1. (BFB 2.2.1) Let \( f(x) = x^4 + 2x^2 - x - 3 \). Show that each of the following functions has a fixed point at \( p > 0 \) if and only if \( f(p) = 0 \):

   (a) \( g_1(x) = (3 + x - 2x^2)^{1/4} \)
   
   (b) \( g_2(x) = \left( \frac{x+3-x^2}{2} \right)^{1/2} \)
   
   (c) \( g_3(x) = \left( \frac{x+3}{x^2+2} \right)^{1/2} \)
   
   (d) \( g_4(x) = \frac{3x^4+2x^2+3}{4x^3+4x-1} \)

2. Consider the fixed point iteration

   \[ x_{n+1} = \frac{-x_n^2 - c}{2b}, \] (*)&

where \( b \) and \( c \) are fixed real parameters.

   (a) If \( x_n \rightarrow x \), what does \( x \) solve?

   (b) Analyze and sketch the region of \((b, c)\) values where (*)& converges at a rate \( O(2^{-n}) \) or better from an interval of starting values \( x_0 \) near \( x \).

3. Consider the fixed point iteration

   \[ x_{n+1} = -b - c \frac{x_n}{x_n^2} = g(x_n) \] (**)

   (a) Show that \(|g'(x)| \leq \frac{1}{2} \) whenever \( x^2 \geq 2|c| \).

   (b) Show that \( g(x)^2 \geq 2|c| \) whenever \( x^2 \geq 2|c| \) and \( b^2 \geq \frac{9}{2}|c| \).

   (c) Draw the region of \((b, c)\)-space where (**) converges at a rate at least \( O(2^{-n}) \) from any starting point \( x_0 \) with \( x_0^2 \geq 2|c| \).

4. (BFB 2.3.31) Derive the formula for Newton’s method from the graphical description: \( p_n \) is the \( x \)-intercept of the line tangent to the graph of \( f \) at the point \((p_{n-1}, f(p_{n-1}))\).

5. Fix \( a > 0 \) and consider the fixed point iteration

   \[ x_{n+1} = x_n(2 - ax_n). \] (***)

   (a) Show that if \( x_n \rightarrow x \) then \( x = \frac{1}{a} \) or \( x = 0 \).

   (b) Find an interval \((\alpha, \beta)\) containing \( \frac{1}{a} \) such that (***)) converges to \( 1/a \) whenever \( x_0 \in (\alpha, \beta) \).

   (c) Find the rate of convergence in (b).
6 (BFB 2.4.13) Given $f$, let

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{f''(x)}{2f'(x)} \left( \frac{f(x)}{f'(x)} \right)^2.$$  

If $f(p) = 0 \neq f'(p)$, then $g'(p) = g''(p) = 0$. Thus the iterative method $q_n = g(q_{n-1})$ will generally yield cubic ($\alpha = 3$) convergence. Extend Example 1 of Section 2.3 to compare quadratic and cubic convergence.

7 (a) Write down Newton’s method in the form

$$x_{k+1} = g(x_k)$$

for solving

$$f(x) = x^2 - 2bx + b^2 - d^2 = 0$$

where $b > 0$ and $d > 0$ are parameters. (b) Show that $|g'(x)| \leq 1/2$ whenever $|x - b| \geq d/\sqrt{2}$. (c) Show that $|g(x) - b| \geq d/\sqrt{2}$ whenever $|x - b| \geq d/\sqrt{2}$. (d) Sketch the graph of $f(x)$ with the roots of $f(x) = 0$ and the intervals of $x$ where Newton’s method is guaranteed to converge.

8 Implement Newton’s method in a Matlab or Octave program `newton.m` of the form

```matlab
function r = newton(x0, f, p, n)
    % x0: initial estimate of the root
    % f: function and derivative handle [ y, yp ] = f(x, p)
    % p: parameters to pass through to f
    % n: number of steps

    Use `newton.m` to solve the equation

    $$f(x) = \frac{1}{x} + \ln x - 2 = 0$$

    for $x > 0$. Characterize the convergence as linear or quadratic by tabulating the number of correct bits at each step of the iteration.

9 Use `newton.m` to solve the equation

$$f(x) = (x - 0.111)^3 = 0$$

Characterize the convergence as linear or quadratic by tabulating the number of correct bits at each step of the iteration. Explain your results.

10 Use `newton.m` to solve the equation

$$f(x) = \arctan(x - 0.111) = 0$$

for a diverse selection of starting values. Find starting values which lead to convergence, divergence and oscillation.