Question 1  (a) Compute an orthonormal basis for the column space of

\[ A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \\ 1/3 & 1/4 \end{bmatrix} = QR \]

Solution : Suppose \( A = [v, w] \), let \( u_1 = v \). Then

\[ u_2 = w - \frac{\langle w, v \rangle}{\langle v, v \rangle} v = \begin{bmatrix} -.051 \\ .0578 \\ .0663 \end{bmatrix}. \]

Then we normalize to get

\[ e_1 = \begin{bmatrix} .8571 \\ .4286 \\ .2857 \end{bmatrix}, e_2 = \begin{bmatrix} -.5016 \\ .5685 \\ .6521 \end{bmatrix}. \]

(b) find the orthonormal and upper-triangular matrices \( Q \) and \( R \).

Solution : Either \( Q = [e_1, e_2] \) whereupon \( R = Q^* A \) or compute the QR factorization in Matlab. Either way,

\[ A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \\ 1/3 & 1/4 \end{bmatrix} = QR = \begin{bmatrix} .8571 & -.5016 \\ .4286 & .5685 \\ .2857 & .6521 \end{bmatrix} \begin{bmatrix} 1.1667 & .6429 \\ 0 & .1017 \end{bmatrix}. \]

(c) Compute the orthogonal projection \( P \) onto the range of \( A \).

Solution : Either \( P = QQ^* \), \( P = e_1 e_1^* + e_2 e_2^* \), or

\[ P = A(A^* A)^{-1} A^*. \]

To show that this is indeed an orthogonal projection note that \( P = P^2 = P^* \).

To show that it is onto the range of \( A \), let \( u = Aw \in Range(A) \). Then

\[ Pu = A(A^* A)^{-1} A^* Aw = Aw = u. \]

Explicitly,

\[ P = \begin{bmatrix} .9863 & .0822 & -.0822 \\ .0822 & .5068 & .4932 \\ -.0822 & .4932 & .5068 \end{bmatrix}. \]
Question 2  Find $a_0$ and $a_1$ minimizing
\[ F(a_0, a_1) = \int_0^1 |a_0 + a_1 x - e^{ix}|^2 \, dx. \]

Solution : This is equivalent to finding the vector $a_0 + a_1 x$ closest to $f(x) = e^{ix}$ in $L^2(0,1)$, which is equivalent to projecting $e^{ix}$ onto the span of 1 and $x$. Letting $v_1(x) = 1, v_2(x) = x$ then

\[ V^*V = [ <v_i, v_j> ] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}. \]

Then the projection $P$ is given by
\[ P = V(V^*V)^{-1}V^* = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix} \begin{bmatrix} 1^* \\ x^* \end{bmatrix}. \]

So to compute $Pf$ we first compute
\[ 1^*f = <f, 1> = \int_0^1 e^{ix} \, dx = -ie^i + i \]
and
\[ x^*f = <f, x> = \int_0^1 xe^{ix} \, dx = -ie^i + e^i - 1. \]

That means
\[ Pf(x) = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} e^i(-6 + 2i) + 6 + 4i \\ e^i(12 - 6i) - 12 - 6i \end{bmatrix} \]
so that $a_0 = e^i(-6 + 2i) + 6 + 4i$ and $a_1 = e^i(12 - 6i) - 12 - 6i$.

Question 3  (a) Find an orthonormal basis for the 3-dimensional subspace of $L^2(-1,1)$ spanned by 1, $x$ and $x^2$.
(b) Interpret as a $QR$ factorization.

