2 Consider the iteration
\[ x_{n+1} = \frac{x_n^3 + 3ax_n}{3x_n^2 + a}. \]

(a) What is it intended to compute? (b) Given \( a = 2 \) and \( x_0 = 1 \), compute \( x_1 \) and \( x_2 \). (c) Define and determine the order of convergence of this iteration.

4 (a) Derive a numerical integration formula
\[ \int_0^1 f(x)dx = w_0f(0) + w_1f(1) + w_2f(2) \]
which is exact for polynomials of as high degree \( d \) as possible, and determine the maximal degree \( d \). (b) Without any additional work, determine an equally accurate rule of the form
\[ \int_0^1 f(x)dx = u_0f(-1) + u_1f(0) + u_2f(1). \]
(c) Show that
\[ \int_0^h f(x)dx = h(w_0f(0) + w_1f(h) + w_2f(2h)) + O(h^{d+1}) \]
as \( h \to 0 \). (d) Use (a), (b), and (c) to build a quadrature formula with error \( O(h^d) \) on an arbitrary interval \([a, b]\) divided into \( n > 1 \) subintervals of length \( h = (b - a)/n \).

5 (a) Write down the Newton and Lagrange forms of the quadratic interpolant \( p(x) \) to a function \( f \) at three points \( a, b \) and \( c \). (b) Give a formula for the error \( p(x) - f(x) \) if \( f \) is a nice function with all derivatives bounded. Explain why your error formula makes sense in terms of dimensions, zeroes and the derivatives which appear versus the degree of polynomial used. (c) Specialize to \( f(x) = R - 1/x \) and evaluate the coefficients in the Newton representation of \( p(x) \). (d) Use (c) to express \( p \) in the power form \( p(x) = q_0 + q_1x + q_2x^2 \). (e) How would you use the formula of (d) to derive an iterative method for finding \( 1/R \)?

6 Suppose \( A \) is a square invertible matrix. (a) Define the condition number \( \kappa(A) \). (b) Suppose \( E \) is a matrix the same size as \( A \) and
\[ \kappa(A) \frac{\|E\|}{\|A\|} \leq \epsilon \leq \frac{1}{2}. \]
Show that \( A + E \) is invertible. (c) Show that
\[ \frac{\|(A + E)^{-1} - A^{-1}\|}{\|A^{-1}\|} \leq 2\epsilon. \]
8 Prove that any model of floating-point arithmetic which requires that the floating-point result of the multiplication \( x \ast y \) be given by the exact result correctly rounded satisfies the relative error bound
\[
\frac{|x \ast y - \text{fl}(x \ast y)|}{|x \ast y|} \leq \epsilon
\]
as long as no overflow or underflow occurs and \( x \ast y \neq 0 \).

(9) Given 10 values \( f(x_j) \) at distinct points \( 0 < x_j < \pi \), show that there are unique coefficients \( c_j \) such that
\[
f(x_j) = c_1 \sin(x_j) + \cdots + c_{10} \sin(10x_j)
\]
for \( j = 1 : 10 \). (Hint: apply linear algebra, then differentiate something \( 2n \) times and let \( n \to \infty \).)

(10) Suppose a quadrature rule
\[
\int_0^1 f(x)dx = \sum_{j=1}^n W_j f(X_j) + E_n(f)
\]
has error term \( E_n(f) = C_n f^{(2n)}(\xi) \) for some unknown point \( \xi \in [0,1] \). (a) Find points and weights of a quadrature rule on \([0,h]\) with
\[
\int_0^h f(x)dx = \sum_{j=1}^n w_j f(x_j) + O(h^{2n+1}f^{(2n)})
\]
as \( h \to 0 \). (b) Describe a quadrature rule on an arbitrary interval \([a,b]\) with
\[
\int_a^b f(x)dx = \sum_{j=1}^N u_j f(z_j) + O(h^{2n})
\]
where \( N = np \) and \( h = (b-a)/p \).

(12) Define the condition number \( \kappa(A) \) and use it to bound the change in the solution \( x \) of \( Ax = b \) when \( A \) is changed to \( A + E \) with \( E \) small.

(13) Calculate \( f[0,1,2,3] \) where \( f(x) = 1/(1+x) \).

