1 (cf. SB 4.1) (a) Show that
\[ \|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \]

(b) Show that
\[ \|x\|_2 \leq \sqrt{n}\|x\|_\infty \]
and
\[ \|x\|_1 \leq \sqrt{n}\|x\|_2, \]
and find a case for each inequality where equality holds.

(c) Find computable formulas for \( \|A\|_1 \) and \( \|A\|_\infty \).

(d) For any vector norm \( \|\| \) show that the induced matrix norm satisfies
\[
\frac{1}{\|A^{-1}\|} = \min_{y \neq 0} \frac{\|Ay\|}{\|y\|}
\]
for any nonsingular matrix \( A \).
2 (cf. SB 4.2) For any vector norm $\|\|$ and a fixed nonsingular matrix $D$ let

$$\|x\|_D = \|Dx\|$$

(a) Show that $\|\|_D$ is a vector norm.
(b) Show that $m\|x\| \leq \|x\|_D \leq M\|x\|$ where

$$m = \frac{1}{\|D^{-1}\|}$$

and

$$M = \|D\|$$

(c) Find a formula for the matrix norm $\|A\|_D$ induced by $\|\|_D$ in terms of the matrix norm induced by $\|\|$.
(d) Show that the condition number in the $D$-norm can be made arbitrarily large by choice of $D$.
(e) Bound the $\infty$-norm condition number above and below by the 2-norm condition number.
3 Show that IEEE standard floating point arithmetic computes the inner product $x^T y$ of two $n$-vectors $x$ and $y$ with backward error bounded by $2n\epsilon$. Can you bound the forward relative error?
4 (cf. GGK 3.19) Suppose $A$ is a nonsingular matrix. Let $y$ be a unit vector such that $\|A^{-1}\|_2 = \|A^{-1}y\|_2$ and let $x = A^{-1}y/\|A^{-1}y\|_2$. Let $E = -Ax x^T$.

(a) Show that $(A + E)x = 0$ so that $A + E$ is singular.

(b) Show that $\|E\|_2 = 1/\|A^{-1}\|_2$ and conclude that $1/\kappa_2(A)$ is the relative distance from $A$ to the nearest singular matrix.
Let
\[ H_{ij} = \int_0^1 t^i t^j dt, \quad 0 \leq i, j \leq n \]
be the elements of the \( n + 1 \times n + 1 \) Hilbert matrix \( H \). Let
\[ P_i(t) = \sum_{j=0}^{i} p_{ij} t^j, \quad 0 \leq i, j \leq n, \]
define the coefficients \( p_{ij} \) of the orthonormal Legendre polynomials on \([0, 1]\), so that
\[ \int_0^1 P_i(t) P_j(t) dt = \delta_{ij}, 0 \leq i, j \leq n. \]
Find an invertible lower triangular matrix \( L \) such that \( H^{-1} = L^T L \).