Solution : Let $V = \begin{bmatrix} 1 & x & x^2 \end{bmatrix}$. Then we directly compute
\[ V^*V = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix}. \]
We compute the Cholesky decomposition $V^*V = R^*R$ where

$$ R = \begin{bmatrix} \sqrt{2} & 0 & \sqrt{\frac{7}{5}} \\ 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & \sqrt{\frac{8}{45}} \end{bmatrix} $$

where $R$ is as in the $QR$ factorization $V = QR$. Then our orthonormal basis is given by the columns of $Q = VR^{-1}$ where

$$ R^{-1} = \begin{bmatrix} \sqrt{\frac{1}{2}} & 0 & -\sqrt{\frac{5}{8}} \\ 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{45}{8}} \end{bmatrix} . $$

So we can let

$$ e_1 = \sqrt{\frac{1}{2}}, \ e_2 = \sqrt{\frac{3}{2}}x, \ e_3 = \sqrt{\frac{1}{8}}(-\sqrt{5} + \sqrt{45}x^2). $$

**Question 4** Let

$$ H^1 = H^1(0,1) = \{ f \in L^2(0,1) | f' \in L^2(0,1) \} $$

with inner product

$$ < f, g > = \int_0^1 f(x)g(x) + f'(x)g'(x) \ dx. $$

(For simplicity assume all functions are real-valued.)

(a) Show that every $f \in H^1$ is continuous and bounded on $(0,1)$.

**Solution**: Let $f \in H^1$ then using the fundamental theorem of calculus,
we have

\[ |f(x) - f(y)| = \left| \int_x^y f'(s) \, ds \right| \]
\[ \leq \int_x^y |f'(s)| \cdot 1 \, ds \]
\[ \leq \sqrt{\int_x^y |f'(s)|^2 \, ds} \sqrt{\int_x^y 1^2 \, ds} \]
\[ \leq \sqrt{|x - y|} \sqrt{\int_0^1 |f'(s)|^2 \, ds} \]
\[ \leq \sqrt{|x - y|} \| f' \|. \]

Then \( f(x) - f(y) \to 0 \) as \( x \to y \) so \( f \) is uniformly continuous on \((0, 1)\), which thus extends to continuity on \([0, 1]\). Then \( f \) is bounded by the extreme value theorem.

(b) Let \( g \in H^1 \) and suppose also that \( g' \) and \( g'' \) are continuous except at some point \( x_0 \in (0, 1) \). Show that

\[ <f, g> = f(1)g'(1) + f(x_0) \left( g'(x_0^-) - g'(x_0^+) \right) - f(0)g'(0) + \int_0^1 f(x) (g(x) - g''(x)) \, dx \]

for every \( f \in H^1 \).

**Solution**: We first compute the integrals

\[ \int_{x_0}^1 f'(x)g'(x) \, dx = f(x)g'(x) \bigg|_{x_0}^1 - \int_{x_0}^1 f(x)g''(x) \, dx = f(1)g'(1) - f(x_0)g'(x_0^+) - \int_{x_0}^1 f(x)g''(x) \, dx \]

and

\[ \int_0^{x_0} f'(x)g'(x) \, dx = f(x)g'(x) \bigg|_0^{x_0} - \int_0^{x_0} f(x)g''(x) \, dx = f(x_0)g'(x_0^-) - f(0)g'(0) - \int_0^{x_0} f(x)g''(x) \, dx. \]

Thus

\[ <f, g> = f(1)g'(1) - f(x_0)g'(x_0^+) + f(x_0)g'(x_0^-) - f(0)g'(0) + \int_0^1 f(x)(g(x) - g''(x)) \, dx \]
(c) Find \( g \in H^1 \) such that 
\[ <f, g> = f(x_0) \]
for every \( f \in H^1 \).

**Solution:** By the previous exercise, we see that this is equivalent to finding \( g \) such that
\[-g'' + g = 0, \ x \neq x_0 \]
\[ g'(0) = 0 \]
\[ g'(1) = 0 \]
\[ g'(x_0^-) - g'(x_0^+) = 1 \]
\[ g(x_0^-) - g(x_0^+) = 0. \]