(14) Let \( f(x) = \cos(x) \). Derive a bound for the maximum error on \([-1,1]\) when \( f \) is approximated at a point \( x \) by (a) a 3-term Taylor expansion about 0, and (b) quadratic interpolation from the nearest points to \( x \) in an equidistant \( n \)-point grid on \([-1,1]\).

(19) Suppose \( f \) is a nice smooth function and \( r \) is a point where \( f(r) = f'(r) = 0 \) but \( f''(r) \neq 0 \). Determine how fast the iteration
\[
x_{n+1} = x_n - \frac{u(x_n)}{u'(x_n)}
\]
converges near $x = r$ if $u$ is defined by

$$u(x) = \frac{f(x)}{f'(x)}.$$  

(20) Find the LU decomposition of

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 2 & -2 \\ 4 & 5 & 1 \end{bmatrix}$$

and check that $LU = A$.

(22) The secant method can be written in two algebraically equivalent forms

(a)

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}f(x_n)$$

and (b)

$$x_{n+1} = \frac{x_{n-1}f(x_n) - x_nf(x_{n-1})}{f(x_n) - f(x_{n-1})}.$$  

Which is better in floating-point arithmetic and why?

(23) The mean $M$ and variance $V$ of $n$ numbers $x_i$ are defined by

$$M = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad V = \frac{1}{n} \sum_{i=1}^{n} (x_i - M)^2 = \left( \frac{1}{n} \sum_{i=1}^{n} x_i^2 \right) - M^2$$

where the two formulas for $V$ are equivalent in exact arithmetic because

$$\sum_{i=1}^{n} (x_i - M) = 0.$$  

Take $n = 2$ for simplicity, and show that in floating-point arithmetic with machine precision $\epsilon$ the two formulas satisfy different error bounds

$$|V - \text{fl} \left( \frac{1}{2} \sum_{i=1}^{2} (x_i - M)^2 \right) | \leq 5\epsilon|V| + O(\epsilon^2)$$

and

$$|V - \text{fl} \left( \left( \frac{1}{2} \sum_{i=1}^{2} x_i^2 \right) - M^2 \right) | \leq 2\epsilon(x_1^2 + x_2^2 + M^2) + O(\epsilon^2)$$

Which formula has a better guaranteed relative accuracy in floating-point arithmetic?

(25) Compute the Newton form of the cubic polynomial which interpolates the function $f(x) = |x - 1|$ at the points $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ and $x_4 = 0$.

(27) (a) Consider the MATLAB code

3
x=1; q=0;
while x>0
  x=x/2;
  q=q+1;
end;

In binary floating-point arithmetic, what does it evaluate?

(b) Consider the matlab code

x=1;
while 1 + x > 1
  x=x/2;
end;

In binary floating-point arithmetic, what does it evaluate?

(c) Consider the matlab code

function a = f(x,y,z)
  n = length(x);
  for k = 1:n-1
    y(k+1:n) = (y(k+1:n)-y(k:n-1)) ./ (x(k+1:n) - x(1:n-k));
  end
  a = y(n)*ones(size(z));
  for k=n-1:-1:1
    a = (z-x(k)).*a + y(k);
  end
end

In exact arithmetic, what would it evaluate?

(28) (a) Find the Lagrange form of the quadratic polynomial interpolating the function \( f(x) = x^3 \) at the points \( x_1 = 1, x_2 = 2, x_3 = 3 \).

(b) Find a numerical integration rule of the form

\[
\int_0^3 f(x)dx = af(1) + bf(2) + cf(3)
\]

which is exact whenever \( f \) is a polynomial of degree 2. Note that the lower limit of integration is 0, not 1.

(c) Give weights \( w_1, w_2 \) and \( w_3 \) such that the error \( E(h) \) in the approximation

\[
\int_0^{3h} f(x)dx = \sum_{i=1}^{3} w_i f(ih) + E(h)
\]

is of order \( E(h) = O(h^4) \).

