(1a) Find an orthonormal basis $e_1, e_2$ for the range of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix} = [a_1|a_2|a_3]$$

**Solution:** The range of $A$ is spanned by $a_1$ and $a_3$ since $a_2$ is proportional to $a_1$. The Gram matrix of $a_1$ and $a_3$ is given by

$$G = \begin{bmatrix} \langle a_1, a_1 \rangle & \langle a_1, a_3 \rangle \\ \langle a_3, a_1 \rangle & \langle a_3, a_3 \rangle \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = R^T R$$

where

$$R = \begin{bmatrix} \sqrt{3} & 1/\sqrt{3} \\ 0 & 2\sqrt{2}/\sqrt{3} \end{bmatrix}.$$

Thus the columns of

$$Q = [a_1|a_3] R^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -1/2\sqrt{6} \\ 0 & \sqrt{3}/2\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}.$$

give an orthonormal basis of the range of $A$. 
(1b) Find the $3 \times 3$ matrix $P$ which projects orthogonally onto the range of $A$, and verify that $P$ is an orthogonal projection.

Solution:

$$P = QQ^T = \begin{bmatrix}
1/\sqrt{3} & 1/\sqrt{6} \\
1/\sqrt{3} & -2/\sqrt{6} \\
1/\sqrt{3} & 1/\sqrt{6}
\end{bmatrix} \begin{bmatrix}
1/\sqrt{3} & 1/\sqrt{6} \\
1/\sqrt{3} & -2/\sqrt{6} \\
1/\sqrt{3} & 1/\sqrt{6}
\end{bmatrix}^T = \frac{1}{6} \begin{bmatrix}
3 & 0 & 3 \\
0 & 6 & 0 \\
3 & 0 & 3
\end{bmatrix}.$$

To verify that $P$ is an orthogonal projection, we check that $P^2 = P$ and $P^T = P$. 
Let $g$ be a $2\pi$-periodic function with $g$ and the derivative $g'$ both in $L^2(-\pi, \pi)$, and let $u(x, t)$ be the solution of the dispersive wave equation

$$u_t(x, t) = u_{xxx}(x, t)$$

which is $2\pi$-periodic in $x$ and satisfies the initial condition $u(x, 0) = g(x)$. Find the complex Fourier coefficients $\hat{u}(k, t)$ in terms of $\hat{g}(k)$.

**Solution:** Multiply the given equation by $e^{-ikx}$, divide by $1/\sqrt{2\pi}$, integrate from $-\pi$ to $\pi$, and integrate by parts to get

$$\hat{u}_t(k, t) = -ik^3\hat{u}(k, t).$$

Apply the integrating factor $e^{ik^3t}$ and integrate to get

$$\hat{u}(k, t) = e^{-ik^3t}\hat{u}(k, 0).$$

Since $u(x, 0) = g(x)$, this simplifies to

$$\hat{u}(k, t) = e^{-ik^3t}\hat{g}(k).$$
(2b) Show that \( u \) is \( 2\pi \)-periodic in \( t \):

\[
u(x, t + 2\pi) = u(x, t)\]

for \( |x| \leq \pi \) and \( t \geq 0 \). Justify the convergence of any infinite series you employ.

**Solution:** Since

\[
\hat{u}(k, t) = e^{-ik^3t}\hat{g}(k),
\]

incrementing \( t \) by \( 2\pi \) gives

\[
\hat{u}(k, t + 2\pi) = e^{-ik^32\pi}\hat{u}(k, t).\]

Since \( k \) is an integer, \( k^3 \) is also an integer. Hence \( \hat{u}(k, t + 2\pi) = \hat{u}(k, t) \) for each \( t \geq 0 \).

Since \( g \) and \( g' \) are both in \( L^2 \), Parseval implies that

\[
\sum_k |\hat{g}(k)|^2 = \|g\|^2 < \infty
\]

and

\[
\sum_k |\hat{g}'(k)|^2 = \sum_k k^2|\hat{g}(k)|^2 = \|g'\|^2 < \infty
\]

Since \( |\hat{u}(k, t)| = |\hat{g}(k)| \) and \( |\hat{u}_x(k, t)| = |\hat{g}'(k)| \), Parseval implies that \( u \) and \( u_x \) are in \( L^2 \) for each \( t \geq 0 \). Hence the Fourier series of \( u \) converges uniformly for each \( t \geq 0 \) and summing the Fourier series gives the result.
(3a) Compute the complex Fourier coefficients \( \hat{f}(k) \) on the interval \(-\pi < x < \pi\) of the function \( f(x) = 1 \) for \(|x| \leq 1\) and 0 otherwise.

Solution:

\[
\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ikx} \, dx = \frac{\sqrt{2/\pi} \sin(k)}{k}.
\]

For \( k = 0 \) take the limit to get \( \hat{f}(0) = \sqrt{2/\pi} \).
(3b) Prove that
\[ \sum_{k=1}^{\infty} \frac{\sin^2(k)}{k^2} = \frac{\pi - 1}{2}. \]
Justify the convergence of any infinite series that you use.

**Solution:** Since \( f \) is in \( L^2 \), Parseval’s equality says
\[ \int_{-1}^{1} 1^2 dx = \sum_k |\hat{f}(k)|^2 \]
or
\[ 2 = \frac{2}{\pi} + 2 \sum_{k=1}^{\infty} \frac{2 \sin^2(k)}{k^2}. \]
Simplifying gives the result.

**Extra credit:** When is
\[ \sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(x) dx? \]

By the Poisson sum formula,
\[ \sum_{n=-\infty}^{\infty} f(n) = \sqrt{2\pi} \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k) \]
where now \( \hat{f} \) is the Fourier transform of \( f \). Since
\[ \hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) dx \]
we will have
\[ \sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(x) dx \]
whenever \( \hat{f}(k) = 0 \) for \( |k| \geq 2\pi \).