We seek the solution in the form
\[ g(x) = \begin{cases} 
A(e^x + e^{-x}) & \text{if } x < x_0 \\
B(e^{-x} + e^{1-x}) & \text{if } x > x_0 
\end{cases} \]
so that \( g'(0) = g'(1) = 0 \) automagically. The jump conditions at \( x = x_0 \) give two linear equations in two unknowns \( A \) and \( B \):
\[ A(e^{x_0} + e^{-x_0}) = B(e^{x_0-1} + e^{1-x_0}) \]
and
\[ A(e^{x_0} - e^{-x_0}) = B(e^{x_0-1} - e^{1-x_0}) + 1. \]
Adding and subtracting gives
\[ 2Ae^{x_0} = 2Be^{x_0-1} + 1 \]
and
\[ 2Ae^{-x_0} = 2Be^{1-x_0} - 1. \]
Solving the 2x2 linear system gives
\[ A = \frac{1}{2(e^1 - e^{-1})} (e^{1-x_0} + e^{x_0-1}) \]
and
\[ B = \frac{1}{2(e^1 - e^{-1})} (e^{-x_0} + e^{x_0}). \]
Simplifying gives the symmetric formula
\[ g(x) = \frac{1}{2(e^1 - e^{-1})} \begin{cases} 
(e^{x_0-1} + e^{1-x_0})(e^x + e^{-x}) & \text{if } x < x_0 \\
(e^{x_0} + e^{-x_0})(e^{x-1} + e^{1-x}) & \text{if } x > x_0
\end{cases} \]

In terms of \( x_\leq = \min(x, x_0) \) and \( x_\geq = \max(x, x_0) \) this is just
\[ g(x) = \frac{1}{\sinh(1)} \cosh(x_\leq) \cosh(1 - x_\geq), \]
a formula which makes everything obvious.

**Question 5** Given \( n + 1 \) distinct points \( 0 < x_0 < x_1 < \cdots < x_n < 1 \), let \( P_n \) be the linear operator which takes \( f \in H^1 \) into the unique degree-\( n \) polynomial
\[ p_n(x) = P_n f(x) = \sum_{j=0}^{n} L_j(x) f(x_j) \]
which interpolates the \( n + 1 \) values \( f(x_j) \). Here \( L_j(x) \) are the degree-\( n \) polynomials satisfying
\[ L_i(x_j) = \delta_{ij}. \]

(a) Show that \( P_n \) is a projection.

**Solution**: We claim \( P_n^2 = P_n \). Let \( f \in H^1 \). Then \( P_n f \) is the unique polynomial of degree \( n \) such that \( f(x_j) = P_n f(x_j) \). Then \( P_n^2 f = P_n (P_n f) \) is the unique polynomial of degree \( n \) such that \( P_n^2 f(x_j) = P_n f(x_j) = f(x_j) \). Since \( P_n^2 f \) is interpolating the same values as \( P_n f \), we have \( P_n^2 f = P_n f \) by uniqueness of interpolation.

(b) Find the adjoint operator \( P_n^* g \) for \( g \in H^1 \).

**Solution**: Let \( g_j \) be such that
\[ <f, g_j> = f(x_j), \forall f \in H^1. \]
By the previous exercise, we see that \( g_j \) is given piecewise by functions of the form \( Ae^x + Be^{-x} \). By definition
\[ P_n f(x) = \sum_{j=0}^{n} L_j(x) <f, g_j>. \]
Then for any \( g \in H^1 \) we have

\[
< P_n f, g > = \sum_{j=0}^n L_j < f, g_j >, g >
\]

\[
= \sum_{j=0}^n < f, g_j > < L_j, g >
\]

\[
= \sum_{j=0}^n g_j < L_j, g >
\]

\[
= < f, P^*_n g > .
\]

Thus

\[
P^*_n g(x) = \sum_{j=0}^n g_j(x) < L_j, g >
\]

\[
= \sum_{j=0}^n g_j(x) \int_0^1 L_j(y)g(y) + L'_j(y)g'(y)dy
\]

\[
= \int_0^1 \sum_{j=0}^n g_j(x)L_j(y)g(y) + \sum_{j=0}^n g_j(x)L'_j(y)g'(y)dy.
\]

(c) Show that \( P_n \) is not an orthogonal projection.

