1 (BFB 1.3.3) The Maclaurin series
\[ P_n(x) = \sum_{i=1}^{n} \frac{(-1)^{i+1} x^{2i-1}}{2i-1} \]
for the arctangent function converges for \(-1 < x \leq 1\).

a. Use the fact that \(\tan \pi/4 = 1\) to determine the number \(n\) of terms of the series that need to be summed to ensure that \(|4P_n(1) - \pi| < 10^{-3}\).

b. Suppose the C++ programming language requires the value of \(\pi\) to be within \(10^{-10}\). How many terms of the series would we need to sum to obtain this degree of accuracy?

2 (BFB 1.3.5) Another formula for computing \(\pi\) can be deduced from the identity \(\pi/4 = 4 \arctan(1/15) - \arctan(1/239)\). Determine the number of terms that must be summed to ensure an approximation to \(\pi\) to within \(10^{-3}\).

3 (BFB 2.1.19) Let \(\{p_n\}\) be the sequence defined by
\[ p_n = \sum_{k=1}^{n} \frac{1}{k}. \]
Show that \(\{p_n\}\) diverges even though \(\lim_{n \to \infty} (p_n - p_{n-1}) = 0\).

4 Fix integer \(n \geq 1\), \(n\) points \(x_i\) with \(|x_i| \leq 1\), \(n\) points \(y_j\) with \(|y_j| \leq 1\), \(n\) coefficients \(f_j\), and \(n\) coefficients \(g_j\).

(a) Fix integer \(k \geq 0\). Design an algorithm for evaluating
\[ f(x) = \sum_{j=1}^{n} f_j (x y_j)^k \]
at \(n\) points \(x_i\), in \(O(n)\) operations.

(b) Find a degree-8 polynomial \(P(x)\) with
\[ |P(x) - \cos(x)| \leq 10^{-6} \]
on the interval \(|x| \leq 1\).

(c) Design an algorithm for approximating
\[ g(x) = \sum_{j=1}^{n} g_j \cos(x y_j) \]
at \(n\) points \(x_i\) in \(O(n)\) operations, with absolute error bounded by
\[ 10^{-6} \sum_{j=1}^{n} |g_j|. \]
5 Design an algorithm to evaluate
\[ f(x) = \frac{e^x - 1 - x}{x^2} \]
in IEEE double precision arithmetic, to 12-digit accuracy for all machine numbers \(|x| \leq 1\).

6 The Fibonacci numbers \(f_n\) are defined by
\[ f_{n+2} = f_{n+1} + f_n \]
with \(f_0 = 0\) and \(f_1 = 1\). (a) Show that
\[ \frac{f_{n+1}}{f_n} \to \phi = \frac{1 + \sqrt{5}}{2} \]
as \(n \to \infty\). (b) Determine the rate of convergence of \(f_{n+1}/f_n\) to \(\phi\).

7 Figure out exactly what sequence of intervals is produced by bisection for solving \(x = 0\) with initial interval \([a_0, b_0] = [-1, 2]\). How many steps will it take to get maximum accuracy in IEEE standard floating point arithmetic?

8 Implement a MATLAB function bisection.m of the form
\[
\text{function } [r, h] = \text{bisection}(a, b, f, p, t) \\
\text{\% a: Beginning of interval [a, b]} \\
\text{\% b: End of interval [a, b]} \\
\text{\% f: function handle y = f(x, p)} \\
\text{\% p: parameters to pass through to f} \\
\text{\% t: User-provided tolerance for interval width}
\]
At each step \(j = 1\) to \(n\), carefully choose \(m\) as in geometric mean bisection (watch out for zeroes!). Replace \([a, b]\) by the smallest interval with endpoints chosen from \(a, m, b\) which keeps the root bracketed. Repeat until a \(f\) value exactly vanishes, \(b - a \leq t\max(|a|, |b|),\) or \(b\) and \(a\) are adjacent floating point numbers, whichever comes first. Return the final approximation to the root \(r\) and a \(3 \times n\) history matrix \(h[1:3,1:n]\) with column \(h[1:3,j] = (a, b, f(m))\) recorded at step \(j\). Try to make your implementation as foolproof as possible.

9 (BFB 2.1.7) a. Sketch the graphs of \(y = x\) and \(y = 2 \sin x\). b. Use bisection.m to find an approximation to within \(10^{-5}\) to the first positive value of \(x\) with \(x = 2 \sin x\). Report the number of steps, the final result, and the absolute and relative errors.

10 Use bisection.m to solve the equation
\[ f(x) = \frac{1}{x} + \ln x - 2 = 0 \]
for \(x > 0\). Report the number of steps, the final result, and the absolute and relative errors.
11 Use bi\text{\texttt{section.m}} to solve the equation

\[ f(x) = x^3 = 0 \]

on the interval \([-1, 2]\). Report the number of steps, the final result, and the absolute and relative errors.

12 Use bi\text{\texttt{section.m}} to solve the equation

\[ f(x) = \arctan(x) = 0 \]

on the interval \([-1, 2]\). Report the number of steps, the final result, and the absolute and relative errors.