \]

which should yield a periodic orbit \((\cos(t), \sin(2t))\) with period \(T = 2\pi\). (Convert to a 4 \times 4 first-order autonomous system and solve linear equations for the stages.)