**Solution**: By the previous problem, \( P^*_n g \) is a linear combination of functions which are piecewise given by \( Ae^x + Be^{-x} \). Thus \( P^*_n g \) satisfies the differential equation \( f'' = f \) away from the points \( x_0, \ldots, x_n \). However, no nonzero polynomial can satisfy this differential equation. Since the range of \( P_n \) consists of polynomials, it follows that the ranges of \( P_n \) and \( P^*_n \) are not equal, so \( P^*_n \neq P_n \), so \( P_n \) is not an orthogonal projection.

(d) Find a basis \( \{e_0, e_1, e_2, e_3\} \) for the range of \( P_3 \) which is orthogonal in the \( H^1 \) inner product.

**Solution**: We choose the standard basis \( \{1, x, x^2, x^3\} \) and compute the Gram
matrix

\[
V^*V = \begin{bmatrix}
\langle 1, 1 \rangle & \langle 1, x \rangle & \langle 1, x^2 \rangle & \langle 1, x^3 \rangle \\
\langle x, 1 \rangle & \langle x, x \rangle & \langle x, x^2 \rangle & \langle x, x^3 \rangle \\
\langle x^2, 1 \rangle & \langle x^2, x \rangle & \langle x^2, x^2 \rangle & \langle x^2, x^3 \rangle \\
\langle x^3, 1 \rangle & \langle x^3, x \rangle & \langle x^3, x^2 \rangle & \langle x^3, x^3 \rangle \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

\[
= R^*R
\]

where

\[
S = R^{-1} = \begin{bmatrix}
1 & -0.48038 & 0.28630 & -0.01701 \\
0 & 0.96077 & -1.71780 & 2.24532 \\
0 & 0 & 1.71780 & -6.63390 \\
0 & 0 & 0 & 4.42260 \\
\end{bmatrix}
\]

Then our orthonormal basis is given by

\[
e_1(x) = 1 \\
e_2(x) = s_{12} + s_{22}x \\
e_3(x) = s_{13} + s_{23}x + s_{33}x^2 \\
e_4(x) = s_{14} + s_{24}x + s_{34}x^2 + s_{44}x^3.
\]

(e) Find the orthogonal projection \( Q_3 \) onto the range of \( P_3 \). Express \( Q_3 \) as an integrodifferential operator

\[
Q_3 f(x) = \int_0^1 K(x, y) f(y) + K'(x, y) f'(y) \, dy
\]

and compute the kernels \( K \) and \( K' \) in terms of \( \{e_0, e_1, e_2, e_3\} \).
Solution: As usual,

\[ Q_3f(x) = \sum_{j=1}^{4} e_j(x) < f, e_j > \]

\[ = \sum_{j=1}^{4} e_j(x) \int_0^1 f(y)e_j(y) + f'(y)e'_j(y)dy \]

\[ = \int_0^1 \left( \sum_{j=1}^{4} e_j(x)e_j(y) \right)f(y) + \left( \sum_{j=1}^{4} e_j(x)e'_j(y) \right)f'(y)dy \]

\[ = \int_0^1 K(x,y)f(y) + K'(x,y)f'(y)dy. \]

(f) Show that \( q = Q_3f \) minimizes the \( H^1 \) norm \( \|q - f\| \) over \( q \) in the range of \( P_3 \).

Solution: We show generally for an inner product space \( V \) and a finite dimensional subspace \( U \subset V \) that the projection \( u = \pi(v) \) of a vector \( v \in V \) is the point that minimizes \( \|u - \pi(v)\| \). First observe the orthogonal decomposition

\[ v - u = (v - \pi(v)) - (u - \pi(v)). \]

This is an orthogonal decomposition because \( v - \pi(v) \in U^\perp \) and \( u - \pi(v) \in U \). Then

\[ \|v - u\|^2 = \|v - \pi(v)\|^2 + \|u - \pi(v)\|^2 \geq \|v - \pi(v)\|^2 \]

and minimization occurs when \( u = \pi(v) \).