1. (BFB 1.3.3) The Maclaurin series

\[ P_n(x) = \sum_{i=1}^{n} (-1)^{i+1} \frac{x^{2i-1}}{2i - 1} \]

for the arctangent function converges for \(-1 < x \leq 1\).

a. Use the fact that \(\tan \frac{\pi}{4} = 1\) to determine the number \(n\) of terms of the series that need to be summed to ensure that \(|4P_n(1) - \pi| < 10^{-3}\).

b. Suppose the C++ programming language requires the value of \(\pi\) to be within \(10^{-10}\). How many terms of the series would we need to sum to obtain this degree of accuracy?

2. (BFB 1.3.5) Another formula for computing \(\pi\) can be deduced from the identity \(\pi/4 = 4 \arctan(1/5) - \arctan(1/239)\). Determine the number of terms that must be summed to ensure an approximation to \(\pi\) to within \(10^{-3}\).

3. (BFB 2.1.19) Let \(\{p_n\}\) be the sequence defined by

\[ p_n = \sum_{k=1}^{n} \frac{1}{k}. \]

Show that \(\{p_n\}\) diverges even though \(\lim_{n \to \infty} (p_n - p_{n-1}) = 0\).

4. Fix integer \(n \geq 1\), \(n\) points \(x_i\) with \(|x_i| \leq 1\), \(n\) points \(y_j\) with \(|y_j| \leq 1\), \(n\) coefficients \(f_j\), and \(n\) coefficients \(g_j\).

a. Fix integer \(k \geq 0\). Design an algorithm for evaluating

\[ f(x) = \sum_{j=1}^{n} f_j (xy_j)^k \]

at \(n\) points \(x_i\), in \(O(n)\) operations.

b. Find a degree-8 polynomial \(P(x)\) with

\[ |P(x) - \cos(x)| \leq 10^{-6} \]

on the interval \(|x| \leq 1\).

c. Design an algorithm for approximating

\[ g(x) = \sum_{j=1}^{n} g_j \cos(xy_j) \]

at \(n\) points \(x_i\) in \(O(n)\) operations, with absolute error bounded by

\[ 10^{-6} \sum_{j=1}^{n} |g_j|. \]
5 Design an algorithm to evaluate

\[ f(x) = \frac{e^x - 1 - x}{x^2} \]

in IEEE double precision arithmetic, to 12-digit accuracy for all machine numbers \(|x| \leq 1\).

6 The Fibonacci numbers \(f_n\) are defined by

\[ f_{n+2} = f_{n+1} + f_n \]

with \(f_0 = 0\) and \(f_1 = 1\). (a) Show that

\[ \frac{f_{n+1}}{f_n} \to \varphi = \frac{1 + \sqrt{5}}{2} \]

as \(n \to \infty\). (b) Determine the rate of convergence of \(f_{n+1}/f_n\) to \(\varphi\).

7 Figure out exactly what sequence of intervals is produced by bisection for solving \(x = 0\) with initial interval \([a_0, b_0] = [-1, 2]\). How many steps will it take to get maximum accuracy in IEEE standard floating point arithmetic?

8 Implement a MATLAB function \texttt{bisection.m} of the form

\begin{verbatim}
function [r, h] = bisection(a, b, f, p, t)
    % a: Beginning of interval [a, b]
    % b: End of interval [a, b]
    % f: function handle y = f(x, p)
    % p: parameters to pass through to f
    % t: User-provided tolerance for interval width
    At each step \(j = 1\) to \(n\), carefully choose \(m\) as in geometric mean bisection (watch out for zeroes!). Replace \([a, b]\) by the smallest interval with endpoints chosen from \(a, m, b\) which keeps the root bracketed. Repeat until a \(f\) value exactly vanishes, \(b - a \leq t \max(|a|, |b|)\), or \(b\) and \(a\) are adjacent floating point numbers, whichever comes first. Return the final approximation to the root \(r\) and a \(3 \times n\) history matrix \(h[1:3,1:n]\) with column \(h[1:3,j] = (a, b, f(m))\) recorded at step \(j\). Try to make your implementation as foolproof as possible.
\end{verbatim}

9 (BFB 2.1.7) a. Sketch the graphs of \(y = x\) and \(y = 2 \sin x\). b. Use \texttt{bisection.m} to find an approximation to within \(10^{-5}\) to the first positive value of \(x\) with \(x = 2 \sin x\). Report the number of steps, the final result, and the absolute and relative errors.

10 Use \texttt{bisection.m} to solve the equation

\[ f(x) = \frac{1}{x} + \ln x - 2 = 0 \]

for \(x > 0\). Report the number of steps, the final result, and the absolute and relative errors.
11 Use `bisection.m` to solve the equation

\[ f(x) = x^3 = 0 \]

on the interval \([-1, 2]\). Report the number of steps, the final result, and the absolute and relative errors.

12 Use `bisection.m` to solve the equation

\[ f(x) = \arctan(x) = 0 \]

on the interval \([-1, 2]\). Report the number of steps, the final result, and the absolute and relative errors.