1  (BFB 1.3.3) The Maclaurin series

\[ P_n(x) = \sum_{i=1}^{n} (-1)^{i+1} \frac{x^{2i-1}}{2i - 1} \]

for the arctangent function converges for \(-1 < x \leq 1\).

a. Use the fact that \(\tan \pi/4 = 1\) to determine the number \(n\) of terms of the series that need to be summed to ensure that \(|4P_n(1) - \pi| < 10^{-3}\).

b. Suppose the C++ programming language requires the value of \(\pi\) to be within \(10^{-10}\). How many terms of the series would we need to sum to obtain this degree of accuracy?

2  (BFB 1.3.5) Another formula for computing \(\pi\) can be deduced from the identity \(\pi/4 = 4 \arctan(1/5) - \arctan(1/239)\). Determine the number of terms that must be summed to ensure an approximation to \(\pi\) to within \(10^{-3}\).

3  (BFB 2.1.19) Let \(\{p_n\}\) be the sequence defined by

\[ p_n = \sum_{k=1}^{n} \frac{1}{k}. \]

Show that \(\{p_n\}\) diverges even though \(\lim_{n \to \infty} (p_n - p_{n-1}) = 0\).

4  Fix integer \(n \geq 1\), \(n\) points \(x_i\) with \(|x_i| \leq 1\), \(n\) points \(y_j\) with \(|y_j| \leq 1\), \(n\) coefficients \(f_j\), and \(n\) coefficients \(g_j\).

(a) Fix integer \(k \geq 0\). Design an algorithm for evaluating

\[ f(x) = \sum_{j=1}^{n} f_j(xy_j)^k \]

at \(n\) points \(x_i\), in \(O(n)\) operations.

(b) Find a degree-8 polynomial \(P(x)\) with

\[ |P(x) - e^{ix}| \leq 10^{-6} \]

on the interval \(|x| \leq 1\).

(c) Design an algorithm for approximating

\[ g(x) = \sum_{j=1}^{n} g_j e^{ixy_j} \]

at \(n\) points \(x_i\) in \(O(n)\) operations, with absolute error bounded by

\[ 10^{-6} \sum_{j=1}^{n} |g_j|. \]
5 Design an algorithm to evaluate
\[ f(x) = \frac{e^x - 1 - x}{x^2} \]
in IEEE double precision arithmetic, to 12-digit accuracy for all machine numbers \(|x| \leq 1\).

6 The Fibonacci numbers \(f_n\) are defined by
\[ f_{n+2} = f_{n+1} + f_n \]
with \(f_0 = 0\) and \(f_1 = 1\). (a) Show that
\[ \frac{f_{n+1}}{f_n} \to \varphi = \frac{1 + \sqrt{5}}{2} \]
as \(n \to \infty\). (b) Determine the rate of convergence of \(f_{n+1}/f_n\) to \(\varphi\).

7 Figure out exactly what sequence of intervals is produced by bisection for solving \(x = 0\) with initial interval \([a_0, b_0] = [-1, 2]\). How many steps will it take to get maximum accuracy in IEEE standard floating point arithmetic?

8 Implement a MATLAB function `bisection.m` of the form

```matlab
function [r, h] = bisection(a, b, f, p, t)
% a: Beginning of interval [a, b]
% b: End of interval [a, b]
% f: function handle y = f(x, p)
% p: parameters to pass through to f
% t: User-provided tolerance for interval width

At each step \(j = 1\) to \(n\), carefully choose \(m\) as in geometric mean bisection (watch out for zeroes!). Replace \([a, b]\) by the smallest interval with endpoints chosen from \(a, m, b\) which keeps the root bracketed. Repeat until a \(f\) value exactly vanishes, \(b - a \leq t \min(|a|, |b|)\), or \(b\) and \(a\) are adjacent floating point numbers, whichever comes first. Return the final approximation to the root \(r\) and a \(3 \times n\) history matrix \(h[1:3, 1:n]\) with column \(h[1:3, j] = (a, b, f(m))\) recorded at step \(j\). Try to make your implementation as foolproof as possible.

9 (BFB 2.1.7) a. Sketch the graphs of \(y = x\) and \(y = 2 \sin x\). b. Use `bisection.m` to find an approximation to within \(10^{-5}\) to the first positive value of \(x\) with \(x = 2 \sin x\). Report the number of steps, the final result, and the absolute and relative errors.

10 Use `bisection.m` to solve the equation
\[ f(x) = \frac{1}{x} + \ln x - 2 = 0 \]
for \(x > 0\). Report the number of steps, the final result, and the absolute and relative errors.
11  Use \texttt{bisection.m} to solve the equation

\[ f(x) = (x - 0.111)^3 = 0 \]

on the interval \([-1, 2]\). Report the number of steps, the final result, and the absolute and relative errors.

12  Use \texttt{bisection.m} to solve the equation

\[ f(x) = \arctan((x - 0.111)) = 0 \]

on the interval \([-1, 2]\). Report the number of steps, the final result, and the absolute and relative errors.