**Question 1**  Consider a differential equation 

\[ y'(t) = f(t, y(t)), \]

where \( f \) satisfies the condition 

\[ (u - v)(f(t, u) - f(t, v)) \leq 0 \]

for all \( u \) and \( v \).

(a) Suppose \( U(t) \) and \( V(t) \) are exact solutions. Show that 

\[ |U(t) - V(t)| \leq |U(0) - V(0)| \]

for all \( t \geq 0 \).

(b) Suppose \( W \) satisfies a perturbed differential equation 

\[ W'(t) = f(t, W(t)) + r(t) \]

for \( t \geq 0 \). Show that 

\[ |U(t) - W(t)| \leq |U(0) - W(0)| + \int_0^t |r(s)| ds \]

for \( t \geq 0 \).

(c) Show that two numerical solutions \( u_n \) and \( v_n \) generated by implicit Euler (e.g. with different initial values) satisfy 

\[ |u_n - v_n| \leq |u_0 - v_0| \]

for all \( n \geq 0 \).

(d) Show that the local truncation error \( \tau_{n+1} \) of the implicit Euler method 

\[ u_{n+1} = u_n + hf(t_{n+1}, u_{n+1}) \]

is given by 

\[ \tau_{n+1} = \frac{y_{n+1} - y_n}{h} - f(t_{n+1}, y_{n+1}) = -\frac{h}{2} y''(\zeta) \]

where \( y_n = y(t_n) \) is the exact solution and \( \zeta \) is an unknown point.

(e) Show that the numerical solution \( u_n \) generated by implicit Euler with \( u_0 = y_0 \) satisfies 

\[ |u_n - y_n| \leq nh\tau \]

for \( 0 \leq nh < \infty \), where \( \tau = Mh/2 \) and \( |y''| \leq M \).

**Question 2**  Define a family of implicit Runge-Kutta methods parametrized by order \( p \), by applying up to \( p - 1 \) passes of deferred correction to \( p \) steps of the implicit Euler method. I.e. starting from \( u_1^n = u_n \) define the uncorrected solution by solving 

\[ u_{n+1}^1 = u_{n+1}^1 + hf(t_{n+1}, u_{n+1}^1) \]
for $0 \leq j \leq p - 1$. Let $u(t) = U_1(t)$ be the degree-$p$ polynomial that interpolates the $p + 1$ values $u_{n+j}^1$ at the $p$ points $t = t_{n+j}$ for $0 \leq j \leq p$. Solve the error equation from problem set 10 by the implicit Euler method, yielding approximate errors $e_{n+1}^1, e_{n+2}^1, \ldots, e_{n+p}^1$. Produce a second-order accurate corrected solution

$$u_{n+j}^2 = u_{n+j}^1 + e_{n+j}^1$$

for $1 \leq j \leq p$. Repeat the procedure to produce $u_{n+j}^3, \ldots, u_{n+j}^p$. Finally let $u_{n+p} = u_{n+p}^p$.

(a) Verify that $p = 1$ gives the implicit Euler method. Taylor expand $k_1(h)$. Show that your method has local truncation error $\tau = O(h)$ and find the coefficient of the $O(h)$ term.

(b) For $p = 2$ express your method as a 4-stage implicit Runge-Kutta method in the form

$$k_i = f(t_n + 2hc_i, u_n + 2h \sum_{j=1}^{4} a_{ij}k_j)$$

for $1 \leq i \leq 4$,

$$u_{n+2}^2 = u_n + 2h \sum_{i=1}^{4} b_i k_i.$$ 

Find all the constants $c_i, a_{ij}$ and $b_j$ and arrange them in a Butcher array.

**Question 3**  Consider the linear initial value problem

$$y' = -L(y(t) - \varphi(t)) + \varphi'(t)$$

$y(0) = y_0$

where $\varphi(t) = \cos(30t)$.

(a) Solve the initial value problem exactly.

(b) Write a matlab script which uses the method you derived in question 2 with $p = 2$ to solve the initial value problem with $y(0) = 2$ for $0 \leq t \leq 1$ with $L = 10^k$ for $k = 1$ to 5. For each $L$ use $h = 10^{-j}$ with $j = 1$ to 5. Tabulate the errors. Plot an accurate solution for each $L$. 