Question 1  (a) Show that

\[ \int_0^1 x^{-x} dx = \sum_{n=1}^{\infty} n^{-n} \]

(b) Use the sum in (a) to evaluate the integral in (a) to 12-digit accuracy.
(c) Evaluate the integral in (a) by Romberg integration. Estimate how many function evaluations Romberg integration will require to achieve 12-digit accuracy. Explain the agreement or disagreement of your results with theory.

Question 2  In class we proved the Euler-Maclaurin summation formula

\[ \int_0^1 f(x) dx = \frac{1}{2} (f(0) + f(1)) + \sum_{m=1}^{\infty} b_m \left( f^{(2m-1)}(1) - f^{(2m-1)}(0) \right) \]

for some unknown constants \( b_m \) independent of \( f \).

(a) Find a formula for \( b_m \) by evaluating both sides for \( f(x) = e^{\lambda x} \) where \( \lambda \) is a parameter.
(b) Compute \( b_1, b_2, b_3, \ldots, b_{10} \).

Question 3  (a) Use the Euler-Maclaurin formula to show that

\[ \sum_{j=1}^{n} j^k = P_{k+1}(n) \]

is a degree-\((k+1)\) polynomial in \( n \). Example:

\[ \sum_{j=1}^{n} j = \frac{n(n + 1)}{2}. \]

(b) Use the results of question 2 to find \( P_{k+1} \) for \( 2 \leq k \leq 10 \).
(c) Use polynomial interpolation to find \( P_{k+1} \) for \( 2 \leq k \leq 10 \) and compare with the results from (b).