1 (BFB 3.1.1) Let \( x_0 = 0, x_1 = 0.6, \) and \( x_2 = 0.9. \) Construct polynomials \( P_{jd} \) of degree \( d \) interpolating each \( f_j(x) \) at \( x_0 \) through \( x_d \) for

\[
\begin{align*}
f_1(x) &= \cos(x) \\
f_2(x) &= \sqrt{1 + x} \\
f_3(x) &= \ln(1 + x) \\
f_4(x) &= \tan(x)
\end{align*}
\]

and \( d = 1 \) through 2. Report the absolute error in each \( P_{jd} \) at \( x = 0.45. \)

2 Approximate \( \sqrt{2} \) by the quartic polynomial \( P(x) \) which interpolates a function \( f(x) \) at five points \( p = [x_0, x_1, x_2, x_3, x_4]. \) Use

(a) \( f(x) = 2^x, \) \( p = [-2, -1, 0, 1, 2], \) and

(b) \( f(x) = \sqrt{x}, \) \( p = [0, 1, 3, 4, 5]. \)

Compare the accuracy of the results in (a) and (b) with the error estimate for polynomial interpolation.

3 (BFB 3.3.20) (a) Show that the interpolating polynomials

\[
P(x) = 3 - 2(x + 1) + 0(x + 1)(x) + (x + 1)(x)(x - 1)
\]

and

\[
Q(x) = -1 + 4(x + 2) - 3(x + 2)(x) + (x + 2)(x)(x + 1)(x - 1)
\]

both interpolate the data \( f(x) = [-1, 3, 1, -1, 3,] \) at \( x = [-2, -1, 0, 1, 2]. \)

(b) Why does part (a) not violate the uniqueness property of interpolating polynomials?

4 (BFB 3.3.21) Given

\[
P(x) = f[x_0] + f[x_0, x_1](x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \cdots
\]

use \( P_n(x_2) \) to show that \( a_2 = f[x_0, x_1, x_2]. \)

5 Let \( \lambda_k \) be \( n + 1 \) distinct real numbers. Let \( t_j \) be \( n + 1 \) distinct real numbers.

(a) Show that

\[
a(t) = \sum_{k=0}^{n} a_k e^{\lambda_k t}
\]

can vanish for all real \( t \) only if \( a_0 = a_1 = \cdots = a_n = 0. \)

(b) Show that for the exponential interpolation problem

\[
a(t_j) = \sum_{k=0}^{n} a_k e^{\lambda_k t_j} = f_j \quad 0 \leq j \leq n
\]

there exists a unique solution \( a(t) \) for any data values \( f_j. \)

(c) For equally spaced \( \lambda_k = -k/n, \) find an explicit formula and an error estimate for \( a(t). \)

(d) Interpolate the function

\[
f(t) = \frac{1}{1 + t^6}
\]

at \( n + 1 \) equidistant points on \([0, 5]\) by your formula from (c) and tabulate the error for \( n = 3, 5, 9, 17, 33. \)
6 For equidistant points $x_j = j, 0 \leq j \leq n$, $n$ even, let

$$\omega(x) = (x - x_0)(x - x_1) \ldots (x - x_n)$$

Use Stirling’s formula to estimate the ratio $\omega(1/2)/\omega(n/2 + 1/2)$ for large $n$. Define and explain the Runge phenomenon.

7 Interpolate the function

$$f(x) = \frac{1}{1 + x^6}$$
on the interval $[0, 5]$ at

(a) $n + 1$ equidistant points $x_k = 5k/n$,
(b) $n + 1$ Chebyshev points $x_k = (5 + 5 \cos((2k + 1)\pi/(2n + 2))) / 2$, and
(c) geometrically distributed points with $x_0 = 0, x_1 = 5, x_{j+1} = (x_j + x_{j-1})/2$. Use $n = 3, 5, 9, 17, 33$ points. For each case

(1) tabulate the maximum error over 1000 random points $y_k \in [0, 5],$
(2) plot $\ln(1 + |\omega(x)|) = \ln(1 + |(x - x_0)(x - x_1) \ldots (x - x_n)|)$, and
(3) identify subregions of the intervals $[0, 5]$ and $[-1, 6]$ where $|\omega(x)|$ is exceptionally large.