Question 1  (a) For arbitrary real $s$ find the exact solution of the initial value problem
\[ y'(t) = \frac{1}{2} \left( y(t) + y(t)^3 \right) \]
with $y(0) = s > 0$.
(b) Show that the solution blows up when $t = \log(1 + 1/s^2)$.

Question 2  (a) Find the general solution of the difference equation
\[ u_{j+2} = u_{j+1} + u_j. \]
(b) Find all initial values $u_0$ and $u_1$ such that $u_j$ remains bounded by a constant as $j \to \infty$.

Question 3  (a) Write, test and debug a matlab function
\[
\text{function } y = \text{euler}(a, b, ya, f, n) \\
\text{\hspace{1cm}} \% a,b: interval endpoints with a < b \\
\text{\hspace{1cm}} \% n: number of steps with h = (b-a)/n \\
\text{\hspace{1cm}} \% ya: vector y(a) of initial conditions \\
\text{\hspace{1cm}} \% f: function handle f(t, y) to integrate (y is a vector) \\
\text{\hspace{1cm}} \% y: output approximation to the vector y(b) \\
\text{\hspace{1cm}} \]
which approximates the final solution $y(b)$ of the initial value problem
\[ y' = f(t, y) \]
\[ y(a) = ya \]
by Euler’s method
\[ y_{n+1} = f(t_n, y_n). \]
(b) Use euler.m to approximate the solution at $T = 4\pi$ of the initial value problem
\[ x' = u \]
\[ y' = v \]
\[ u' = -x/(x^2 + y^2) \]
\[ v' = -y/(x^2 + y^2) \]
with initial conditions $x = 1$, $y = 0$, $u = 0$, $v = 1$ at $t = 0$ which cause the solution to move in a unit circle forever. Measure the maximum error
\[ E_N = \max(|x_N - \cos t_N|, |y_N - \sin t_N|, |u_N + \sin t_N|, |v_N - \cos t_N|) \]
after 2 revolutions ($T = 4\pi$) with time steps $h = T/N$ for $N = 100, 200, \ldots, 800$. Estimate the constant $C$ such that the error behaves like $Ch$. Measure the CPU time for each run and estimate the total CPU time necessary to obtain the solution to three–digit, six–digit and twelve–digit accuracy. Plot the solutions.
(c) Use euler.m to verify conclusion (b) of problem 1.