Question 1  Let $p$ be a positive integer and
\[ f(x) = 2^x \]
for $0 \leq x \leq 2$.
   (a) Find a formula for the $p$th derivative $f^{(p)}(x)$.
   (b) For $p = 0, 1, 2$ find a formula for the polynomial $H_p$ of degree $2p + 1$
   such that
\[ H_p^{(k)}(x_j) = f^{(k)}(x_j) \]
for $0 \leq k \leq p$, $0 \leq j \leq 1$, $x_0 = 0$, $x_1 = 2$.
   (c) For general $p$ prove that
\[ |f(x) - H_p(x)| \leq \left( \frac{1}{p+1} \right)^{2p+2} \]
for $0 \leq x \leq 2$.
   (d) Show that one step of Newton’s method for solving
\[ g(y) = x \ln 2 - \ln y = 0 \]
starting from $y_0 = H_4(x)$ gives $y_1 = f(x) = 2^x$ to full double precision
accuracy for $0 \leq x \leq 2$.

Question 2  (See BF p. 192.) For integer $k \geq 4$ let
\[ p_k = k \sin \left( \frac{\pi}{k} \right) \quad P_k = k \tan \left( \frac{\pi}{k} \right) \]
   (a) Show that $p_4 = 2\sqrt{2}$ and $P_4 = 4$.
   (b) Show that
\[ P_{2k} = \frac{2p_k P_k}{p_k + P_k} \quad p_{2k} = \sqrt{p_k P_{2k}} \]
for $k \geq 4$.
   (c) Approximate $\pi$ within $10^{-4}$ by computing $p_k$ and $P_k$ until $P_k - p_k < 10^{-4}$.
   (d) Use Taylor series to show that
\[ \pi = p_k + \sum_{j=1}^{\infty} q_j k^{-2j} \quad \pi = P_k + \sum_{j=1}^{\infty} Q_j k^{-2j} \]
for some constants $q_j$ and $Q_j$.
   (e) Use extrapolation with $h = 1/k$ to approximate $\pi$ within $10^{-12}$.  

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