1A Let \( x_n \) be a sequence of real numbers defined by \( x_0 = 5 \) and
\[
x_{n+1} = 2 + \frac{1}{x_n} = g(x_n).
\]
Assume \( x_n \to x \) for some \( x \geq 2 \) as \( n \to \infty \). Show that
\[
|x_{n+1} - x| \leq \frac{1}{4} |x_n - x|
\]
for all \( n \).

Solution: If \( x_n \to x \) then by continuity of \( g \)
\[
x = 2 + \frac{1}{x}
\]
so subtraction gives
\[
x_{n+1} - x = \frac{1}{x_n} - \frac{1}{x} = \frac{x - x_n}{xx_n}.
\]
Since \( x_0 = 5 \geq 2 \geq 0 \),
\[
x_{n+1} = 2 + \frac{1}{x_n} \geq 2
\]
by induction and
\[
|x_{n+1} - x| \leq \frac{1}{4} |x - x_n|.
\]
1B In floating point arithmetic, $x_n$ is approximated by $y_n$ satisfying

$$y_{n+1} = \text{fl}(x_{n+1}) = \left(2 + \frac{1}{y_n}(1 + \delta_n)\right)(1 + \delta'_n)$$

where $|\delta_n| \leq \epsilon$ and $|\delta'_n| \leq \epsilon$.

(a) Show that

$$|y_{n+1} - x| \leq \frac{1}{4}|y_n - x| + 3\epsilon + O(\epsilon^2)$$

for all $n$, and

(b) describe the behavior of $y_n$ as $n \to \infty$.

Solution:

(a) As in (1A), subtraction gives

$$y_{n+1} - x = \frac{x - y_n}{xy_n} + 2\delta'_n + \frac{2}{y_n}\delta_n + O(\epsilon^2).$$

Since $y_n \geq 2 \geq 0$, we have $1/y_n \leq 1/2$. Thus

$$|y_{n+1} - x| \leq \frac{1}{4}|x - y_n| + 2\epsilon + \frac{2}{y_n}\epsilon + O(\epsilon^2) \leq \frac{1}{4}|x - y_n| + 3\epsilon + O(\epsilon^2).$$

(b) As $n \to \infty$, $y_n$ converges to within $4\epsilon$ of $x$ and then bounces around:

$$|y_n - x| \leq a\epsilon$$

implies

$$|y_{n+1} - x| \leq \left(\frac{1}{4}a + 3\right)\epsilon = a\epsilon$$

if $a = 4$. 
2A Let $H(x)$ be the cubic polynomial interpolating $f(0)$, $f'(0)$, $f(1)$, and $f'(1)$.

(a) Give a formula for the error $f(x) - H(x)$ which includes the $p$th-order derivative $f^{(p)}(\xi)$, evaluated at an unknown point $\xi$.

(b) Specify the value of $p$ and explain why it is inevitable.

**Solution:**

(a) 

$$f(x) - H(x) = C_4 f^{(4)}(\xi) (x - 0)^2(x - 1)^2$$

where

(b) $f^{(4)}$ ensures that the error vanishes whenever $f$ is a cubic polynomial (since the Hermite interpolation polynomial is unique) and is therefore inevitable.

Determine $C_4$ if necessary by testing on $f(x) = x^2(x - 1)^2$ for which $H(x) = 0$ (by uniqueness) and $f^{(4)}(\xi) = 4!$ so

$$x^2(x - 1)^2 - 0 = C_4 4! x^2(x - 1)^2$$

and $C_4 = 1/4!$. 
2B For the specific function \( f(x) = x^4 \),
(a) build the divided difference table,
(b) find the Newton form of \( H(x) \),
(c) evaluate \( H(1/2) \), and
(d) show that your error formula from (2A) is satisfied at \( x = 1/2 \).

Solution:
(a) The difference table is constructed with \( f'(x_j) = 4x^3_j \) in place of \( f[x_j, x_{j+1}] \) whenever \( x_{j+1} = x_j \):

\[
\begin{array}{c|c|cccc}
 j & x_j & f[x_j] & f[x_j, x_{j+1}] & f[x_j, x_{j+1}, x_{j+2}] & f[x_j, x_{j+1}, x_{j+2}, x_{j+3}] \\
 0 & 0 & 0 & 0' & 1 & 2 \\
 1 & 0 & 0 & 1 & 3 \\
 2 & 1 & 1 & 4' \\
 3 & 1 & 1 \\
\end{array}
\]

(b) Thus reading along the top row,
\[
H(x) = 0 + 0(x - 0) + 1(x - 0)^2 + 2(x - 0)^2(x - 1)
\]

and
(c) \( H(1/2) = (1/2)^2 + 2(1/2)^2(-1/2) = 0 \).

