Euler-Maclaurin formula Integrate by parts as in Taylor expansion:

\[
\int_0^1 f(x)dx = \int_0^1 \frac{d}{dx}(x-1/2)f(x)dx \\
= (1/2)(f(0) + f(1)) - \int_0^1 (x-1/2)f'(x)dx \\
= (1/2)(f(0) + f(1)) - \int_0^1 \frac{1}{2}(x-1/2)^2f'(x)dx \\
= (1/2)(f(0) + f(1)) - \frac{1}{2}(1/2)^2(f'(1) - f'(0)) + \int_0^1 \frac{1}{3!}(x-1/2)^3f''(x)dx \\
= (1/2)(f(0) + f(1)) - \frac{1}{2!(1/2)^2}(f'(1) - f'(0)) + \frac{1}{3!}(1/2)^3(f''(1) + f''(0)) - \int_0^1 \frac{1}{dx^4!}(x-1/2)^4f'''(x)dx
\]

and so forth. Rearrange to get an error formula for the trapezoidal rule:

\[
(1/2)(f(0) + f(1)) = \int_0^1 f(x)dx + \frac{1}{2!}(1/2)^2(f'(1) - f'(0)) - \frac{1}{3!}(1/2)^3(f''(1) + f''(0)) + \int_0^1 \frac{1}{dx^4!}(x-1/2)^4f'''(x)dx.
\]

Apply the formula to \(f''\) in place of \(f\):

\[
(1/2)(f''(0) + f''(1)) = \int_0^1 f''(x)dx + \frac{1}{2!}(1/2)^2(f'''(1) - f'''(0)) - \frac{1}{3!}(1/2)^3(f''''(1) + f''''(0)) + \cdots
\]

\[
= f'(1) - f'(0) + \frac{1}{2!}(1/2)^2(f'''(1) - f'''(0)) - \frac{1}{3!}(1/2)^3(f''''(1) + f''''(0)) + \int_0^1 \frac{1}{dx^4!}(x-1/2)^4f''''(x)dx.
\]

Key step: Use the result to eliminate the term involving \(f''(1) + f''(0)\) from the previous formula:

\[
(1/2)(f(0) + f(1)) = \int_0^1 f(x)dx + \frac{1}{2!}(1/2)^2(f'(1) - f'(0)) + \frac{1}{2!}(1/2)^2(f'''(1) - f'''(0)) + \int_0^1 \frac{1}{dx^4!}(x-1/2)^4f''''(x)dx.
\]

Now imagine repeating the elimination infinitely often. The result would be to eliminate all the terms with plus signs between even derivatives of \(f\) and leave an infinite series of the form

\[
(1/2)(f(0) + f(1)) = \int_0^1 f(x)dx + b_1(f'(1) - f'(0)) + b_2(f''(1) - f''(0)) + b_3(f'''(1) - f'''(0)) + \cdots
\]

with some unknown constants \(b_1 = 1/12, b_2, b_3, \ldots\), multiplying differences of odd-numbered derivatives of \(f\). The Euler-Maclaurin summation formula follows by compounding:

\[
\frac{1}{2}f(0) + f(1) + f(2) + \cdots + f(n-1) + \frac{1}{2}f(n) = \int_0^n f(x)dx + b_1(f'(n) - f'(0)) + b_2(f''(n) - f''(0)) + \cdots
\]

because the differences of derivatives all telescope, canceling the interior terms. Conclusion: The error in the trapezoidal rule depends only on the derivatives of the integrand at the endpoints of the domain of integration. For example, the trapezoidal rule integrates a smooth periodic function over a full period with great accuracy.

It follows that the order of accuracy (degree of precision) of the trapezoidal rule can be increased by endpoint corrections which change the weights only near the endpoints of the interval. Such corrections can be derived by coupling the Euler-Maclaurin formula with finite difference approximations to the derivatives, or by polynomial interpolation as follows.

Let’s use cubic interpolation to derive a fourth-order endpoint corrected trapezoidal rule. To do this, we interpolate four successive function values \(f_0, f_1, f_2, f_3\) to integrate over the interval \([1, 2]\). Since the Lagrange basis functions are

\[
L_0(x) = (x - 1)(x - 2)(x - 3)/(0 - 1)(0 - 2)(0 - 3) = (x - 1)(x - 2)(x - 3)/(-6)
\]
\[ L_1(x) = \frac{(x - 0)(x - 2)(x - 3)}{(1 - 0)(1 - 2)(1 - 3)} = \frac{(x - 0)(x - 2)(x - 3)}{2} \]
\[ L_2(x) = \frac{(x - 0)(x - 1)(x - 3)}{(2 - 0)(2 - 1)(2 - 3)} = \frac{(x - 0)(x - 1)(x - 3)}{(-2)} \]
\[ L_3(x) = \frac{(x - 0)(x - 1)(x - 2)}{(3 - 0)(3 - 1)(3 - 2)} = \frac{(x - 0)(x - 1)(x - 2)}{6} \]

The resulting rule is
\[
\int_1^2 f(x)dx = w_0 f(0) + w_1 f(1) + w_2 f(2) + w_3 f(3)
\]
where
\[
w_0 = \int_1^2 L_0(x)dx = -\frac{1}{24} = w_3
\]
and
\[
w_1 = \int_1^2 L_1(x)dx = \frac{13}{24} = w_2.
\]

At the end intervals such as [0, 1] we do not have \( f(-1) \) so we drop to quadratic interpolation with
\[
L_0(x) = \frac{(x - 1)(x - 2)}{(0 - 1)(0 - 2)} = \frac{(x - 1)(x - 2)}{2}
\]
\[
L_1(x) = \frac{(x - 0)(x - 2)}{(1 - 0)(1 - 2)} = \frac{(x - 0)(x - 2)}{(-1)}
\]
\[
L_2(x) = \frac{(x - 0)(x - 1)}{(2 - 0)(2 - 1)} = \frac{(x - 0)(x - 1)}{2}
\]

The resulting rule is
\[
\int_0^1 f(x)dx = w_0 f(0) + w_1 f(1) + w_2 f(2)
\]
where
\[
w_0 = \frac{10}{24}, \quad w_1 = \frac{16}{24}, \quad w_2 = \frac{-2}{24}.
\]

Putting it all together gives the fourth-order endpoint corrected trapezoidal rule
\[
\int_0^1 f(x)dx = \frac{h}{24} \left( 9f(0) + 23f(h) + 28f(2h) + 24f(3h) + 24f(4h) + \cdots + 9f(nh) \right)
\]