Problem 1  Suppose that a numerical integration rule
\[ \int_0^1 f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i) \]
with points \( x_i \in [0, 1] \) is exact for polynomials of degree \( d \). Suppose that a function \( f \) can be approximated by a polynomial \( p \) of degree \( d \) to accuracy \( \epsilon \) on the interval \([0, 1] \):
\[ \max_{0 \leq x \leq 1} |f(x) - p(x)| \leq \epsilon. \]
Show that the error in the numerical integration rule applied to integrate \( f \) is bounded by
\[ |\int_0^1 f(x)dx - \sum_{i=1}^{n} w_i f(x_i)| \leq \Omega \epsilon \]
where
\[ \Omega = 1 + \sum_{i=1}^{n} |w_i|. \]

Solution: Choose a polynomial \( p \) which approximates \( f \) within \( \epsilon \) on the interval \([0, 1] \). Since the rule is exact for \( p \), we have
\[ |\int_0^1 f(x)dx - \sum_{i=1}^{n} w_i f(x_i)| = |\int_0^1 f(x)dx - p(x)dx + \sum_{i=1}^{n} w_i (p(x_i) - f(x_i))| \]
\[ \leq \int_0^1 |f(x) - p(x)|dx + \sum_{i=1}^{n} |w_i||p(x_i) - f(x_i)| \]
\[ \leq \Omega \epsilon. \]

Problem 2 (a) Write out the Lagrange form of the quadratic polynomial \( p(x) \) interpolating values \( f_1, f_2 \) and \( f_3 \) at points \( x_1, x_2 \) and \( x_3 \).

Solution:
\[ p(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} f_1 + \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} f_2 + \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} f_3 \]
\[ = \varphi_1(x)f_1 + \varphi_2(x)f_2 + \varphi_3(x)f_3 \]

(b) Give a formula for the error \( f(x) - p(x) \) in terms of \( f \) and the \( x_i \)'s.

Solution:
\[ f(x) - p(x) = \frac{f^{(3)}(\zeta)}{3!}(x - x_1)(x - x_2)(x - x_3) \]

(c) Assume IEEE standard floating-point arithmetic with machine precision \( \epsilon < 1/200 \). Assume that the values \( f_i \) are nonzero. Show that evaluating \( p \) at any point \( x \) gives the exact value at \( x \) of the quadratic polynomial \( \hat{p} \) which interpolates values \( \hat{f}_i \) satisfying
\[ \frac{|f_i - \hat{f}_i|}{|f_i|} \leq 40\epsilon, \]

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at points \( x_1, x_2 \) and \( x_3 \). You may use the fact that
\[
\prod_{i=1}^{n} (1 + \delta_i)^{\sigma_i} = 1 + \Delta
\]
where \( |\Delta| \leq n\epsilon/(1 - n\epsilon) \) if each \( \delta_i \leq \epsilon \), each \( \sigma_i = \pm 1 \), and \( n\epsilon < 1 \).

**Solution:** In floating-point arithmetic, the \( i \)th operation commits relative error \( \delta_i \) bounded by the machine precision \( \epsilon \), so
\[
\text{fl}(p(x)) = \left( \frac{(x - x_2)(1 + \delta_1)(x - x_3)(1 + \delta_2)(1 + \delta_3)}{(x_1 - x_2)(1 + \delta_4)(x_1 - x_3)(1 + \delta_5)(1 + \delta_6)} \right) (1 + \delta_7)f_1(1 + \delta_8) + \frac{(x - x_1)(1 + \delta_9)(x - x_2)(1 + \delta_{10})(1 + \delta_{11})}{(x_2 - x_1)(1 + \delta_{12})(x_2 - x_3)(1 + \delta_{13})(1 + \delta_{14})} (1 + \delta_{15})f_2(1 + \delta_{16})(1 + \delta_{17}) + \frac{(x - x_1)(1 + \delta_{18})(x - x_2)(1 + \delta_{19})(1 + \delta_{20})}{(x_3 - x_1)(1 + \delta_{21})(x_3 - x_2)(1 + \delta_{22})(1 + \delta_{23})} (1 + \delta_{24})f_3(1 + \delta_{25})(1 + \delta_{26})
\]
where each \( |\Delta_i| \leq 10\epsilon/(1 - 10\epsilon) \leq 20\epsilon \). Defining \( \hat{f}_i = f_i(1 + \Delta_i) \) gives the result (with a factor of 2 to spare).

**Problem 3 (a)** Find a numerical integration rule of the form
\[
\int_0^3 f(x)dx = af(0) + bf(1) + cf(2)
\]
which is exact whenever \( f \) is a polynomial of degree 2. Note that the upper limit of integration is 3, not 2.