(29) (a) Find the power form of the linear polynomial \( p(x) \) interpolating the function \( f(x) = x^2 \) at the points \( x_1 = 1 \) and \( x_2 = 2 \).
(b) Show that the absolute error in evaluating \( p(x) \) at an arbitrary point \( x \) by floating-point arithmetic with machine precision \( \epsilon \) is bounded by
\[
|p(x) - \text{fl}(p(x))| \leq 8(|x| + 1)\epsilon.
\]

(c) Show that \( \text{fl}(p(x)) = p(\hat{x}) \) is equal to the exact value of the polynomial \( p \) at a point \( \hat{x} \) such that \( |x - \hat{x}| \leq 5(1 + |x|)\epsilon \).

(30) Let \( k \) be an arbitrary positive integer and consider interpolating \( \sin(kx) \) by a polynomial of degree 10 at 11 equispaced points on the interval \([0, 1]\). Give a bound for the error \( |\sin(kx) - p(x)| \) as a function of \( k \) when \( x \) lies in the interval \([0.0, 0.1]\).

(31) Let
\[
\varphi_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}
\]
be the \( i \)th Lagrange basis function for interpolation at \( n \) points \( x_1, \ldots, x_n \). Show that
\[
\alpha_k(x) = (1 - 2\varphi'_k(x_k)(x - x_k))\varphi_k(x)^2
\]
and
\[
\beta_k(x) = (x - x_k)\varphi_k(x)^2
\]
are basis functions for Hermite interpolation of a function \( f \) and its derivative \( f' \) at the points \( x_1, \ldots, x_n \).

(32) Suppose a function \( f \) can be evaluated with relative error bounded by \( \delta < 1/2 \). We approximate its derivative by the forward difference quotient
\[
D_h f(x) = \frac{f(x + h) - f(x)}{h}.
\]

(a) Use Taylor expansion to show that in exact arithmetic
\[
|D_h f(x) - f'(x)| \leq \frac{M_2 h^2}{2}
\]
whenever \( |f''(x)| \leq M_2 \) for all \( x \).

(b) Show that in floating-point arithmetic with machine epsilon \( \epsilon < 1/2 \) the error in evaluating the approximation is bounded by \( 6(M_0 + M_1)(\epsilon + \delta)/h \) whenever \( |f(x)| \leq M_0 \) and \( |f'(x)| \leq M_1 \) for all \( x \).

(c) Find \( h \) (as a function of \( \epsilon + \delta, M_0, M_1 \) and \( M_2 \)) which gives the best error bound on the approximation of \( f'(x) \) by \( \text{fl}(D_h f(x)) \).

(33) Consider two sets of data
\[
x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \quad y_1 = 1, \quad y_2 = 12, \quad y_3 = 13,
\]
and
\[ \hat{x}_1 = 3, \quad \hat{x}_2 = 2, \quad \hat{x}_3 = 1, \quad \hat{y}_1 = 13, \quad \hat{y}_2 = 12, \quad \hat{y}_3 = 1. \]
Prove or disprove: the same quadratic polynomial \( p(x) \) interpolates both sets of data.

(34) Describe three techniques for proving the following theorem: given \( n \) distinct points \( x_i \in R \) and \( n \) values \( y_i \in R \), there is a unique polynomial \( p \) of degree \( n - 1 \) such that \( p(x_i) = y_i \) for \( i = 1 : n \).

(36) Verify that \( p(x) = 3 + 2(x-1) + 4(x-1)(x+2) \) and \( q(x) = 4x^2 + 6x - 7 \) both interpolate \( q(x) \) at \( x = 1, -2 \) and 0. Explain why this does not contradict the uniqueness of polynomial interpolation.

(37) The polynomial \( p(x) = x^4 - x^3 + x^2 - x + 1 \) has values 31, 5, 1, 1, 11, 61 at the points \(-2, -1, 0, 1, 2, 3\). Find a degree-5 polynomial \( q(x) \) with values 31, 5, 1, 1, 11, and 30 at the same points.

(38) Let \( p(x) \) have degree \( n \). What are the divided differences \( p[x_1, x_2, \ldots, x_{n+2}] \) for \( n + 2 \) distinct points \( x_i \)? Prove it.

(40) Let \( |\delta_i| \leq \delta \) for \( i = 1 : n \), where \( n\delta < 1/2 \).
(a) Show that
\[ \prod_{j=1}^{n} (1 + \delta_i) = 1 + \Delta \]
where \( |\Delta| \leq 2n\delta \).
(b) Show that
\[ \prod_{j=1}^{n} (1 + \delta_i)^{-1} = 1 + \Delta \]
where \( |\Delta| \leq n\delta/(1 - n\delta) \).