(d) Since \( f^{(4)}(\xi) = 4! \), the error formula
\[
f(1/2) - H(1/2) = \frac{1}{16} - 0 = \frac{1}{4!} 4!(1/2)^4 = \frac{1}{16}
\]
is satisfied.
3A
(a) Find constants \(a, b\) and \(c\) such that the numerical integration rule
\[
\int_0^1 f(t) \, dt = af(-1) + bf(0) + cf(1)
\]
is exact whenever \(f\) is a quadratic polynomial. (Hint: Integrate Lagrange basis polynomials or solve a linear system.)
(b) Find constants \(a', b'\) and \(c'\) such that the numerical integration rule
\[
\int_0^1 f(t) \, dt = a'f(0) + b'f(1) + c'f(2)
\]
is exact whenever \(f\) is a quadratic polynomial. (Hint: Change variables, integrate Lagrange basis polynomials, or solve a linear system.)

Solution:
(a) The Lagrange basis functions integrate to
\[
a = \int_0^1 L_{-1}(t)dt = \int_0^1 \frac{(t-0)(t-1)}{(-1-0)(-1-1)}dt = \frac{1}{2} \int_0^1 t(t-1)dt = -\frac{1}{12},
\]
\[
b = \int_0^1 L_0(t)dt = \int_0^1 \frac{(t-(1))(t-1)}{(0-(-1))(0-1)}dt = \int_0^1 1-t^2dt = \frac{2}{3},
\]
\[
c = \int_0^1 L_1(t)dt = \int_0^1 \frac{(t-(1))(t-0)}{(1-(-1))(1-0)}dt = \frac{1}{2} \int_0^1 t(t+1)dt = \frac{5}{12}.
\]
Check (or alternate solution): \(a + b + c = 1, -a + c = 1/2, a + c = 1/3\) so quadratics are exact.
(b) Change variables \(t = 1 - s\) so
\[
\int_0^1 f(t)dt = \int_0^1 f(1-s)ds = af(1-(-1)) + bf(1-0) + cf(1-1) = cf(0) + bf(1) + af(2)
\]
and
\[
a' = c, \quad b' = b, \quad c' = a.
\]
Check (or alternate solution): \(a' + b' + c' = 1, b' + 2c' = 1/2, b' + 4c' = 1/3\) so quadratics are exact.
Find weights $w_0, w_1, w_2$ which make the numerical integration rule

$$
\int_0^{Nh} f(x)dx = h\sum_{j=0}^{N-1} f(jh+th)dt = h\sum_{j=0}^{N} w_jf(jh), \quad w_3 = \cdots = w_{N-3} = 1,
$$

accurate to order $O(h^3)$.

**Solution:** Use the primed weights on the leftmost interval $j = 0$ and the unprimed weights on the rest:

$$
\int_0^{Nh} f(x)dx = h(a' f(0) + b' f(h) + c' f(2h)) + af(0) + bf(h) + cf(2h) + a f(h) + b f(2h) + cf(3h) + a f(2h) + b f(3h) + cf(4h) + \cdots
$$

so

$$
w_0 = a' + a = \frac{4}{12}, \quad w_1 = b' + b + a = \frac{15}{12}, \quad w_2 = c' + c + b + a = \frac{11}{12}.
$$

Quick check: $w_0 + w_1 + w_2 = 30/12$.

Alternate solution: Use primed weights on all the intervals except the rightmost one:

$$
w_0 = \frac{5}{12}, \quad w_1 = \frac{13}{12}, \quad w_2 = \frac{12}{12}
$$

General solution: there is a 1-parameter family of rules

$$
w_0 = \frac{5 - t}{12}, \quad w_1 = \frac{13 + 2t}{12}, \quad w_2 = \frac{12 - t}{12}
$$

which integrate piecewise quadratic polynomials exactly for any $t \in R$. These rules use primed weights for $j = 0$, unprimed weights for $j = N - 1$, and a linear combination in between. The rules derived above have $t = 1$ and $t = 0$ respectively. As a special case, we derived a symmetric rule in class with $t = 1/2$ and weights

$$
w_0 = \frac{9}{24}, \quad w_1 = \frac{28}{24}, \quad w_2 = \frac{23}{24},
$$

where the symmetry turns out to give one extra order of exactness.