**Solution:** If \( f(x) \) is a polynomial of degree 2, then its quadratic interpolant \( p \) at the points 0, 1 and 2 is exactly equal to \( f \). Thus
\[
\int_0^3 f(x)dx = \int_0^3 \varphi_1(x)f(0) + \varphi_2(x)f(1) + \varphi_3(x)f(2)dx = af(0) + bf(1) + cf(2)
\]
where
\[
a = \int_0^3 \varphi_1(x)dx = \int_0^3 \frac{(x - 1)(x - 2)}{(0 - 1)(0 - 2)} dx = \frac{1}{2} \left( \frac{1}{3} x^3 - \frac{1}{2} x^2 + 2x \right) \bigg|_0^3 = \frac{3}{4}
\]
\[
b = \int_0^3 \varphi_2(x)dx = \int_0^3 \frac{(x - 0)(x - 2)}{(1 - 0)(1 - 2)} dx = -\frac{1}{1} \left( \frac{1}{3} x^3 - \frac{1}{2} x^2 \right) \bigg|_0^3 = 0
\]
\[
c = \int_0^3 \varphi_3(x)dx = \int_0^3 \frac{(x - 0)(x - 1)}{(2 - 0)(2 - 1)} dx = \frac{1}{2} \left( \frac{1}{3} x^3 - \frac{1}{2} x^2 \right) \bigg|_0^3 = \frac{9}{4}
\]
Note that \( a + b + c = 3 \) as a check: the constant function 1 integrates to 3 as it should. Since this rule integrates quadratics exactly, its error must be bounded by
\[
| \int_0^3 f(x)dx - af(0) - bf(1) - cf(2) | \leq CM_3
\]
for some constant \( C \), whenever \( |f^{(3)}(x)| \leq M_3 \) for all \( x \).
(b) Assume we know the \( a, b \) and \( c \) from part (a). Find weights \( w_0, w_1 \) and \( w_2 \) such that the absolute error \( E(h) \) in the approximation

\[
\int_0^{3h} g(x)\,dx = \sum_{i=0}^{2} w_i g(ih) + E(h)
\]

is of order \( E(h) = O(h^4) \).

**Solution:** Let \( f(t) = g(ht) \) where \( 0 \leq t \leq 3 \). Then we know from part (a) that

\[
\int_0^3 f(t)\,dt = af(0) + bf(1) + cf(2) + Cf^{(3)}(\zeta).
\]

On the other hand,

\[
\int_0^3 f(t)\,dt = \int_0^3 g(ht)\,dt = \frac{1}{h} \int_0^{3h} g(x)\,dx
\]

and

\[
f'(t) = hg'(ht), \quad f''(t) = h^2g''(ht), \quad f'''(t) = h^3g'''(ht). \]

Thus multiplying through by \( h \) gives

\[
\int_0^{3h} g(x)\,dx = hag(0) + hbg(h) + chc(2h) + O(Ch^4M_3)
\]

and

\[
w_0 = ha = 3h/4, \quad w_1 = hb = 0, \quad w_2 = hc = 9h/4.
\]

**Extra Credit Problem 4** Let \( p(x) \) be the polynomial of degree \( n - 1 \) which interpolates \( f(x) \) at \( n \) points \( x_i \). Show that the derivatives satisfy

\[
f'(x_i) - p'(x_i) = \frac{1}{n!} f^{(n)}(\xi) \prod_{j \neq i} (x_i - x_j)
\]

(for some unknown point \( \xi \) in the interval \( \min_i x_i, \max_i x_i \)) at each interpolation point \( x_i \).

**Solution:** We know that the error in polynomial interpolation is given by

\[
f(x) - p(x) = \frac{f^{(n)}(\zeta_x)}{n!} (x - x_1)(x - x_2) \cdots (x - x_n)
\]

where \( \zeta_x \) depends continuously on \( x \). Thus the definition of the derivative gives

\[
f'(x_i) - p'(x_i) = \lim_{h \to 0} \frac{f(x_i + h) - f(x_i) - p(x_i + h) + p(x_i)}{h}
\]

\[
= \lim_{h \to 0} \frac{f^{(n)}(\zeta_{x_i+h})(x_i + h - x_1) \cdots (x_i + h - x_i) \cdots (x_i + h - x_n)}{h}
\]

\[
\rightarrow \frac{1}{n!} f^{(n)}(\zeta_{x_i}) \prod_{j \neq i} (x_i - x_j)
\]

as \( h \to 0 \).