(41) Prove that the \( n \times n \) matrix with 4’s down the diagonal, \(-1\)’s adjacent to the diagonal and 0’s elsewhere is invertible for any \( n \).

(42) Suppose \( x \) and \( y \) are floating-point numbers and the floating-point subtraction \( \text{fl}(x - y) \) evaluates to 0. Prove or give a counterexample: \( x = y \).

(43) Define the machine precision \( \epsilon \) for floating-point arithmetic. Prove that the model of floating-point arithmetic which requires that the floating-point result of the addition \( x + y \) be given by the exact result correctly rounded satisfies the relative error bound
\[ \frac{|x + y - \text{fl}(x + y)|}{|x + y|} \leq \epsilon \]
as long as no overflow or underflow occurs and \( x + y \neq 0 \).
(45) Show that for any function \( f \) on the interval \([-1, 1]\) and any nonzero \( s \), there exists a unique function of the “exponential spline” form
\[
p(t) = a + bt + c \cosh(st) + d \sinh(st)
\]
such that
\[
p(-1) = f(-1), \quad p(+1) = f(+1), \quad p'(-1) = f'(-1), \quad p'(+1) = f'(+1).
\]

(46) Show that for any function \( f \) on the interval \([-1, 1]\) and any \( s \in (0, \pi) \), there exists a unique function of the “trigonometric spline” form
\[
p(t) = a + bt + c \cos(st) + d \sin(st)
\]
such that
\[
p(-1) = f(-1), \quad p(+1) = f(+1), \quad p'(-1) = f'(-1), \quad p'(+1) = f'(+1).
\]

48. (a) Write out the Lagrange form of the quadratic polynomial \( p(x) \) interpolating values \( f_1, f_2 \) and \( f_3 \) at points \( x_1, x_2 \) and \( x_3 \).
(b) Give a formula for the error \( f(x) - p(x) \) in terms of \( f \) and the \( x_i \)'s.
(c) Assume IEEE standard floating-point arithmetic with machine precision \( \epsilon < 1/200 \). Assume that the values \( f_i \) are nonzero. Show that evaluating \( p \) at any point \( x \) gives the exact value at \( x \) of the quadratic polynomial \( \hat{p} \) which interpolates values \( \hat{f}_i \) satisfying
\[
\frac{|f_i - \hat{f}_i|}{|f_i|} \leq 40\epsilon,
\]
at points \( x_1, x_2 \) and \( x_3 \). You may use the fact that
\[
\prod_{i=1}^{n} (1 + \delta_i)^{\sigma_i} = 1 + \Delta
\]
where \( |\Delta| \leq n\epsilon/(1 - n\epsilon) \) if each \( \delta_i \leq \epsilon \), each \( \sigma_i = \pm 1 \), and \( n\epsilon < 1 \).

49. (a) Find a numerical integration rule of the form
\[
\int_{0}^{3} f(x)dx = af(0) + bf(1) + cf(2)
\]
which is exact whenever \( f \) is a polynomial of degree 2. Note that the upper limit of integration is 3, not 2.
(b) Assume we know the \( a, b \) and \( c \) from part (a). Find weights \( w_0, w_1 \) and \( w_2 \) such that the absolute error \( E(h) \) in the approximation
\[
\int_{0}^{3h} g(x)dx = \sum_{i=0}^{2} w_ig(ih) + E(h)
\]
is of order $E(h) = O(h^4)$.

50. Let $p(x)$ be the polynomial of degree $n - 1$ which interpolates $f(x)$ at $n$ points $x_i$. Show that the derivatives satisfy

$$f'(x_i) - p'(x_i) = \frac{1}{n!} f^{(n)}(\xi) \prod_{j \neq i} (x_i - x_j)$$

(for some unknown point $\xi$ in the interval $[\min_i x_i, \max_i x_i]$) at each interpolation point $x_i$.

53. Let $P(x)$ be the quadratic polynomial that interpolates $f(x) = 1/(1+x^2)$ at $x = 0, 1$ and $-1$.

(a) Use Newton interpolation to evaluate $P$ at $x = 2$.

