Let \( H_{ij} = 1/(i + j - 1) \) for \( 1 \leq i, j \leq n \) be the elements of the \( n \times n \) Hilbert matrix \( H \). and \( QR = H \) be its QR factorization. Compute \( Q \) and \( R \) for \( n = 1 : 20 \) by four methods:

(a) Cholesky factorization \( R^T R = H^T H \) and \( Q = HR^{-1} \),

(b) Classical Gram-Schmidt,

(c) Modified Gram-Schmidt,

(d) Matlab’s built-in (Householder).

Plot \( \|Q^T Q - I\|_2 \) and \( \|QR - H\|_2 / \|H\|_2 \) vs. \( n \) and discuss.
2. (cf. GGK 6.6) Construct a matrix $P$ which projects onto the plane $x_1 + x_2 + x_3 = 0$ in $\mathbb{R}^3$. 
3 (cf. GGK 6.8) Construct an orthogonal matrix which reflects across the plane \( x_1 = x_3 \) in \( \mathbb{R}^3 \).
4  (cf. GGK 3.7) Let

\[ A = \begin{bmatrix} 1 & 2 & 6 \\ 1 & 3 & 7 \\ 1 & 4 & 7 \\ 1 & 5 & 9 \end{bmatrix}. \]

Find a Householder vector \( u \) so that \( H = I - 2uu^T \) reduces the first column of \( A \) to zero below the diagonal:

\[ HA = \begin{bmatrix} \sigma & x & x \\ 0 & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix}. \]
Suppose $Q$ is orthogonal and $R$ is upper triangular but $QR \neq A$. Find explicit formulas for updated factors $Q+\delta Q$ and $R+\delta R$ that reduce the residual $\rho = \sqrt{\|Q^TQ - I\|_2^2 + \|QR - A\|_2^2}$ quadratically. You may use the matrix-matrix function $U = \text{uth}(A)$ defined to have elements $U_{ij} = A_{ij}$ for $i < j$, $U_{ij} = A_{ij}/2$ for $i = j$, and $U_{ij} = 0$ for $i > j$. 