**Question 1** Fix $t > 0$ and let

\[ f(x) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}. \]

(a) Compute $\hat{f}(k)$.

**Solution** : We compute the following integral first

\[ I = \int_{\mathbb{R}} e^{-x^2} \, dx. \]

Then

\[ I^2 = \int_{\mathbb{R}^2} e^{-x^2-y^2} \, dx \, dy = \int_0^{2\pi} \int_0^\infty e^{-r^2} r \, dr \, d\theta = \pi - e^{-u}|_0^\infty = \pi. \]

So $I = \sqrt{\pi}$. Now we compute a slightly more complicated integral

\[ I(k) = \int_{\mathbb{R}} e^{-x^2} e^{-ikx} \, dx. \]

We differentiate under the integral and integrate by parts

\[ I'(k) = \int_{\mathbb{R}} e^{-x^2} (-ix) e^{-ikx} \, dx = \frac{-i}{2} \int_{\mathbb{R}} (2x) e^{-x^2} e^{-ikx} \, dx = \frac{-i}{2} \int_{\mathbb{R}} e^{-x^2} (-ik) e^{-ikx} \, dx = \frac{-k}{2} I(k). \]

Thus we need to solve the differential equation $I' = -\frac{k}{2} I$. That means

\[ (\ln(I))' = \frac{I'}{I} = -\frac{k}{2}, \]

so that

\[ \ln(I) = -\frac{k^2}{4} + C \]

and

\[ I = Me^{-\frac{k^2}{4}}. \]

Setting $k = 0$ gives $M = \sqrt{\pi}$ so that

\[ I(k) = \sqrt{\pi} e^{-\frac{k^2}{4}}. \]
Now we compute \( \hat{f}(k) \). We have

\[
\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4} e^{-ikx} \, dx.
\]

Letting \( u = \frac{x}{2\sqrt{t}} \) we get

\[
\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-u^2} e^{-2iku\sqrt{t}} \, du = \frac{1}{\sqrt{2\pi}} I(2k\sqrt{t}) = \frac{1}{\sqrt{2\pi}} \sqrt{\pi e^{-4k^2t/4}} = \frac{1}{\sqrt{2\pi}} e^{-k^2t}.
\]

(b) Compute \( \hat{f}(k) \) by a different method.

Solution: We first introduce the gamma function

\[
\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, dx.
\]

Then \( \Gamma(1) = \int_0^\infty e^{-x} \, dx = 1 \) and

\[
\Gamma(n + 1) = \int_0^\infty x^n e^{-x} \, dx = n \int_0^\infty x^{n-1} e^{-x} \, dx = n\Gamma(n).
\]

Thus it follows that \( \Gamma(n) = (n-1)! \) for \( n \) a positive integer. In general it follows that

\[
z\Gamma(z) = \Gamma(z + 1).
\]

We are also interested in

\[
\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}} \, dx.
\]

Letting \( u = \sqrt{x} \) then \( dx = 2udu \), so

\[
\Gamma\left(\frac{1}{2}\right) = \int_0^\infty 2e^{-u^2} \, du = \sqrt{\pi}.
\]

Then

\[
\Gamma\left(\frac{1}{2}\right) = \frac{2n-1}{2} \cdots \frac{1}{2} \sqrt{\pi} = \frac{(2n)!}{2^{2n}n!} \sqrt{\pi}.
\]

We recompute \( I(k) \) with a Taylor series and then proceed as in the previous section. First,

\[
I(k) = \int_{\mathbb{R}} e^{-y^2} e^{-iky} \, dy = \sum_{n=0}^\infty \frac{1}{n!} \int_{\mathbb{R}} e^{-y^2} (iky)^n \, dy.
\]
Now the odd terms are all 0, and the even terms become integrals on $[0, \infty)$.

$$I(k) = 2 \sum_{n=0}^{\infty} \frac{(-ik)^{2n}}{(2n)!} \int_0^\infty e^{-y^2} y^{2n} \, dy.$$ 