(b) Bound the relative error $e(x) = |f(x) - P(x)|/|f(x)|$ on the interval $1 \leq x \leq 1 + h$ for any $h \geq 0$. Why does your bound increase as $h$ increases?

(c) State the general formula which gives the error at a point $t$ in the degree-$n$ polynomial $P$ which interpolates a smooth function $f$ at $n + 1$ points $t_j$ in $[a, b]$. Separate the error into a product of three factors and explain why two of them are inevitable. Find a test function which determines the third factor and explain why.

54. Suppose that a numerical integration rule with nodes $x_j \in [0, 1]$, weights $w_j$, and error

$$E_p(f) = \int_0^1 f(x)dx - \sum_{j=1}^p w_j f(x_j)$$

is exact for polynomials of degree $d$: $E_p(f) = 0$ whenever $f$ is a polynomial of degree $d$.

(a) Assume $f$ can be approximated by a degree-$d$ polynomial within error $\epsilon$ on the whole interval $[0, 1]$. Prove that $|E_p(f)| \leq \Lambda \epsilon$ where $\Lambda = 1 + \sum_{j=1}^p |w_j|$. What is the best possible value of $\Lambda$ and when does it occur?

(b) If the nodes $x_j$ are equidistant on $[0, 1]$, with $x_1 = 0$ and $x_p = 1$, what is the biggest possible value of $d$? What are the resulting rules called?

(c) If you get to choose the nodes $x_j$ anywhere in $[0, 1]$, how big can you make $d$? What are the resulting rules called?

55. Suppose that a numerical integration rule with nodes $x_j \in [0, 1]$ and weights $w_j$ has an error of the form

$$\int_0^1 f(x)dx - \sum_{j=1}^p w_j f(x_j) = C f^{(q)}(q)$$

for some unknown $\xi$.

(a) Prove that the nodes $t_j = hx_j$ in the interval $[0, h]$ and the weights $u_j = hw_j$ satisfy the error estimate

$$\int_0^h g(t)dt - \sum_{j=1}^p u_j g(t_j) = Ch^{q+1} g^{(q)}(\xi)$$
for some unknown $\xi$.

(b) For arbitrary $a$ and $b$ and $N = np$, write down formulas for nodes $s_j \in [a, b]$ and weights $v_j$ such that

$$\int_a^b g(s)ds - \sum_{j=1}^N v_j g(s_j) \leq C|b - a|h^q \max_{a \leq s \leq b} |g^{(q)}(s)|$$

where $h = (b - a)/n$.

(c) Find the weights $v_j$ and nodes $s_j$ on an arbitrary interval $[a, b]$ for the special case with $p = 2$ equidistant nodes $x_1 = 0$, $x_2 = 1$ and $q = 2$.

58. (a) Use Gaussian elimination to find a permutation matrix $P$, a unit lower triangular matrix $L$, and an upper triangular matrix $R$ such that $PA = LR$ where

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}.$$  

(b) Repeat with

$$A = \begin{bmatrix} 1 & 2 \\ 300 & 4 \end{bmatrix}.$$  

(c) Find vectors $u$ and $v$ such that

$$L = \begin{bmatrix} 1 & 0 \\ 300 & 1 \end{bmatrix} = I + uv^T.$$  

Express $L^{-1}$ in terms of $u$ and $v$.

61. (a) Use Gaussian elimination with partial pivoting to find a permutation matrix $P$, a unit lower triangular matrix $L$, and an upper triangular matrix $R$ such that $PA = LR$ where

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & 3 \\ 10^6 & 0 & -3 \end{bmatrix}.$$  

63. (a) Write down Newton’s method for finding the root $c$ of a single scalar equation $f(x) = 0$, and simplify it for the specific problem of finding the positive cube root $c$ of a number $c^3 = a > 0$. (I.e. write it in an algebraic form similar to the iteration $x_{n+1} = (1/2)(x_n + a/x_n)$ for finding $\sqrt[3]{a}$.)

(b) Verify that for $a = 27$ and $x_0 = 4$, the error decreases after one Newton step: $|c - x_1| < |c - x_0|$.

