1 (BFB 3.4.5) (a) Use the values of \( f(x) = \sin(x) \) and \( f'(x) = \cos(x) \) at \( x = [0.30, 0.32, 0.35] \), with five-digit rounding arithmetic, to construct the Hermite interpolating polynomial that approximates \( \sin(0.34) \).

(b) Determine an error bound for the approximation in part (a) and compare it to the actual error.

(c) Add data at \( x = 0.33 \) and redo the calculations.

2 (BFB 3.4.10) Let \( z_0 = z_1 = x_0, z_2 = z_3 = x_1 \). Form the divided-difference table

\[
\begin{array}{cccc}
  z_0 &=& x_0 & f[z_0] = f(x_0) \\
  z_1 &=& x_0 & f[z_1] = f(x_0) \\
  z_2 &=& x_1 & f[z_2] = f(x_1) \\
  z_3 &=& x_1 & f[z_3] = f(x_1)
\end{array}
\]

Show that the cubic Hermite polynomial \( H_3(x) \) can also be written as

\[
f[z_0] + f[z_0, z_1](x-x_0) + f[z_0, z_1, z_2](x-x_0)^2 + f[z_0, z_1, z_2, z_3](x-x_0)^2(x-x_1).
\]

3 (See BFB 3.4.11) (a) Show that \( H_{2n+1}(x) \) is the unique polynomial \( p \) agreeing with \( f \) and \( f' \) at \( x_0, \ldots, x_n \). (Hint: Find a square system of linear equations that determine the coefficients of \( p \) in some basis for degree-(2\( n \)+1) polynomials. Show that a (possibly non-unique) solution always exists. Use linear algebra.)

(b) Derive the error term in Theorem 3.9. (Hint: Use the same method as in the Lagrange error derivation, Theorem 3.3, defining

\[
g(t) = f(t) - H_{2n+1}(t) - \frac{(t-x_0)^2 \cdots (t-x_n)^2}{(x-x_0)^2 \cdots (x-x_n)^2}(f(x) - H_{2n+1}(x))
\]

and using the fact that \( g'(t) \) has \( 2n + 2 \) distinct zeroes in \( [a, b] \).)

(c) Separate the error into three factors and explain why each factor is inevitable.

4 Let \( p \) be a positive integer and

\[
f(x) = 2^x
\]

for \( 0 \leq x \leq 2 \).

(a) Find a formula for the \( p \)th derivative \( f^{(p)}(x) \).
(b) For $p = 0, 1, 2$ find a formula for the polynomial $H_p$ of degree $2p + 1$ such that

$$H_p^{(k)}(x_j) = f^{(k)}(x_j)$$

for $0 \leq k \leq p$, $0 \leq j \leq 1$, $x_0 = 0$, $x_1 = 2$.

(c) For general $p$ prove that

$$|f(x) - H_p(x)| \leq \left( \frac{1}{p+1} \right)^{2p+2}$$

for $0 \leq x \leq 2$.

(d) Show that one step of Newton’s method for solving

$$g(y) = x \ln 2 - \ln y = 0$$

starting from $y_0 = H_4(x)$ gives $y_1 = f(x) = 2^x$ to full double precision accuracy for $0 \leq x \leq 2$.

5 (BFB 4.1.13) Use the following data and the knowledge that the first five derivatives of $f$ are bounded on $[1, 5]$ by 2, 3, 6, 12 and 23 respectively, to approximate $f'(3)$ as accurately as possible. Find a bound for the error.

$$x = [1, 2, 3, 4, 5], \quad f(x) = [2.4142, 2.6734, 2.8974, 3.0976, 3.2804]$$

6 Let $n \geq m \geq 0$, $a \in \mathbb{R}$, and $n + 1$ distinct interpolation points $x_0, x_1, \ldots, x_n$. Let $\delta_{nk}^m(a)$ be the differentiation coefficients

$$\delta_{nk}^m(a) = \left( \frac{d}{dx} \right)^m L_k^n(x)|_{x=a}$$

such that the degree-$n$ polynomial $p(x)$ which interpolates $n + 1$ values $f_j$ at $n + 1$ points $x_j$ satisfies

$$p^{(m)}(a) = \sum_{k=0}^{n} \delta_{nk}^m(a)f_k.$$ 

(a) Derive the recurrence relation

$$\delta_{nk}^m(a) = \frac{m}{x_k - x_n} \delta_{n-1,k}^{m-1}(a) + \frac{a - x_n}{x_k - x_n} \delta_{n-1,k}^m(a).$$
for $0 \leq k \leq n - 1$.

(b) Write a Matlab code which evaluates $\delta^m_{nk}(a)$ for $0 \leq m \leq M$, given $n$ and the points $a$ and $x_j$.

(c) Validate your coefficients $\delta^m_{nk}(a)$ by verifying $O(h^{n-m})$ accuracy for the $m$th derivative of $f(x) = e^x$ evaluated at $n + 1$ equidistant points $x_j = jh$.

(d) Fix interpolation points $x_j$ and form an $(n + 1) \times (n + 1)$ matrix $A_m$ of differentiation coefficients with

$$(A_m)_{ij} = \delta^m_{nk}(x_i).$$

Is $A_m = A_1^m$? Why or why not?