Question 1  The position \((x(t), y(t))\) of a satellite orbiting around the earth and moon is described by the second-order system of ordinary differential equations
\[
x'' = x + 2y' - b \frac{x + a}{(x + a)^2 + y^2}^{3/2} - a \frac{x - b}{(x - b)^2 + y^2}^{3/2}
\]
\[
y'' = y - 2x' - b \frac{y}{(x + a)^2 + y^2}^{3/2} - a \frac{y}{(x - b)^2 + y^2}^{3/2}
\]
where \(a = 0.012277471\) and \(b = 1 - a\). When the initial conditions
\[
x(0) = 0.994 \\
x'(0) = 0 \\
y(0) = 0 \\
y'(0) = -2.00158510637908
\]
are satisfied, there is a periodic orbit with period \(T = 17.06521656015796\).

(a) Convert this problem to a 4 × 4 first-order system \(u' = f(t, u), u(0) = u_0\), by introducing
\[
u = [x, x', y, y'] = [u_1, u_2, u_3, u_4]
\]
as a new vector unknown function and defining \(f\) appropriately.

(b) Choose a rectangular region in the 4-dimensional \(u\) space which includes the initial point \(u_0\), and find a Lipschitz constant \(L\) for \(f\) on that region.

Question 2  Define a family of explicit Runge-Kutta methods parametrized by order \(p\), by extrapolating Euler’s method. I.e. for \(q = 1\) to \(p\) define stages \(k_{qj}\) for \(1 \leq j \leq q\) by
\[
k_{q1} = f(t_n, u_n),
\]
\[
k_{q2} = f(t_n + \frac{1}{q}h, u_n + h \frac{1}{q}k_{q1}),
\]
\[
k_{q3} = f(t_n + \frac{2}{q}h, u_n + h(\frac{1}{q}k_{q1} + \frac{1}{q}k_{q2})),
\]
and so forth. Each of the \(p\) approximate solutions
\[
u_{n+1}^q = u_n + \frac{1}{q}h(k_{q1} + \cdots + k_{qq})
\]
is the result of \(q\) steps of Euler with step size \(h/q\), so
\[
u_{n+1}^q = U_p(h/q) + O(h^p)
\]
where \(U_p(h)\) is a polynomial of degree \(p - 1\) in \(h\). Apply Richardson extrapolation (i.e. degree \(p - 1\) polynomial interpolation from \(u_{n+1}^1\) at \(h\), \(u_{n+1}^2\) at \(h/2\), \(u_{n+1}^3\) at \(h/3\), and so forth,
to \( u_{n+1} \) at \( h = 0 \) to the collection of approximate solutions \( u^q_{n+1} \) to design a Runge-Kutta method

\[
    u_{n+1} = U_p(0) = u_n + h \left( \sum_{q=1}^{p} \sum_{j=1}^{q} b_{qj} k_{qj} \right)
\]

which is accurate of order \( p \). (a) Verify that \( p = 1 \) gives Euler’s method and \( p = 2 \) gives the explicit midpoint rule

\[
    u_{n+1} = u_n + hf(t_n + \frac{1}{2} h, u_n + \frac{1}{2} f(t_n, u_n)).
\]

(b) Write out the constants \( b_{qj} \) that you get for \( p = 3 \). (c) For arbitrary \( p \), verify that your method gives the correct Taylor expansion up to order \( p − 1 \) when you solve \( y' = y \).

**Question 3**  (a) Write, test and debug a matlab function

```matlab
function yb = rich(a, b, ya, f, p, n)
    % a,b: interval endpoints with a < b
    % ya: vector y(a) of initial conditions
    % f: function handle f(t, y) to integrate (y is a vector)
    % p: order of accuracy of the method
    % n: number of steps with h = (b-a)/n
    % yb: output approximation to the final solution vector y(b)

    which approximates the final solution vector \( y(b) \) of the vector initial value problem

    \[
    y' = f(t, y)
    \]
    \[
    y(a) = y_a
    \]

    by the numerical solution vector \( u_n \) of the method you derived in problem 2, with \( h = (b - a)/n \) and \( u_0 = y_a \).

    (b) Use rich.m with a selection of orders \( p \) and step sizes \( N \) to approximate the final solution vector \( u(T) \) of the initial value problem derived in problem 1.

    Tabulate the errors

    \[
    E_{pN} = \max_{1 \leq j \leq 4} \left| u_j(T) - u_j(0) \right|
    \]

    with time steps \( h = T/N \) for \( N = 100, 200, \ldots \). Estimate the constant \( C_p \) such that the error behaves like \( C_p h^p \).

    Measure the CPU time for each run and estimate the total CPU time necessary to obtain an orbit which is periodic to three–digit, six–digit and twelve–digit accuracy.

    Plot some inaccurate solutions and some accurate solutions and draw conclusions about values of the order \( p \) which give three, six or twelve digits of accuracy at minimal cost (i.e. CPU time).

    Find a rectangular region in the 4-dimensional \( u \) space which includes an accurate orbit, and find an approximate Lipschitz constant \( L \) for \( f \) on that region. Draw conclusions about uniqueness of the periodic orbit.