(c) Prove that $x_n \to c$ as $n \to \infty$ if $1 < c < x_0$. (Hint: to save algebra, do the general error estimate first, and specialize to the cube root at the last minute.)

(d) Replace $f'(x_n)$ in Newton’s method by the forward difference approximation $D_h f(x) = (f(x+h) - f(x))/h$, for some small $h$ (comparable to the error
\( e_n = c - x_n \) at each step). Show that the resulting iteration still converges quadratically (for a general function \( f \) with \( f'' \) bounded) if \( h = O(|e_n|) \) as \( n \to \infty \).

(e) Would you expect floating-point error in computer evaluation of \( f \) to totally ruin the conclusion of part (d)?

64. Let \( P(x) \) be the quadratic polynomial that interpolates \( f(x) = \cos(x) \) at \( x = 9.0, 9.1 \) and \( 9.2 \) (in radians).

(a) Write down the Lagrange and Newton formulas for \( P \).

(b) Bound the relative error \( e(x) = \frac{|f(x) - P(x)|}{|f(x)|} \) on the interval \( 0.9 \leq x \leq 9.2 \).

(c) Bound the relative error \( e(x) = \frac{|f(x) - P(x)|}{|f(x)|} \) on the interval \( 9.09 \leq x \leq 9.11 \). Why is your bound so much smaller or larger than the bound in (a)?

67. Let \( P(x) \) be the quadratic polynomial that interpolates \( f(x) = x^3 \) at \( x = 1, 2 \) and 3.

(a) Write down the Lagrange and Newton formulas for \( P \).

(b) Bound the relative error \( e(x) = \frac{|f(x) - P(x)|}{|f(x)|} \) on the interval \( 1 \leq x \leq 3 \).

(c) Bound the relative error \( e(x) = \frac{|f(x) - P(x)|}{|f(x)|} \) on the interval \( 1.9 \leq x \leq 2.1 \). Why is your bound so much smaller or larger than the bound in (a)?

70. Consider the equation \( x - 1 - 1/x = 0 \).

(a) What are its solutions?

(b) For which solutions will the fixed point iteration

\[
x_{n+1} = 1 + 1/x_n
\]

converge if started close enough? For which will Newton’s method converge if started close enough?

(c) In each case where convergence occurs, will it be linear or quadratic?

71.

(a) Suppose \( c_i \) and \( x_i \) are chosen so that \( 0 \leq x_i \leq h \) and

\[
\int_0^h p(x) \, dx = \sum_{i=0}^n c_i p(x_i)
\]

whenever \( p \) is a polynomial of degree \( \leq n \). Prove that

\[
\int_0^h f(x) \, dx = \sum_{i=0}^n c_i f(x_i) + O(h^{n+2})
\]

whenever \( f \) has \( n + 1 \) bounded continuous derivatives.
(b) Find a numerical integration formula

\[ \int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2) \]

which is exact for 1, \(x\), \(x^2\) and \(x^3\). Use this formula to find a fifth-order formula on the interval \([0, h]\).

73. Find a numerical integration formula of the form

\[ \int_0^h f(x)dx = Af(0) + B f(h) + C(f'(0) - f'(h)) + O(h^q) \]

which has as high order \(q\) as possible. What is \(q\)?

76. Prove that floating point arithmetic with machine epsilon \(\epsilon\) produces an inner product (summed in the natural order) satisfying

\[ \text{fl}(x^T y) = x^T (y + e) \]

where

\[ |e_i| \leq 2n\epsilon|y_i|, \]

as long as \(n\epsilon \leq 1/2\). Under what conditions on \(x\) and \(y\) does this bound guarantee small relative error in \(x^T y\)?

78. Let \(A\) be a number in the range \(0.25 \leq A \leq 1\) and consider the Newton iteration

\[ x_{k+1} = (x_k + A/x_k)/2 \]

for computing \(x_\infty = \sqrt{A}\). Assume that the initial guess \(x_0 = (1 + 2A)/3\) has error less than 0.05 for any such \(A\).

(a) Prove that the error \(e_k = x_k - \sqrt{A}\) satisfies

\[ e_{k+1} = e_k^2/2x_k. \]

(b) Determine the number \(k\) of steps necessary to guarantee sixteen-digit accuracy in \(\sqrt{A}\).

