Problem 1 Let $\lambda_k$ be $n+1$ distinct real numbers. Let $t_j$ be $n+1$ distinct real numbers.

(a) Show that

$$a(t) = \sum_{k=0}^{n} a_k e^{\lambda_k t}$$

can vanish for all real $t$ only if $a_0 = a_1 = \cdots = a_n = 0$.

(b) Show that for the exponential interpolation problem

$$a(t_j) = \sum_{k=0}^{n} a_k e^{\lambda_k t_j} = f_j \quad 0 \leq j \leq n$$

there exists a unique solution $a(t)$ for any data values $f_j$.

(c) Interpolate the function

$$f(t) = \frac{1}{1 + t^6}$$

by $n+1$ exponentials with $\lambda_k = -k/n$, $k = 0$ through $n$, at $n+1$ equidistant points $t_j = 5j/n$ for $j = 0$ through $n$ on the interval $[0, 5]$ by and tabulate the error for $n = 3, 5, 9, 17, 33$. 
Problem 2 For equidistant points \( x_j = j, 0 \leq j \leq n, n \) even, let

\[
\omega(x) = (x - x_0)(x - x_1) \ldots (x - x_n)
\]

Use Stirling’s formula to estimate the ratio \( \omega(1/2)/\omega(n/2 + 1/2) \) for large \( n \). Define and explain the Runge phenomenon.
Problem 3 Interpolate the function

\[ f(x) = \frac{1}{1 + x^6} \]

on the interval \([0, 5]\) at

(a) \(n + 1\) equidistant points \(x_k = 5k/n\), and

(b) \(n + 1\) Chebyshev points \(x_k = (5 + 5 \cos((2k + 1)\pi/(2n + 2)))/2\).

Use \(n = 3, 5, 9, 17, 33\) and for each case

(1) tabulate the maximum error over 1000 random points \(y_k \in [0, 5]\), and

(2) plot \(\ln(1 + |\omega(x)|) = \ln(1 + |(x - x_0)(x - x_1)\ldots(x - x_n)|)\).
Problem 4 (See BBF 3.4.11) (a) Show that $H_{2n+1}(x)$ is the unique polynomial $p$ agreeing with $f$ and $f'$ at $x_0,\ldots,x_n$. (Hint: Find a square system of linear equations that determine the coefficients of $p$ in some basis for degree-$(2n+1)$ polynomials. Show that a (possibly non-unique) solution always exists. Use linear algebra.)

(b) Derive the error term in Theorem 3.9. (Hint: Use the same method as in the Lagrange error derivation, Theorem 3.3, defining

$$g(t) = f(t) - H_{2n+1}(t) - \frac{(t-x_0)^2 \cdots (t-x_n)^2}{(x-x_0)^2 \cdots (x-x_n)^2} (f(x) - H_{2n+1}(x))$$

and using the fact that $g'(t)$ has $2n + 2$ distinct zeroes in $[a,b]$.)

(c) Separate the error into three factors and explain why each factor is inevitable.
**Problem 5** Let \( p \) be a positive integer and
\[
f(x) = 2^x
\]
for \( 0 \leq x \leq 2 \).
(a) Find a formula for the \( p \)th derivative \( f^{(p)}(x) \).
(b) For \( p = 0, 1, 2 \) find a formula for the polynomial \( H_p \) of degree \( 2p + 1 \) such that
\[
H_p^{(k)}(x_j) = f^{(k)}(x_j)
\]
for \( 0 \leq k \leq p, 0 \leq j \leq 1, x_0 = 0, x_1 = 2 \).
(c) For general \( p \) prove that
\[
|f(x) - H_p(x)| \leq \left( \frac{1}{p+1} \right)^{2p+2}
\]
for \( 0 \leq x \leq 2 \).
(d) Show that one step of Newton’s method for solving
\[
g(y) = x \ln 2 - \ln y = 0
\]
starting from \( y_0 = H_4(x) \) gives \( y_1 = f(x) = 2^x \) to almost double precision accuracy for \( 0 \leq x \leq 2 \).
Problem 6 Let \( n \geq m \geq 0 \), \( a \in \mathbb{R} \), and \( n + 1 \) distinct interpolation points \( x_0, x_1, \ldots, x_n \). Let \( \delta_{nk}^m(a) \) be the differentiation coefficients

\[
\delta_{nk}^m(a) = \left( \frac{d}{dx} \right)^m L_k^n(x)|_{x=a}
\]

such that the degree-\( n \) polynomial \( p(x) \) which interpolates \( n + 1 \) values \( f_j \) at \( n + 1 \) points \( x_j \) satisfies

\[
p^{(m)}(a) = \sum_{k=0}^{n} \delta_{nk}^m(a)f_k.
\]

(a) Derive the recurrence relation

\[
\delta_{nk}^m(a) = \frac{m}{x_k - x_n} \delta_{n-1,k}^{m-1}(a) + \frac{a - x_n}{x_k - x_n} \delta_{n-1,k}^m(a)
\]

for \( 0 \leq k \leq n - 1 \).

(b) Write a Matlab code which evaluates \( \delta_{nk}^m(a) \) for \( 0 \leq m \leq M \), given \( n \) and the points \( a \) and \( x_j \).

(c) Validate your coefficients \( \delta_{nk}^m(a) \) by verifying \( O(h^{n-m}) \) accuracy for the \( m \)th derivative of \( f(x) = e^x \) evaluated at \( n + 1 \) equidistant points \( x_j = jh \).

(d) Fix interpolation points \( x_j \) and form an \( (n + 1) \times (n + 1) \) matrix \( A_m \) of differentiation coefficients with

\[
(A_m)_{ij} = \delta_{nj}^m(x_i).
\]

Is \( A_m = A_1^m \)? Why or why not?