Now we let $u = y^2$ to get

$$I(k) = \sum_{n=0}^{\infty} \frac{(-ik)^{2n}}{(2n)!} \int_0^\infty e^{-u} u^{n-\frac{1}{2}} \, du = \sum_{n=0}^{\infty} \frac{(-ik)^{2n}}{(2n)!} \Gamma(n + \frac{1}{2}).$$

Combining with our previous result gives

$$I(k) = \sum_{n=0}^{\infty} \frac{(-ik)^{2n}}{(2n)!} \int_0^\infty e^{-u} u^{n-\frac{1}{2}} \, du = \sum_{n=0}^{\infty} \frac{(-ik)^{2n}}{(2n)!} \Gamma(n + \frac{1}{2}) \frac{1}{2^{2n} n!} \sqrt{\pi} = \sqrt{\pi} e^{-\frac{k^2}{4}}.$$

Now that we’ve computed $I(k)$ as in a), we can proceed as we did in a) to obtain $\hat{f}(k)$.

**Question 2** Fix $A$ and $\alpha > 0$ and let $h(x) = Ae^{-\alpha x}$ for $x \geq 0$ and 0 otherwise.

(a) Compute $\hat{h}(k)$.

**Solution**: We compute directly

$$\hat{h}(k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty Ae^{-(\alpha + ik)x} \, dx = \frac{A}{(\alpha + ik)\sqrt{2\pi}}.$$

(b) Let

$$f(x) = e^{-x} (\sin 5x + \sin 3x + \sin x + \sin 40x)$$

for $0 \leq x \leq \pi$ and 0 otherwise. Compute $\hat{f}(k)$.

**Solution** First let $s_m(x) = e^{-x} \sin(mx)$ for $0 \leq x \leq \pi$. If we let $f_m(x) = \sin(mx)$. Then

$$\hat{s}_m(k) = \hat{f}_m(k - i)$$

since $e^{-x} = e^{-i(-i)x}$.

Now we compute

$$\hat{f}_m(k) = \frac{1}{\sqrt{2\pi}} \int_0^\pi e^{ikx - i\frac{\pi}{2}} e^{-imx} \, dx = \frac{1}{2i\sqrt{2\pi}} \left( \frac{e^{i\pi} e^{-ik\pi} - 1}{i(m-k)} + \frac{e^{-i\pi} e^{-ik\pi} - 1}{i(m+k)} \right).$$
Simplifying gives
\[
\hat{f}_m(k) = \frac{1}{2i\sqrt{2\pi}}((-1)^m e^{-ik\pi} - 1) \left( \frac{1}{m-k} + \frac{1}{m+k} \right) = \frac{1}{\sqrt{2\pi}} \frac{m}{k^2 - m^2}((-1)^m e^{-ik\pi} - 1).
\]

Then our answer is
\[
\sum_{k \in \{1, 3, 5, 40\}} \hat{f}_m(k - i).
\]

(c) Plot \(h \ast f(x)\) for \(0 \leq x \leq \pi\) and find interesting values of \(A\) and \(\alpha\). Discuss.

**Solution**: Since we want to preserve the signal, we want \(\int_{\mathbb{R}} h(x)dx = 1\). Then we want to choose \(A = \alpha\). We plot various levels of \(\alpha\). For larger values of \(\alpha\), the high frequencies are damped somewhat but the overall shape is unchanged. For smaller values of \(\alpha\), the lower frequency signals are filtered out together with the higher frequencies, changing the shape of the curve noticeably. Here are the graphs for \(\alpha = 10, 5, 1\).

![Graphs for different values of alpha](image)

**Question 3** Fix \(a > 0\) and let \(f(x) = \sin^2 ax\) for \(|x| \leq \pi\) and 0 otherwise.

(a) Compute \(\hat{f}(k)\).

