1 Math 128b, Spring 2020: Problem Set 6

Exercise 1 Given 2n distinct real numbers \( s_1, s_2, \ldots, s_n \) and \( t_1, t_2, \ldots, t_n \), define the \( n \times n \) Cauchy matrix \( C = C(t, s) \) by \( C_{ij} = 1/(t_i - s_j) \). Express the Lagrange interpolation formula

\[
 p(t_i) = \sum_{j=1}^{n} L_j(t_i)f_j
\]

where

\[
 L_j(t) = \prod_{k \neq j} \frac{t - s_k}{s_j - s_k}
\]

in terms of diagonal and Cauchy matrices.

Exercise 2 Find diagonal matrices \( D_L \) and \( D_R \) such that \( C(t, s)^{-1} = D_L C(s, t) D_R \).

Exercise 3 Write a program to apply a Cauchy matrix \( C \) to an arbitrary n-vector \( g \) by the fast multipole method (see the handout) with \( p = O(\log \epsilon) \) expansion terms. For levels \( l = 2 \) to \( L = O(\log n) \),

1. Partition the \( s \)'s and \( t \)'s into cells \( S \) and \( T \) of size \( 2^{-l} \) and reverse index them.

2. Compute moments

\[
 S_k = \sum_{s_j \in S} g_j (s_j - \sigma)^k
\]

for \( 0 \leq k \leq p \) about the center \( \sigma \) of each interval \( S \).

3. For each interval \( S \), accumulate degree-p Taylor coefficients

\[
 T_m = \sum_{S} \sum_{k=0}^{p} \binom{m+k}{k} (\tau - \sigma)^{m-k-1} S_k
\]

in each well-separated level-\( l \) interval \( T \) with center \( \tau \) which is not accounted for on coarser levels.

4. Evaluate degree-p Taylor polynomials

\[
 p_T(t) = \sum_{m=0}^{p} T_m(\tau - t)^m
\]

at \( t_i \in T \).

Finally, add direct interactions between level-\( L \) cells that are not well-separated.

Test your program on \( n \) randomly generated points \( s_j \) and \( t_i \) on the interval [0,1] with \( p = 10, 20 \) and \( 30 \) expansion terms. Tabulate or plot errors and timings for \( n \) varying from 100 to 10000 or so. Verify the \( O(n \log n \log \epsilon) \) work required for accuracy \( \epsilon \).