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1A Let $x_n$ be a sequence of real numbers defined by $x_0 = 5$ and

$$x_{n+1} = 2 + \frac{1}{x_n} = g(x_n).$$

Assume $x_n \to x$ for some $x \geq 2$ as $n \to \infty$. Show that

$$|x_{n+1} - x| \leq \frac{1}{4}|x_n - x|$$

for all $n$. 
1B In floating point arithmetic, $x_n$ is approximated by $y_n$ satisfying

$$y_{n+1} = \text{fl}(x_{n+1}) = \left( 2 + \frac{1}{y_n} (1 + \delta_n) \right) (1 + \delta'_n)$$

where $|\delta_n| \leq \epsilon$ and $|\delta'_n| \leq \epsilon$.

(a) Show that

$$|y_{n+1} - x| \leq \frac{1}{4} |y_n - x| + 3\epsilon + O(\epsilon^2)$$

for all $n$, and

(b) describe the behavior of $y_n$ as $n \to \infty$. 


2A Let $H(x)$ be the cubic polynomial interpolating $f(0)$, $f'(0)$, $f(1)$, and $f'(1)$.

(a) Give a formula for the error $f(x) - H(x)$ which includes the $p$th-order derivative $f^{(p)}(\xi)$, evaluated at an unknown point $\xi$.

(b) Specify the value of $p$ and explain why it is inevitable.
2B For the specific function $f(x) = x^4$,
(a) build the divided difference table,
(b) find the Newton form of $H(x)$,
(c) evaluate $H(1/2)$, and
(d) show that your error formula from (2A) is satisfied at $x = 1/2$. 
3A

(a) Find constants $a$, $b$ and $c$ such that the numerical integration rule
\[ \int_0^1 f(t) \, dt = af(-1) + bf(0) + cf(1) \]
is exact whenever $f$ is a quadratic polynomial. (Hint: Integrate Lagrange basis polynomials or solve a linear system.)

(b) Find constants $a'$, $b'$ and $c'$ such that the numerical integration rule
\[ \int_0^1 f(t) \, dt = a'f(0) + b'f(1) + c'f(2) \]
is exact whenever $f$ is a quadratic polynomial. (Hint: Change variables, integrate Lagrange basis polynomials, or solve a linear system.)
3B Find weights $w_0, w_1, w_2$ which make the numerical integration rule

$$\int_0^{Nh} f(x) \, dx = \sum_{j=0}^{N-1} h \int_0^1 f(jh+th) \, dt = h \sum_{j=0}^N w_j f(jh), \quad w_3 = \cdots = w_{N-3} = 1,$$

accurate to order $O(h^3)$. 