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(1a) Write down the Newton form of the linear interpolant \( p(x) \) to a function \( f \) at two points \( a \) and \( b \).
(1b) Give a formula for the error \( f(x) - p(x) \) in (1a). Separate the error into three factors and explain why each one is inevitable.
(1c) Specialize to \( f(x) = \pi - 1/x \), keeping \( a \) and \( b \) arbitrary, and evaluate the Newton form of \( p(x) \).
(1d) Set $a = x_{n-1}$, $b = x_n$ and solve $p(x_{n+1}) = 0$ to invent a two-step iteration

$$x_{n+1} = g(x_n, x_{n-1})$$

for computing $1/\pi$ without division.
(2a) Suppose that a numerical integration rule with \( q \geq 2 \) nodes \( x_j \in [0, 1] \), \( q \) weights \( w_j \), and error

\[
E_q(f) = \int_0^1 f(x)dx - \sum_{j=1}^{q} w_j f(x_j)
\]

is exact for polynomials of degree \( d \): \( E_q(f) = 0 \) whenever \( f \) is a polynomial of degree \( d \). Assume a function \( f \) can be approximated by some unknown degree-\( d \) polynomial \( p(x) \) within absolute error \( \epsilon \):

\[
|f(x) - p(x)| \leq \epsilon
\]

for \( 0 \leq x \leq 1 \). Prove that

\[
|E_q(f)| \leq \left( 1 + \sum_{j=1}^{q} |w_j| \right) \epsilon.
\]
(2b) Suppose floating-point arithmetic evaluates \( f(x_j) \) as \( \hat{f}(x_j) \) with absolute error \( |f(x_j) - \hat{f}(x_j)| = |a_j| \leq \epsilon \). Show that

\[
|\hat{E}_q(f)| = \left| \int_0^1 f(x)dx - \sum_{j=1}^{q} w_j \hat{f}(x_j) \right| \leq \left( 1 + 2 \sum_{j=1}^{q} |w_j| \right) \epsilon.
\]
(2c) Suppose $d \geq 2q - 2$. Show that all the weights $w_j$ are positive and find

$$\sum_{j=1}^{q} |w_j|.$$
(3a) Find an upper triangular matrix $R$ such that $A = R^T R$ where

$$A = \begin{bmatrix} 36 & 0 & -6 \\ 0 & 36 & -6 \\ -6 & -6 & 38 \end{bmatrix}.$$
(3b) Is $A$ positive definite? Why or why not?
(4a) Consider the ordinary differential equation

\[ y' = f(t, y) \]

with initial condition \( y(0) = y_0 \), and the implicit numerical method

\[ u_{n+1} = u_n + hf\left( t_n + \frac{h}{2}, u_n + \frac{h}{2} f(t_n + h; u_{n+1}) \right). \]

Write the numerical method as a two-stage Runge-Kutta method and find the Butcher array.
(4b) Use Taylor expansion to show that the local truncation error
\[
\tau_{n+1} = \frac{y_{n+1} - y_n}{h} - f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n + h, y_{n+1})\right)
\]
is $O(h^2)$. 