1 (BFB 1.3.3) The Maclaurin series

\[ P_n(x) = \sum_{i=1}^{n} \frac{(-1)^{i+1} x^{2i-1}}{2i-1} \]

for the arctangent function converges for \(-1 < x \leq 1.

a. Use the fact that \( \tan \frac{\pi}{4} = 1 \) to determine the number \( n \) of terms of the series that need to be summed to ensure that \(|4P_n(1) - \pi| < 10^{-3}\).

b. Suppose the C++ programming language requires the value of \( \pi \) to be within \(10^{-10}\). How many terms of the series would we need to sum to obtain this degree of accuracy?

2 (BFB 1.3.5) Another formula for computing \( \pi \) can be deduced from the identity \( \frac{\pi}{4} = 4 \arctan(\frac{1}{5}) - \arctan(\frac{1}{239}) \). Determine the number of terms that must be summed to ensure an approximation to \( \pi \) to within \(10^{-3}\).

3 (BFB 2.1.19) Let \( \{p_n\} \) be the sequence defined by

\[ p_n = \sum_{k=1}^{n} \frac{1}{k} \]

Show that \( \{p_n\} \) diverges even though \( \lim_{n \to \infty} (p_n - p_{n-1}) = 0 \).

4 Fix integer \( n \geq 1 \), \( n \) points \( x_i \) with \( |x_i| \leq 1 \), \( n \) points \( y_j \) with \( |y_j| \leq 1 \), \( n \) coefficients \( f_j \), and \( n \) coefficients \( g_j \).

(a) Fix integer \( k \geq 0 \). Design an algorithm for evaluating

\[ f(x) = \sum_{j=1}^{n} f_j(x y_j)^k \]

at \( n \) points \( x_i \), in \( O(n) \) operations.

(b) Find a degree-8 polynomial \( P(x) \) with

\[ |P(x) - e^{ix}| \leq 10^{-5} \]

on the interval \( |x| \leq 1 \).

(c) Design an algorithm for approximating

\[ g(x) = \sum_{j=1}^{n} g_j e^{ix y_j} \]

at \( n \) points \( x_i \) in \( O(n) \) operations, with absolute error bounded by

\[ 10^{-5} \sum_{j=1}^{n} |g_j| \].
5 Design an algorithm to evaluate
\[ f(x) = \frac{e^x - 1 - x}{x^2} \]
in IEEE double precision arithmetic, to 12-digit accuracy for all machine numbers \(|x| \leq 1\).

6 The Fibonacci numbers \( f_n \) are defined by
\[ f_{n+2} = f_{n+1} + f_n \]
with \( f_0 = 0 \) and \( f_1 = 1 \). (a) Show that
\[ \frac{f_{n+1}}{f_n} \to \varphi = \frac{1 + \sqrt{5}}{2} \]
as \( n \to \infty \). (b) Determine the rate of convergence of \( f_{n+1}/f_n \) to \( \varphi \).

7 Figure out exactly what sequence of intervals is produced by bisection for solving \( x = 0 \) with initial interval \([a_0, b_0] = [-1, 2]\). How many steps will it take to get maximum accuracy in IEEE standard floating point arithmetic?

8 Implement a MATLAB function `bisection.m` of the form

```matlab
function [r, h] = bisection(a, b, f, p, t)
% a: Beginning of interval [a, b]
% b: End of interval [a, b]
% f: function handle y = f(x, p)
% p: parameters to pass through to f
% t: User-provided tolerance for interval width

At each step \( j = 1 \) to \( n \), carefully choose \( m \) as in geometric mean bisection (watch out for zeroes!). Replace \([a, b]\) by the smallest interval with endpoints chosen from \( a, m, b \) which keeps the root bracketed. Repeat until a \( f \) value exactly vanishes, \( b - a \leq t \min(|a|, |b|) \), or \( b \) and \( a \) are adjacent floating point numbers, whichever comes first. Return the final approximation to the root \( r \) and a \( 3 \times n \) history matrix \( h[1:3, 1:n] \) with column \( h[1:3, j] = (a, b, f(m)) \) recorded at step \( j \). Try to make your implementation as foolproof as possible.

9 (BFB 2.1.7) a. Sketch the graphs of \( y = x \) and \( y = 2 \sin x \). b. Use `bisection.m` to find an approximation to within \( 10^{-5} \) to the first positive value of \( x \) with \( x = 2 \sin x \). Report the number of steps, the final result, and the absolute and relative errors.

10 Use `bisection.m` to solve the equation
\[ f(x) = \frac{1}{x} + \ln x - 2 = 0 \]
for \( x > 0 \). Report the number of steps, the final result, and the absolute and relative errors.
11 Use \texttt{bisection.m} to solve the equation 

\[ f(x) = (x - 0.111)^3 = 0 \]

on the interval \([-1, 2]\). Report the number of steps, the final result, and the absolute and relative errors.

12 Use \texttt{bisection.m} to solve the equation 

\[ f(x) = \arctan((x - 0.111)) = 0 \]

on the interval \([-1, 2]\). Report the number of steps, the final result, and the absolute and relative errors.