**Solution**: We first prove an auxiliary result that for a function \(g(x)\)
\[
\sin(mx)g(x)(k) = \frac{1}{2i\sqrt{2\pi}} e^{imx} g(x) e^{-ikx} - e^{-imx} g(x) e^{-ikx} dx = \frac{1}{2i} (\hat{g}(k - m) - \hat{g}(k + m)).
\]

Then
\[
\sin^2(mx)g(x)(k) = \frac{1}{4} (\hat{g}(k + 2m) - 2\hat{g}(k) + \hat{g}(k - 2m)).
\]

Let \(\chi(x)\) be the indicator function of \([-\pi, \pi]\). Then we’ve shown previously that
\[
\hat{\chi}(k) = \frac{\sqrt{2 \sin(k\pi)}}{k}.
\]
Then since \( f(x) = \chi(x) \cdot \sin^2(ax) \), we have

\[
\hat{f}(k) = \frac{1}{4} \left( -\hat{\chi}(k + 2a) + 2\hat{\chi}(k) - \hat{\chi}(k - 2a) \right).
\]

(b) Compute \( \hat{f}(k) \) by a different method.

We observe that \( \sin^2(ax) = \frac{1}{4} (e^{2iax} - 2 + e^{-2iax}) \) so that

\[
\hat{f}(k) = \frac{1}{4} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} (-e^{2iax} + 2 - e^{-2iax}) e^{-ikx} \, dx
\]

so that the answer is seen to be

\[
\frac{1}{4} (-\hat{\chi}(k + 2a) + 2\hat{\chi}(k) - \hat{\chi}(k - 2a))
\]

as before.

(c) Explain the rate at which \( |\hat{f}(k)| \) decreases as \( |k| \to \infty \).

**Solution:** We compute

\[
\hat{f}(k) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{4} \cdot \left( -\frac{\sin(k\pi + 2a\pi)}{k + 2a} + \frac{2\sin(k\pi)}{k} - \frac{\sin(k\pi - 2a\pi)}{k - 2a} \right).
\]

In general this will be \( O\left(\frac{1}{k}\right) \) since \( f \) is discontinuous, however if \( a \in \mathbb{Z} \) then

\[
\hat{f}(k) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{4} \cdot \sin(k\pi) \left( -\frac{1}{k + 2a} + \frac{2}{k} - \frac{1}{k - 2a} \right) = \sqrt{\frac{2}{\pi}} \cdot \sin(k\pi) \cdot \frac{-2a^2}{k^3 - 4a^2 k}
\]

which is \( O\left(\frac{1}{k^3}\right) \) since the function and its derivative are now continuous.

**Question 4** Let \( \varphi(x) = 1 \) for \( 0 \leq x < 1 \) and 0 otherwise. Show that \( \varphi \ast \varphi(x) = 1 - |x - 1| \) if \( 0 \leq x < 2 \) and 0 otherwise.

**Solution:** By definition

\[
\varphi \ast \varphi(x) = \int_{\mathbb{R}} \varphi(y)\varphi(x-y) \, dy = \int_{0}^{1} \varphi(x-y) \, dy
\]
If $0 \leq x \leq 1$ then

$$\varphi \ast \varphi(x) = \int_0^x 1 \, dy = x.$$ 

Conversely if $1 \leq x \leq 2$ then

$$\varphi \ast \varphi(x) = \int_{x-1}^1 1 \, dy = 2 - x.$$ 

By checking the cases $x < 1$ and $x \geq 1$ we see that $\varphi \ast \varphi(x) = 1 - |x - 1|$.

Intuitively, what is happening here is we are taking the inner product of the functions $\varphi(y)$ and $\varphi(y - x)$, where the second function is gotten by taking the first and reflecting across the $y$–axis and then shifting $x$ units to the right. When $x = 0$, the functions don’t overlap at all. As $x$ increases to $x = 1$, the functions overlap more and more until they overlap completely at $x = 1$. Then as $x$ increases to $x = 2$, they overlap less until they don’t overlap at all. The inner product is just the area of the overlap, so the function increases linearly to 1 then decreases linearly to 0.