1 (a) Given

\[ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq 0 \]

find a \( 2 \times 2 \) orthogonal matrix \( H_\pm \) such that

\[ H_\pm x = \pm \sqrt{x_1^2 + x_2^2} e_1. \]
(b) Given \( x = x_j e_j + x_k e_k \neq 0 \) where \( j \neq k \), find an orthogonal matrix \( H_{\pm} \) such that

\[
H_{\pm} x = \pm \sqrt{x_j^2 + x_k^2} e_j.
\]
(c) Given a tridiagonal matrix

\[
A = \begin{bmatrix}
    a_1 & c_1 & 0 & 0 & \cdots & 0 \\
    b_2 & a_2 & c_2 & 0 & \cdots & 0 \\
    0 & b_3 & a_3 & c_3 & \cdots & 0 \\
    0 & \cdots & 0 & b_{n-1} & a_{n-1} & c_{n-1} \\
    0 & \cdots & 0 & b_n & a_n & \cdots \\
\end{bmatrix}
\]

with \( n \geq 2 \) even, find an orthogonal matrix \( Q \) such that \( QA \) has zeroes in place of even-numbered below-diagonal entries \( b_{2j} \) for \( j = 1 : n/2 \).
(d) Use (c) to devise an algorithm for \( QR \) factorization of a tridiagonal matrix \( A \).
2 Use the pseudoinverse to study high-order quadrature formulas with arbitrary points, as follows. Write a function which accepts an integer \( m > 0 \), an integer \( n \geq m \), \( n+1 \) arbitrary points \( x_i \) and an interval \([a, b]\) and computes \( n+1 \) weights \( w_i \) for a quadrature formula

\[
\int_a^b f(x)dx = \sum_{i=0}^{n} w_i f(x_i).
\]

The weights are to be computed as the minimum 2-norm solution of the underdetermined linear system which expresses the requirement that the formula be exact for \( f(x) = 1, x, \ldots, x^m \) where \( m \) may not be equal to \( n \). Test your function with random, equispaced and Chebyshev points, on the interval \([0, 1]\), for \( n = 2, 4, 8, 16, 32, 64, 128 \) and \( m \) varying from \( n/2 \) to \( n \). In each case, compute the maximum error in integrating \( 1, x, \ldots, x^m, \) and \( f(x) = \cos(10x) \), and compute the quantity \( \kappa = \sum_{i=0}^{n} |w_i| \). Draw general conclusions about formulas of high order \( m \) with \( n \) arbitrary points.
3 Repeat the calculation of problem (2) with the weights computed instead from the linear system which expresses the requirement that the first \( m \) Legendre polynomials \( P_k(x) \) (shifted and scaled to the interval \([a, b]\)) be integrated exactly. The Legendre polynomials on \([a, b]\) can be evaluated by the recurrence

\[
P_{k+1}(x) = ((2k + 1)tP_k(x) - kP_{k-1}(x))/(k + 1)
\]

where \( t = (x - c)/h \) with \( c = (a + b)/2 \) and \( h = (b - a)/2 \). Start with \( P_0 = 1 \) and \( P_1(x) = t \).