84. Let \(a = (a_1, a_2, a_3)^T\) and \(e = (1, 1, 1)^T\). Determine the real number \(x\) which minimizes \(||xe - a||\).

59. (a) Find an upper triangular matrix \(R\) such that \(A = R^T R\) where

\[ A = \begin{bmatrix} 36 & 0 & -6 \\ 0 & 36 & -6 \\ -6 & -6 & 38 \end{bmatrix}. \]
(b) Is $A$ positive definite? Why or why not?
(c) Is $A$ diagonally dominant? Why or why not?
(d) Is $A$ invertible? Why or why not?
(e) Find a permutation matrix $P$, a unit lower triangular matrix $L$ and an upper triangular matrix $U$ such that $PA = LU$. Don’t work too hard.

88. Consider the ordinary differential equation

$$y' = f(t, y)$$

with initial condition $y(0) = y_0$ and the implicit numerical method

$$u_{n+1} = u_n + hf(t_n + h/2, u_n + (1/2)hf(t_{n+1}, u_{n+1}))$$

(a) Show that

$$y'' = f_t(t, y) + f_y(t, y)f(t, y).$$

(b) Write the numerical method as a two-stage Runge-Kutta method and find the Butcher array.

(c) Use Taylor expansion to show that the local truncation error

$$\tau_{n+1} = \frac{y_{n+1} - y_n}{h} - f(t_n + h/2, y_n + (1/2)hf(t_{n+1}, y_{n+1})$$

is $O(h^2)$.

(d) What is the global order of accuracy of the method? Why?

89. We saw that if $f$ satisfies the condition

$$(u - v)(f(t, u) - f(t, v)) \leq 0$$

for all $u$ and $v$, then any two solutions of the differential equation

$$y'(t) = f(t, y(t))$$

approach each other as $t$ increases.

(a) Suppose

$$A = \begin{bmatrix} 18 & 0 & -3 \\ 0 & 18 & -3 \\ 0 & 0 & 18 \end{bmatrix}$$

Show that

$$x^T A x = x^T A^T x > 0$$

for all $x \neq 0$. Note that $A$ is not symmetric.

(b) Show that any two solution vectors $U(t)$ and $V(t)$ of the $3 \times 3$ system

$$y'(t) = -Ay(t)$$
approach each other as $t$ increases.

(c) Show that the numerical solution vector $u_n$ generated by applying implicit Euler to $y' = -Ay$ with $u_0 = y_0$ satisfies

$$|u_n - y_n| \leq nh\tau$$

for $0 \leq nh < \infty$, where $\tau = Mh/2$ and $|y''| \leq M$.

90. Consider an explicit three-stage Runge-Kutta method

$$k_1 = f(t_n + c_1h, u_n),$$
$$k_2 = f(t_n + c_2h, u_n + ha_{21}k_1),$$
$$k_3 = f(t_n + c_3h, u_n + h(a_{31}k_1 + a_{32}k_2)),$$
$$u_{n+1} = u_n + h(b_1k_1 + b_2k_2 + b_3k_3).$$

Suppose the local truncation error of this method is

$$\tau_{n+1} = \frac{y_{n+1} - y_n}{h} - b_1\hat{k}_1 - b_2\hat{k}_2 - b_3\hat{k}_3 = O(h^3).$$

where

$$\hat{k}_i = f(t_n + c_ih, y_n + \sum_{j=1}^{i-1} a_{ij}\hat{k}_j).$$

(a) Show that $x_i = c_i$ and $w_i = b_i$ are the points and weights of some numerical integration rule on $[0, 1]$ which has degree of exactness 3.

(b) Show that $b_1 + b_2 + b_3 = 1$.

(c) Show that $b_1c_1 + b_2c_2 + b_3c_3 = 1/2$.

(d) Show that $b_1c_1^2 + b_2c_2^2 + b_3c_3^2 = 1/3$.

(e) Show that $a_{21} = c_2$.

(f) Show that $a_{31} + a_{32} = c_3$.

(g) For $c_j = j/3$ find $a_{31}$ and $a_{32}$.

(h) Show that using this method to solve $y' = y$ with $u_0 = 1$ gives $u_1 = e^h + O(h^3)$. 

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